CHAPTER 3

[Part 2]

PERMUTATION & COMBINATION

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Permutations vs Combinations

What's the difference?

The **difference** between **combinations** and **permutations** is ordering. With **permutations** we care about the order of the elements, whereas with **combinations** we don't.

Example:

a) "My fruit salad is a combination of apples, grapes and bananas"

Combination => order of the fruits doesn't matter.

b) "The combination of locker number was 5647"

Permutation => The order does matter because if you enter "6547" into your locker, it would not work, nor would "7465". It has to be **exactly** 5647.

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Permutations vs Combinations (cont'd)

- If the order **doesn't** matter, it is a **combination**.
  - Combination means **selection** of things.
  - The word **selection** is used, when the order of things has no importance.

- If the order **does** matter, it is a **Permutation**.
  - Permutation means **arrangement** of things.
  - The word **arrangement** is used, if the order of things is considered.

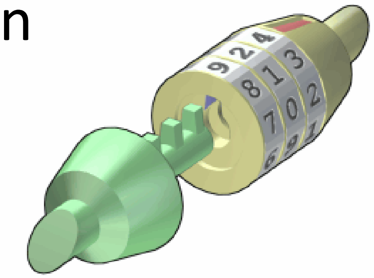
Permutations

Permutations

There are basically two types of permutation:

No Repetition: For example the first three people in a running race. You can't be first and second.

Repetition is Allowed: For example, the permutation lock (in picture). It could be "333".



Permutations – No Repetition

In this case, you have to **reduce** the number of available choices each time.

Example:

What order could 16 pool balls be in, if no repetition (i.e., After choosing, say, number "14" you can't choose it again)?

Solution:

Your first choice would have 16 possibilities, and your next choice would then have 15 possibilities, then 14, 13, etc. The total permutations would be:

$$\mathbf{16 \times 15 \times 14 \times 13 \times \ldots \times 1 = 20{,}922{,}789{,}888{,}000}$$

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Permutations – No Repetition (cont'd)

But maybe you don't want to choose them all, just 3 of them, so that would be only:

$$16 \times 15 \times 14 = 3{,}360$$

In other words, there are 3,360 different ways that 3 pool balls could be selected out of 16 balls.

Permutations – No Repetition (cont'd)

But how do we write that mathematically?

- We use the "factorial function"
- The **factorial function** (symbol: **!**) just means to multiply a series of **descending** natural numbers.

Example:

$4! = 4 \times 3 \times 2 \times 1 = 24$

$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5{,}040$

$1! = 1$

Permutations – No Repetition (cont'd)

- So, if you wanted to select **all** of the billiard balls, the permutations would be:

$$16! = 20{,}922{,}789{,}888{,}000$$

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Permutations – No Repetition (cont'd)

Summary:

- A permutation of n distinct elements $x_1, \ldots, x_n$ is an ordering of the n elements $x_1,..x_n$.
- There are $n!$ permutations of n elements

$$p(n) = n!$$

$$= n \times (n\text{-}1) \times (n\text{-}2)\ldots 2\times 1$$

Example:

How many permutations of three elements?

Solution:

There are 6 permutations of three elements. If the elements are denoted A, B, C, the six permutations are

ABC, ACB, BAC, BCA, CAB, CBA

$$3! = 3 \times 2 \times 1 = 6$$

Example:

How many permutations of letters ABCDEF contain the substring DEF?

DEF	A	B	C

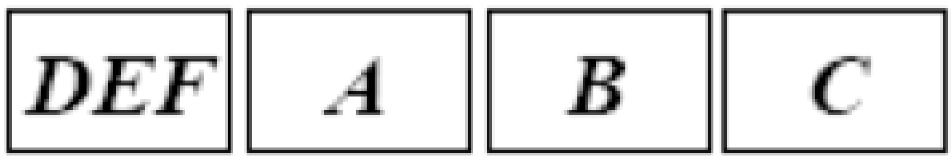
Solution:

There are 24 permutations of 4 elements.

$$4! = 24$$

Example:

How many permutations of letters ABCDEF contain the substring DEF together in any order?



