

CHAPTER 3

[Part 3]

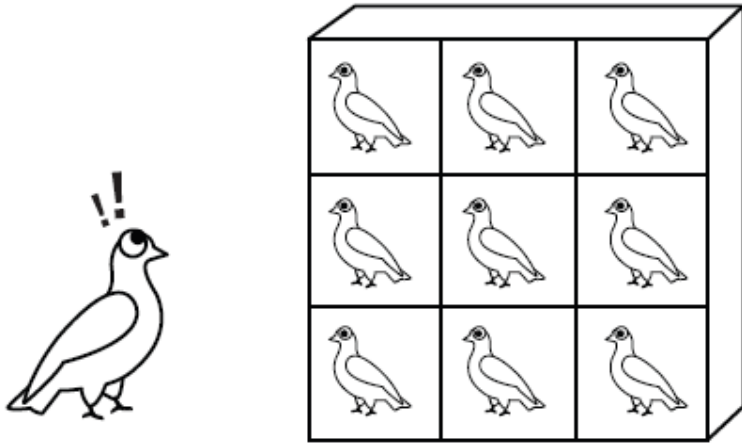
PIGEONHOLE PRINCIPLE

Introduction

- The Pigeonhole Principle is a really simple concept.
- Discovered in the 1800s by Peter Gustav Lejeune Dirichlet. He was the youngest member of the Prussian Academy of Sciences. He worked at number theory and analysis.
- He also came up with a simple little thing that he called The Dirichlet Drawer Principle (or Shoe Box Principle), but that now we call it - **The Pigeonhole Principle.**

Pigeonhole Principle

THE PIGEONHOLE PRINCIPLE



- Imagine 9 pigeonholes and 10 pigeons. A storm comes along, and all of the pigeons take shelter inside the pigeonholes.
- They could be arranged any number of ways. For instance, all 10 pigeons could be inside one hole, and the rest of the holes could be empty.
- What we know for sure, no matter what, is that there is at least one hole that contains more than one pigeon?.

The principle works no matter what the particular number of pigeons and pigeonholes. As long as there are **(N - 1) number of pigeonholes**, and **(N) number of pigeons**, we know there will always be at least two pigeons in one hole.

Pigeonhole Principle – 1st Form

If n pigeons fly into k pigeonholes and $k < n$, some pigeonhole will contains **at least two pigeons**.



Pigeonhole Principle – 1st Form (cont'd)

- The principle tells nothing about how to locate the pigeonhole that contains 2 or more pigeons.
- It only asserts the existence of a pigeonhole containing 2 or more pigeons.
- To apply this principle, one must decide:
 - ✓ Which objects are the pigeons.
 - ✓ Which objects are the pigeonholes.

Example:

1. Among 8 people there are at least two persons who have the same (day) for their birthday.

Pigeonholes: Days (7) – Monday to Sunday

Pigeons: People (8)

2. Among 13 people there are at least two persons whose month of birth is same.

Pigeonholes: Months(12) – January to December

Pigeons: People (13)

Example:

In a party there are n people. Prove that there are at least two persons who know exactly the same number of people.

Solution:

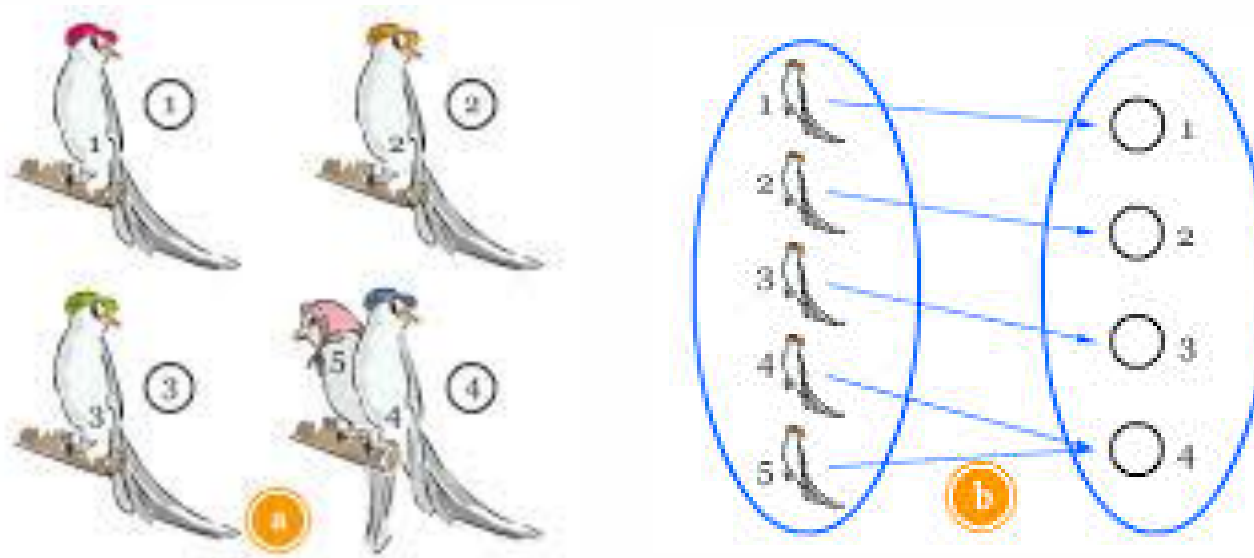
Pigeonholes: How many people a person can know which is at most $n-1$ (need to exclude him/herself)

Pigeons: People (n)

If a person knows i people then the person is put in the i -th box. There are n people. So there must be one box which contains 2 persons.

Pigeonhole Principle – 2nd Form

If f is a function from a finite set $\mathbf{X}$ to a finite set $\mathbf{Y}$ and $|\mathbf{X}| > |\mathbf{Y}|$, then $f(x_1) = f(x_2)$ for some $x_1, x_2 \in \mathbf{X}, x_1 \neq x_2$.



Pigeonhole Principle – 2nd Form (cont'd)

- The 2nd form can be reduced to the 1st form by letting:
 - X = set of pigeons,
 - Y = set of pigeonholes.
- Assign pigeon x to pigeonhole $f(x)$
- By the 1st form principle, at least 2 pigeons;
 $x_1, x_2 \in X$ are assigned to the same pigeonhole,
that is, $f(x_1) = f(x_2)$ for some $x_1, x_2 \in X, x_1 \neq x_2$.

Example:

Let $A = \{1, 2, 3, 4, 5, 6\}$. Show that if we choose any four distinct members of A , then for at least one pair of these four integers their sum is 7.

Solution:

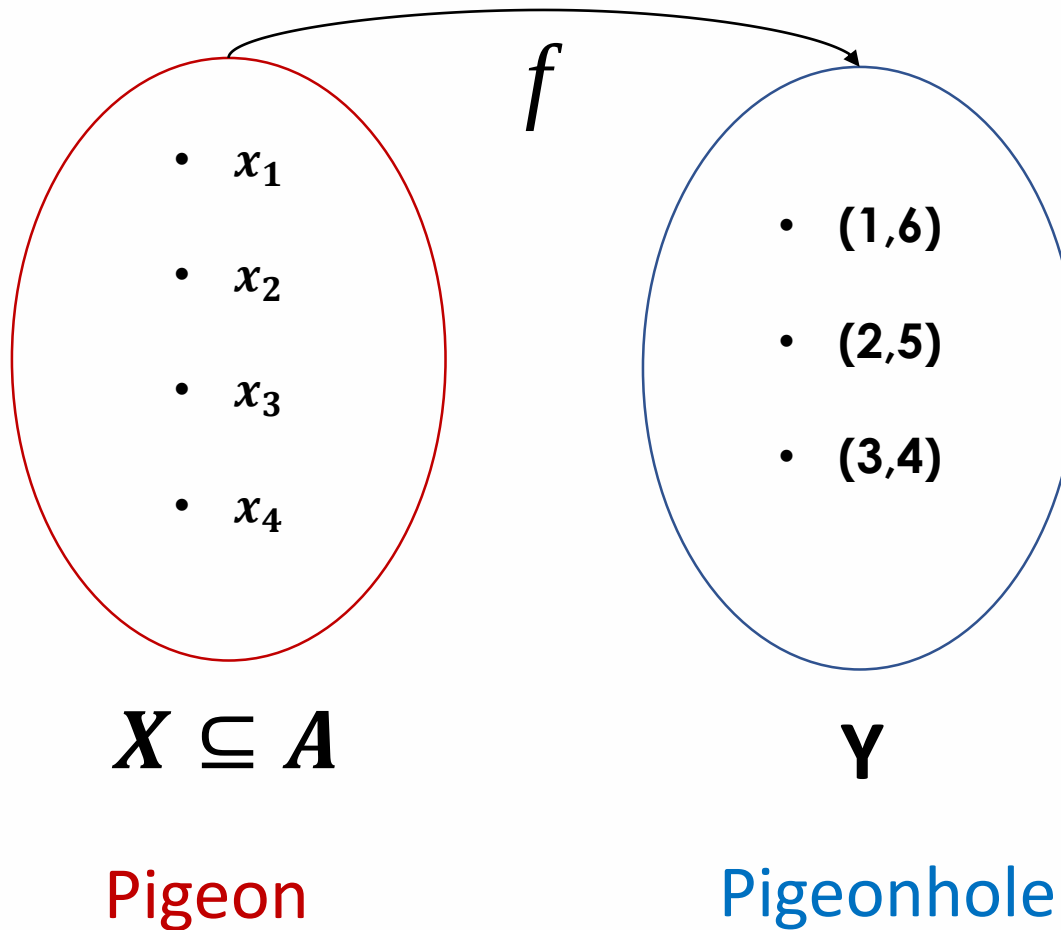
Notice that $(1,6)$, $(2,5)$ and $(3,4)$ are the only pairs of distinct integers such that their sum is 7.

Let $X = \{x_1, x_2, x_3, x_4\}$ be any subset of four distinct elements of A .

Let $Y = \{(1,6), (2,5), (3,4)\}$, a set of 3 distinct elements and a part of A .

$$y_1 = (1,6), y_2 = (2,5), y_3 = (3,4)$$

Example -Solution:

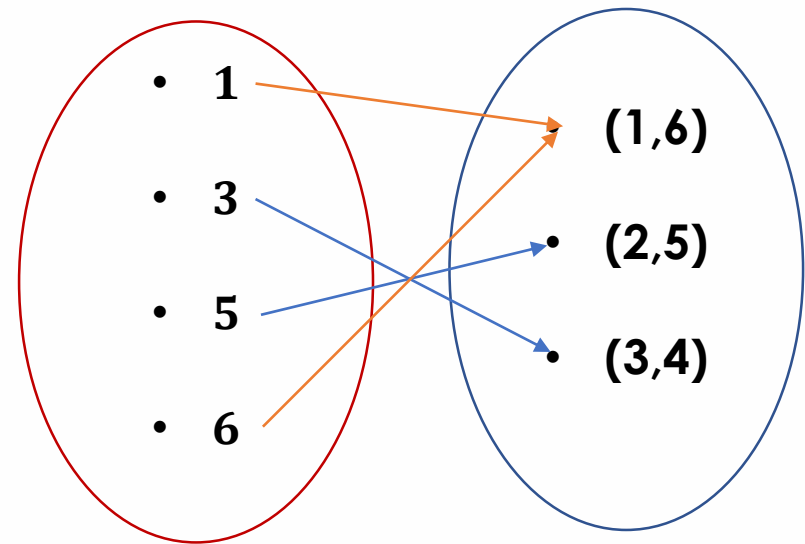


Example -Solution:

Define $f: X \rightarrow Y$ by $f(a) = y_i$ if $a \in y_i$

For example, if $a = 1 \in X$, then $f(1) = (1,6)$

In the case where $X = \{1, 3, 5, 6\}$, see in the figure on right, now $|X| = 4$, $|Y| = 3$. Then by 2nd term of pigeonhole principle – at least two distinct elements of X must be mapped to the same elements of Y .