Solution:

Permutation for DEF: **3! = 6**

Permutation for DEF, A, B, C: **4! = 24**

Therefore, **6 x 24 = 144**

r-Permutations (Permutations without repetition)

- An *r*-permutations of n (distinct) elements $x_1, \ldots, x_n$ is an ordering of the *r*-element subset of $\{x_1,..x_n\}$.
- The number of *r*-permutations of a set of n distinct element is,

$$P(n,r) = {}^{n}P_{r} = {}_{n}P_{r} = \frac{n!}{(n-r)!}$$

Example:

Back to the billiard balls example. If you wanted to select just 3, then you have to stop the multiplying after 14. How do you do that?

There is a neat trick. You divide by **13!**

$$\frac{16\times15\times14\times13\times12\times\ldots}{13\times12\times\ldots}=16\times15\times14=3{,}360$$

Do you see? **16! / 13! = 16 × 15 × 14**

Example (cont'd):

Our "order of 3 out of 16 pool balls example" would be:

$$P(16,3) = \frac{16!}{(16-3)!} = \frac{16!}{13!} = \frac{20,922,789,888,000}{6,227,020,800} = 3,360$$

which is just the same as:

$$\mathbf{16 \times 15 \times 14 = 3,360}$$

Example:

How many ways can first and second place be awarded to 10 people?

Solution:

$$P(10,2) = \frac{10!}{(10-2)!} = \frac{10!}{8!} = \frac{3,628,800}{40,320} = 90$$

which is just the same as: $10 \times 9 = 90$

Example:

How many 2-permutations of **a**, **b**, **c**?

Solution:

2-permutation of **a**, **b**, **c** are,

ab, **ac**, **ba**, **bc**, **ca**, **cb**

$$P(3,2) = \frac{3!}{(3-2)!} = \frac{3!}{1!} = 3 \times 2 = 6$$

Example:

In how many ways can we select a chairperson, vice-chairperson, secretary, and treasurer from a group of 10 persons?

Solution:

$$P(10,4) = \frac{10!}{(10-4)!} = \frac{10!}{6!} = 5040$$

Example:

In how many dance pairs (dance pairs means a pair (W,M), where W stands for a women and M for man), can be formed from a group of 6 women and 10 men?

Solution:

$$P(10,6) = \frac{10!}{(10-6)!} = \frac{10!}{4!} = 151200$$

Example:

In how many ways can six boys and five girls stand in a line so that no two girls are next to each other?

Solution:

- G B_1G B_2G B_3G B_4G B_5G B_6G

$$P(7,5) = \frac{7!}{(7-5)!} = \frac{7!}{2!} = 2520 \qquad P(6,6) = 6! = 720$$

$$P(7,5) \times P(6,6) = 1814400$$

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r-Permutations (Permutations with repetition allowed)

- An *r*-permutations of a set of n (distinct) elements if repetitions are allowed is,

$$P(n, r) = n^r$$

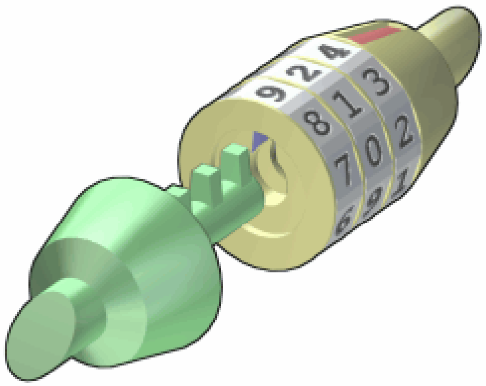
- In other words:
  - ✓There are n possibilities for the first choice,
  - ✓THEN there are n possibilities for the second choice,
  - ✓and so on, multiplying each time.
- Which is easier to write down using an exponent of r:

$$n \times n \times \ldots (r \text{ times}) = n^r$$

Example:

In the lock, there are 10 numbers to choose from (0,1,..9) and you choose 3 of them:

$$10 \times 10 \times \ldots \text{ (3 times)} = 10^3 = 1{,}000 \text{ permutations}$$



Example:

How many five-letters word can be formed from the letters A-Z?

Solution:

$$P(26,5) = 26^5$$

Permutations (n objects, k different types)

For a collection of n objects of k different types, the total number of different arrangements of these n objects is,

$$P(n) = \frac{n!}{(n_1!\, n_2! \ldots n_k!)}$$

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Example:

Find the number of different ways the letters of the word ASSESSMENT can be arranged?

Solution:

There are 10 letters (objects) with 6 types: A(1), S(4), E(2), M(1), N(1), T(1). Total number of different arrangements is,

$$P(10) = \frac{10!}{(1!\,4!\,2!\,1!\,1!\,1!)} = 75600$$

Permutations – Circular Arrangement

- The permutation in a row or along a line has a beginning and an end.
- But there is nothing like beginning or end or first and last in a **circular permutation**.
- In circular permutations, we consider one of the objects as fixed and the remaining objects are arranged as in linear permutation.

Permutations – Circular Arrangement (cont'd)

- There are two cases of circular-permutations:
  - If clockwise and anti clockwise orders **are different**, then total number of circular-permutations is given by

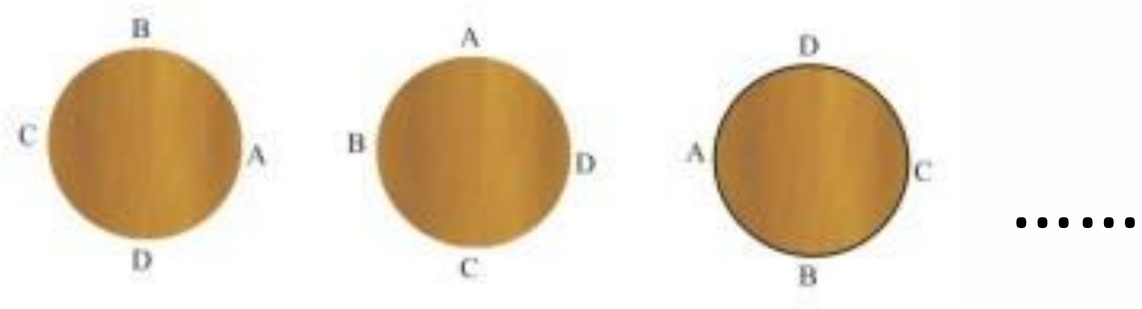
$$(n-1)!$$

  - If clockwise and anti clockwise orders are taken as **not different**, then total number of circular-permutations is given by

$$\frac{(n-1)!}{2!}$$

Permutations – Circular Arrangement (cont'd)

- Let's consider that 4 persons **A**, **B**, **C**, and **D** are sitting around a round table.
- Shifting **A**, **B**, **C**, **D**, one position in anti-clockwise direction, we get the following arrangements:-



- Thus, if 4 persons are sitting at a round table, they can be shifted four times.
- But these four arrangements will be the same, because the sequence of **A**, **B**, **C**, **D**, is same.

Permutations – Circular Arrangement (cont'd)

- But if **A**, **B**, **C**, **D**, are sitting in a row, and they are shifted, then the four **linear-arrangement** will be different.

Permutations – Circular Arrangement (cont'd)

- Hence, the number of circular permutations is

$$P_n = (n-1)!$$

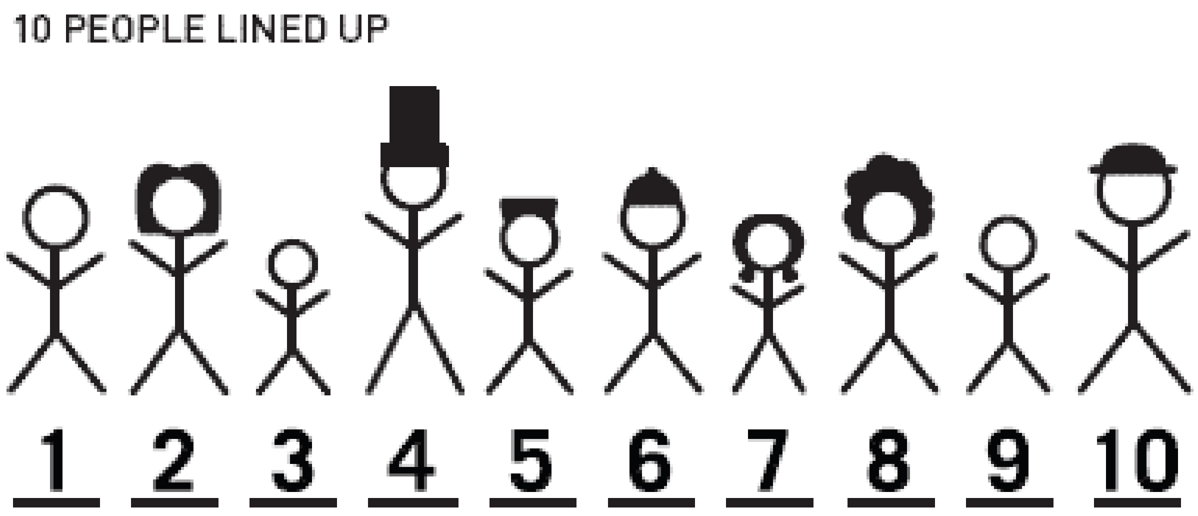
- The number is $(n\text{-}1)!$ instead of the usual factorial, $n!$ since all cyclic permutations of objects are equivalent because the circle can be rotated.