Hence, if we choose any four distinct members of A , then for at least one pair of these four integers, their sum is 7.

Example :

Using instant messaging, every Sunday evening 10 friends communicate with each other. Instant messaging allows a person to open separate window for each person he/she is communicating with. Then at any time at least 2 from 10 friends must be communicating with the same number of friends.

- Let $\mathbf{X} = \{x_1, x_2, \dots, x_{10}\}$ be the set of 10 friends.
- For each x_i , let n_i be the number of friends they are communicating with $i = 1, 2, \dots, 10$
- A person may not be communicating with any person or may be communicating with as many as 9 people. Thus, $0 \leq n_i \leq 9$, $i = 1, 2, \dots, 10$

Example (cont'd):

- If we take $Y = \{0, 1, 2, \dots, 9\}$ then we cannot apply the pigeonhole principle because the number of elements in X and the number of elements in the Y are the same.
- Suppose that one of the friends, say x_i , is not communicating with any other friend. Then, $n_i = 0$. The remaining people can communicate with at most 8 other people. Thus, $0 \leq n_i \leq 8$, $i = 1, 2, \dots, 10$. Then, $Y = \{0, 1, 2, \dots, 8\}$
- Set X is the pigeons and set Y is the pigeonholes. Then $|X| = 10$ and $|Y| = 9$. Then by 2nd term pigeonhole principle, at least two distinct elements of X must be mapped to the same elements of Y .

Pigeonhole Principle – 3rd Form

The Generalized Pigeonhole Principle

For any function, f from a finite set $\mathbf{X}$ with n elements to a finite set $\mathbf{Y}$ with k elements and for any positive integer s : if $s < n/k$, then there exists an element $y \in \mathbf{Y}$ that y is the image of at least $s + 1$ distinct elements of $\mathbf{X}$.



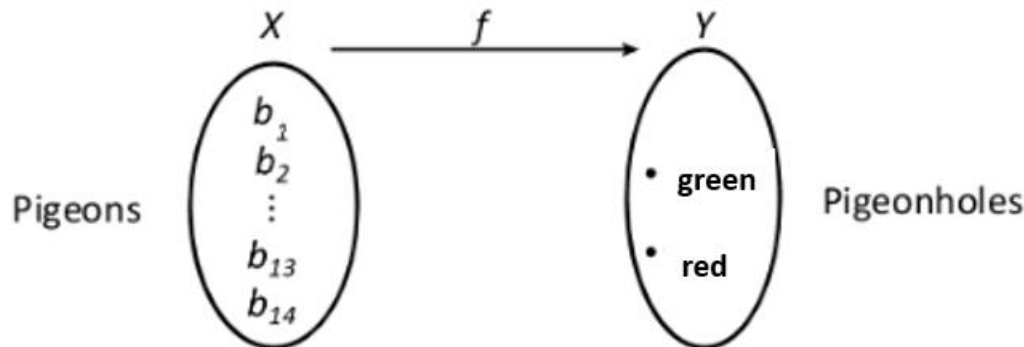
Example:

A box that contains 8 green balls and 6 red balls is kept in a completely dark room. What is the **least number of balls** one must take out from the box so that at least 2 balls will be the same color?

Solution:

Let X be the set of all balls in the box and $Y = \{\text{green}, \text{red}\}$.

Define a function $f: X \rightarrow Y$ by $f(b) = \text{green}$, if the colour of the ball is green and $f(b) = \text{red}$, if the colour of the ball is red.