Permutations – Circular Arrangement (cont'd)

- Thus, the number of permutations of 4 objects in a row = 4!
- Where as the number of circular permutations of 4 objects is (4-1)! = 3!

Example:

Suppose we are expecting ten people for dinner. How many ways can we seat them around a circular table?



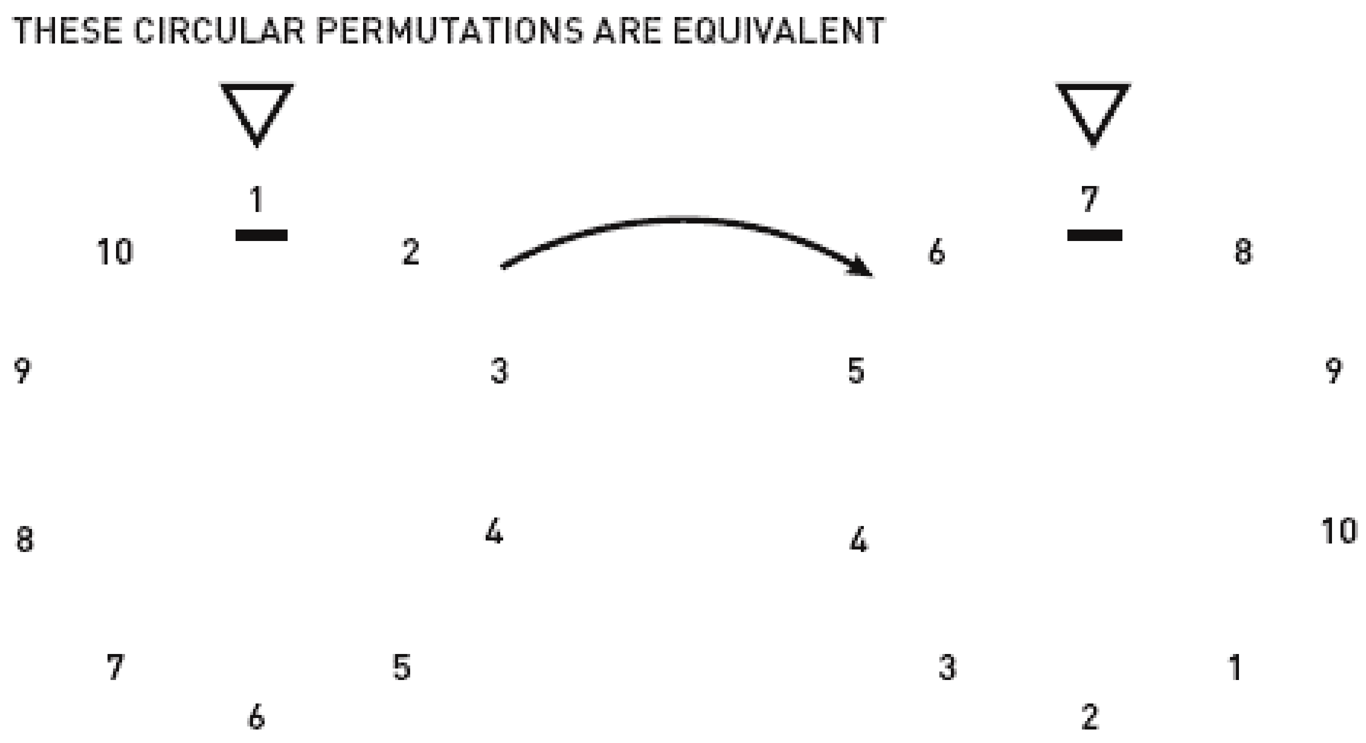
Example - Solution:

- First, let's think about how many ways we can line them up.
- As we indicated above, there will be 10! ways to line up ten guests
- 10 for the first position, 9 for the second, 8 for the third, and so on.

$$10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$$

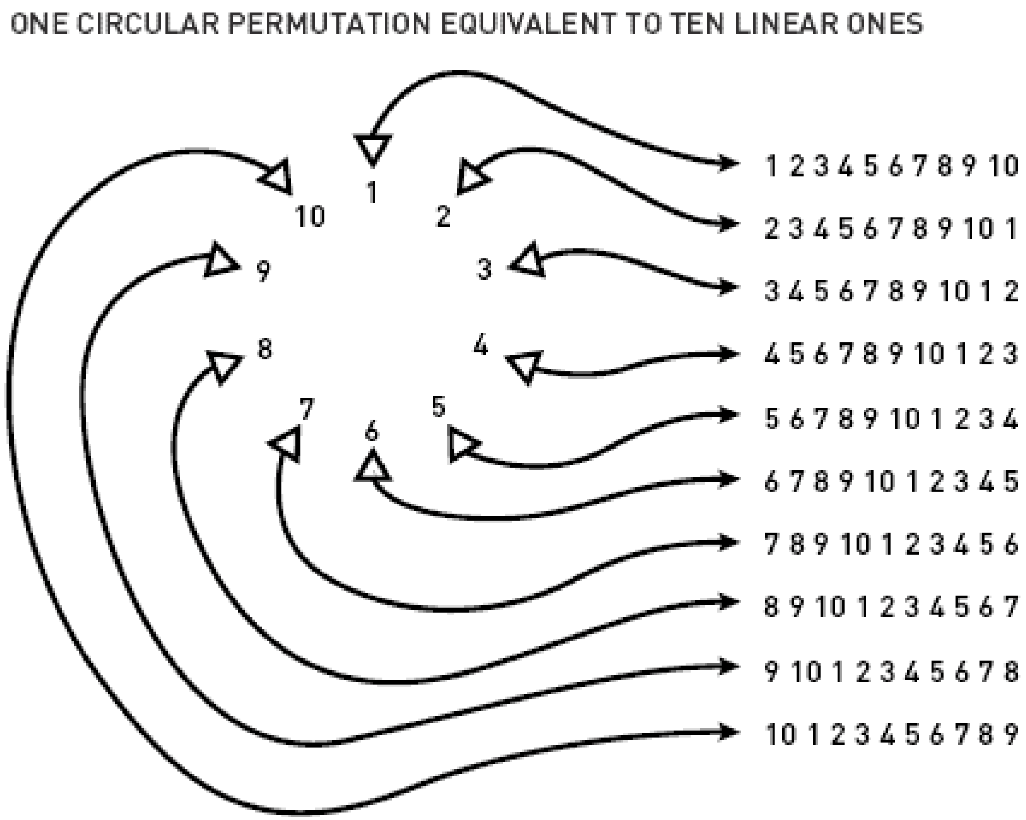
Example – Solution (cont'd):

How does this change if they are seated around a circular table?