Example – Solution (cont'd):

If we take subset **A** of 3 balls of **X**, then $|A| > |Y|$. By the pigeonhole principle, at least two elements of **A** must be assigned the same value in **Y**. Therefore, at least 2 of the balls of **A** must have the same colour.

Alternative:

Let, **A** = subset of **X**; **Y** = {green, red}

$$|A| = x; |Y| = 2; s + 1 = 2 \Rightarrow s = 1$$

$\Rightarrow$ By the pigeonhole principle: $s < \frac{|A|}{|Y|}$

$$\therefore 1 < \frac{x}{2} \Rightarrow 2 < x \Rightarrow \therefore x = 3$$



Pigeonhole Principle – 3rd Form

The Generalized Pigeonhole Principle

Let f be a function from a finite set $\mathbf{X}$ to a finite set $\mathbf{Y}$. Suppose that $|\mathbf{X}| = n$ and $|\mathbf{Y}| = k$.

Let $m = \left\lceil \frac{n}{k} \right\rceil$. Then there are **at least** m values $a_1, a_2, \dots, a_m \in \mathbf{X}$ such that,

$$f(a_1) = f(a_2) = \dots = f(a_m)$$

that must be in one pigeonhole.

Pigeonhole Principle – 3rd Form

The Generalized Pigeonhole Principle

Pigeonhole Principle states a trivial fact as follows:

Theorem 1.1. (Pigeonhole Principle)

When n pigeons are put into k pigeonholes, there exists at least 1 pigeonhole containing not less than $\lceil \frac{n}{k} \rceil$ pigeons and at least 1 pigeonhole containing not more than $\lfloor \frac{n}{k} \rfloor$ pigeons.

- $\lceil x \rceil$ and $\lfloor x \rfloor$ represent the smallest integer not less than x and the greatest integer not greater than x respectively. They are called ‘ceiling function’ and ‘floor function’. For example, $\lceil \pi \rceil = 4$, $\lceil 2 \rceil = 2$, $\lfloor \pi \rfloor = 3$, $\lfloor 2 \rfloor = 2$.

Example:

Suppose that there are 50 people in a room. Show that at least 5 of these people must have their birthday in the same month.

Solution:

Let, pigeon (n) = people = 50 ; pigeonhole (k) = month = 12

$$\text{Thus, } m = \left\lceil \frac{n}{k} \right\rceil = \left\lceil \frac{50}{12} \right\rceil = 5$$

Pigeonhole Principle – 3rd Form

The Generalized Pigeonhole Principle

- A common type of problem asks for the **minimum number** of **pigeon** (n) such that at least m of these pigeon must be in one of **pigeonhole** (k) when these objects are distributes.

- From, $\frac{n}{m} \geq k$

the smallest integer n with $\frac{n}{m} > k - 1$;
 $\therefore n = k(m - 1) + 1$

Example:

What is the minimum number of students required in a course to be sure that **at least six** will receive the same grade, if there are five possible grades: A, B, C, D and F?

Solution:

Pigeon (n) : the number of students.

Pigeonhole (k): the grades, i.e., 5

$$n = ? ; m = 6 ; k = 5$$
$$\therefore n = k(m - 1) + 1 = 26$$

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The art of
being happy
is to be
satisfied with
what you have.

Exercises

Exercise #1:

1. There are 400 students in a programming class. Show that at least 2 of them were born on the same day of a month.
2. Let $A = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$ be a set of seven integers. Show that if these numbers are divided by 6, then at least two of them must have the same remainder.
3. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Show that if you choose any five distinct members of A , then there will be two integers such that their sum is 9.
4. From the integers in the set $\{1, 2, 3, \dots, 19, 20\}$, what is the least number of integers that must be chosen so that at least one of them is divisible by 4?

Exercise #2:

- a) There are 50 baskets of apples. Each basket contains no more than 24 apples. Show that there are at least 3 baskets containing the same number of apples.

- b) 51 numbers are chosen from the integers between 1 and 100 inclusively. Prove that 2 of the chosen integers are consecutive.