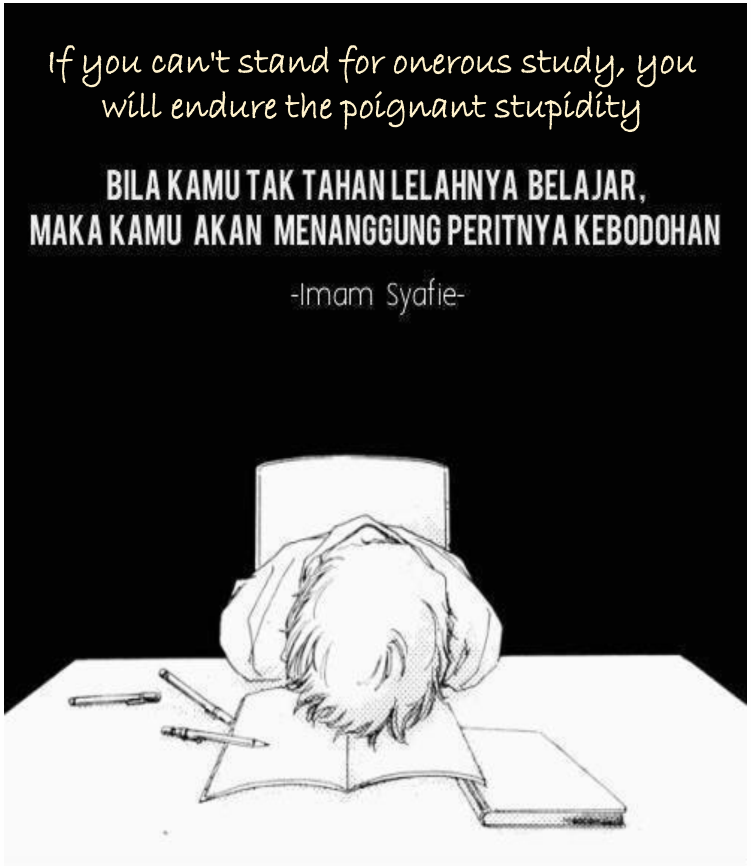
Example – Solution (cont'd):

- Notice that every **circular arrangement** corresponds to ten different linear arrangements.



Example – Solution (cont'd):

- Using our reasoning from before, we can see that the number of circular arrangements is equal to the number of linear arrangements, 10!, divided by ten to compensate for the fact that each circular permutation corresponds to ten different linear ones.
- This gives us the number of ways to arrange ten guests around a table.
- We can generalize this to say that ***n*** elements can be arranged in **(*n*-1)!** ways around a circle.


If you can't stand for onerous study, you will endure the poignant stupidity
BILA KAMU TAK TAHAN LELAHNYA BELAJAR,
MAKA KAMU AKAN MENANGGUNG PERITNYA KEBODOHAN
-Imam Syafie-

Exercises

Exercise #1

How many ways can 10 distinct books be divided among 3 students if Ahmad get 4 books, Nancy and Pamela each get 3 books.

Exercise #2

How many ways can 6 boys and 5 girls stand in a line so that all the boys stand side by side and all girls stand side by side?

G G G G G B B B B B B

or

B B B B B B G G G G G

Exercise #3

Bob, John, Luke and Tim play a tennis tournament. The rules of the tournament are such that at the end of the tournament a ranking will be made and there will be no ties. How many different rankings can there be?

Exercise #4

There is a basket of fruit containing an apple, a banana and an orange and there are five girls who want to eat one fruit. How many ways are there to give three of the five girls one fruit each and leave two of them without a fruit to eat?

Exercise #5

Registration numbers for a vehicle are to be made using three letters (using any letter of the alphabet) followed by four single-digit numbers. For example, JMW4509 is one such registration number. How many such registration numbers are possible if neither letters nor numbers can be repeated?

Exercise #6

In how many ways can five people A, B, C, D, and E be seated around a circular table if,

a) A and B must sit next to each other.

b) A and B must not sit next to each other.

c) A and B must be together and CD must be together.

Combinations

Combinations

There are two types of combinations (remember the order does **not** matter now):

i. **Repetition is Allowed**: such as coins in your pocket (5,5,5,10,10)

ii. **No Repetition**: such as combination of subjects to enroll (Maths., English, Music)

r-Combinations (without repetition)

- Given a set **X**= $\{x_1,..x_n\}$ containing n (distinct) elements.
- An *r*-combinations of **X** is an unordered selection of *r*-element of **X.**
- The number of *r*-combinations of a set of n distinct element is,

$$C(n,r) = {}^{n}C_{r} = {}_{n}C_{r} = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Example:

- Go back to our pool ball example, let say that you just want to know which 3 pool balls were chosen, not the order.
- We already know that 3 out of 16 gave us 3,360 permutations.
- But many of those will be the same to us now, because we don't care what order!

Example (cont'd):

- For example, let say balls 1, 2 and 3 were chosen. These are the possibilities:

Order does matter	Order doesn't matter
1 2 3 1 3 2 2 1 3 2 3 1 3 1 2 3 2 1	1 2 3

- So, the permutations will have 6 times as many possibilities. But, combination will has one only.

Example (cont'd):

- In fact there is an easy way to work out how many ways "1 2 3" could be placed in order. The answer is:

$$3! = 3 \times 2 \times 1 = 6$$

- So, all we need to do is adjust our permutations formula to **reduce it** by how many ways the objects could be in order (because we aren't interested in the order any more):

$$\frac{n!}{(n-r)!} \times \frac{1}{r!} = \frac{n!}{r!(n-r)!}$$

Example (cont'd):

- So, our pool ball example (now without order) is:

$$\frac{16!}{3!(16-3)!} = \frac{16!}{3! \times 13!} = \frac{20,922,789,888,000}{6 \times 6,227,020,800} = 560$$

- Or you could do it this way:

$$\frac{16 \times 15 \times 14}{3 \times 2 \times 1} = \frac{3360}{6} = 560$$

Example:

In how many ways can we select a committee of three from a group of 10 distinct persons?

Solution:

$$C(10,3) = \frac{10!}{3!\,(10-3)!} = \frac{10!}{3!\,7!} = 120$$

Example:

In how many ways can we select a committee of 2 women and 3 men from a group of 5 distinct women and 6 distinct men?

Solution:

To select 2 women: $C(5,2) = \frac{5!}{2!3!} = 10$

To select 3 men: $C(6,3) = \frac{6!}{3!3!} = 20$

$$\therefore 10 \times 20 = 200$$

Example:

How many 8-bit strings contain exactly four 1's?

Solution:

$$C(8,4) = \frac{8!}{4!\,4!} = 70$$

Example:

A committee of six is to be made from 4 students and 8 lecturers. In how many ways can this be done:

i. If the committee contains exactly 3 students?
ii. If the committee contains at least 3 students?

Example - Solution:

i. If the committee contains exactly 3 students?

Select 3 students: $C(4,3) = \frac{4!}{3!1!} = 4$

Select 3 lecturers: $C(8,3) = \frac{8!}{3!5!} = 56$

$\therefore 4 \times 56 = 224$

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Example - Solution:

ii. If the committee contains at least 3 students?

Case (1): 3 students and 3 lecturers

$$\therefore 4 \times 56 = 224$$

Case (2): 4 students and 2 lecturers

Select 4 students: $C(4,4) = 1$

Select 2 lecturers: $C(8,2) = \frac{8!}{2!6!} = 28$

$$\therefore 1 \times 28 = 28$$

Case (1) + Case (2): 224 + 28 = 252

Example:

A student is required to answer 7 out of 12 questions, which are divided into two groups, each containing 6 questions. The student is not permitted to answer more than 5 questions from either group. In how many different ways can the student choose the 7 questions?

Example - Solution:

Number of question from group A		Number of question from group B
5	C(6,5).C(6,2)=90	2
4	C(6,4).C(6,3)=300	3
3	C(6,3).C(6,4)=300	4
2	C(6,2).C(6,5)=90	5

90+300+300+90 = 780

r-Combinations (with repetition allowed)

The number of *r*-combinations of *n* objects with repetitions allowed is,

$$C(n+r-1, r) = \frac{(n+r-1)!}{r!\,(n-1)!}$$

Example:

Let say there are five flavors of ice-cream: **banana, chocolate, lemon, strawberry and vanilla**. You can have three scoops. In how many variations will there be?

Example - Solution:

Let's use letters for the flavors: {b, c, l, s, v}. Example selections would be:

- {c, c, c} (3 scoops of chocolate)
- {b, l, v} (one each of banana, lemon and vanilla)
- {b, v, v} (one of banana, two of vanilla)

Therefore, $n = 5, r = 3$

$$C(5+3-1,3)=C(7,3)=35$$

Example:

A bakery makes 4 different varieties of donuts. Khairin wants to buy 6 donuts. How many different ways can he do it?

Solution:

$n = 4, r = 6$

$$C(4+6-1,6) = \frac{(4+6-1)!}{6!(4-1)!} = \frac{9!}{6!3!} = 84$$

Or, $C(9,6) = 84$

Example:

There is a box containing identical blue, green, pink, yellow, red and dark blue balls. In how many ways we can select 4 balls?

Solution:

$n = 6, r = 4$

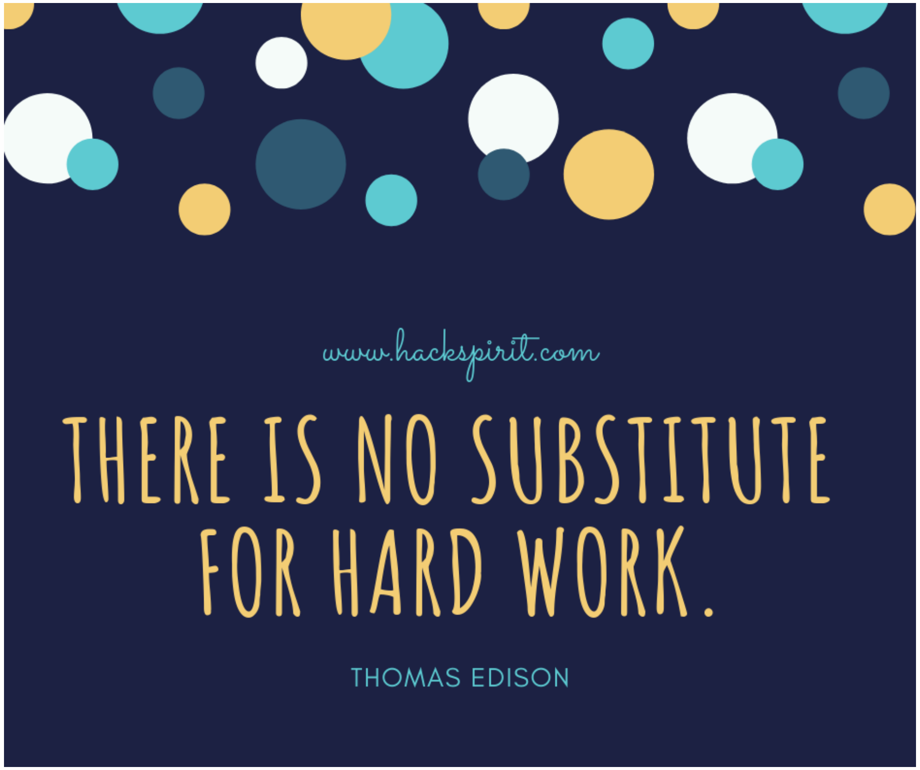
$$C(6+4-1, 4) = \frac{(6+4-1)!}{4!(6-1)!} = \frac{9!}{4!5!} = 126$$

Or, $C(6, 4) = 126$

Summary

Which formula to use?

	Order Matters (Permutations)	Order Does Not Matter (Combinations)
Repetition is allowed	$P_n = n^r$	$C(n+r-1,r) = \dfrac{(n+r-1)!}{r!(n-1)!}$
Repetition is not allowed	$P(n,r) = \dfrac{n!}{(n-r)!}$	$C(n,r) = \dfrac{n!}{r!(n-r)!}$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Exercises

Exercise #1

Ahmad bought a machine to make fresh juice. He has five different fruits: strawberry, oranges, apples, pineapples and lemons. If he only use two fruits, how many different juice drinks can Ahmad make?

Exercise #2

There are 25 people who work in an office together. Five of these people are selected to attend five different conferences. The first person selected will go to a conference in Hawaii, the second will go to New York, the third will go to San Diego, the fourth will go to Atlanta and the fifth will go to Nashville. How many such selection are possible.

Exercise #3

In a group of children; 6 boys and 4 girls, four children are to be selected. In how many different ways can they be selected such that at least one boy should be there?

Exercise #4

There is a shipment of 50 microprocessors of which four are defective.

i) In how many ways can we select a set of 4 microprocessors?

ii) In how many ways can we select a set of 4 microprocessor containing at least 1 defective microprocessor?