

1. Let the universal set be the set \mathbb{R} of all real numbers and let $A = \{x \in \mathbb{R} \mid 0 < x \leq 2\}$,

$B = \{x \in \mathbb{R} \mid 1 \leq x < 4\}$ and $C = \{x \in \mathbb{R} \mid 3 \leq x < 9\}$. Find each of the following:

a) $A \cup C$

b) $(A \cup B)'$

c) $A' \cup B'$

Solutions:

a) $A \cup C$

$$A = (0, 2]$$

$$C = [3, 9)$$

$$\text{Since, } A \cup C = \{x \in \mathbb{R} \mid x \in A \text{ or } x \in C\}$$

$$\text{Thus, } A \cup C = \{x \in \mathbb{R} \mid 0 < x \leq 2 \text{ or } 3 \leq x < 9\}$$

b) $(A \cup B)'$

$$A = (0, 2]$$

$$B = [1, 4)$$

$$\text{Since, } A' = \{x \in \mathbb{R} \mid x \notin A\} \text{ and } A \cup B = \{x \in \mathbb{R} \mid x \in A \text{ or } x \in B\}$$

$$\text{Thus, } (A \cup B)' = \{x \in \mathbb{R} \mid x \notin (0, 4)\}$$

$$(A \cup B)' = \{x \in \mathbb{R} \mid x \leq 0 \text{ or } x \geq 4\}$$

c) $A' \cup B'$

$$A = (0, 2]$$

$$B = [1, 4)$$

$$\text{Since, } A' = \{x \in \mathbb{R} \mid x \notin A\} \text{ and } A \cup B = \{x \in \mathbb{R} \mid x \in A \text{ or } x \in B\}$$

$$\text{Thus, } A' \cup B' = \{x \in \mathbb{R} \mid x \in (-\infty, 0] \text{ or } (2, \infty) \text{ or } (-\infty, 1) \text{ or } [4, \infty)\}$$

$$A' \cup B' = \{x \in \mathbb{R} \mid x \in (-\infty, 1) \text{ or } (2, \infty)\}$$

$$A' \cup B' = \{x \in \mathbb{R} \mid x < 1 \text{ or } x > 2\}$$

2. Draw Venn diagrams to describe sets A, B, and C that satisfy the given conditions.

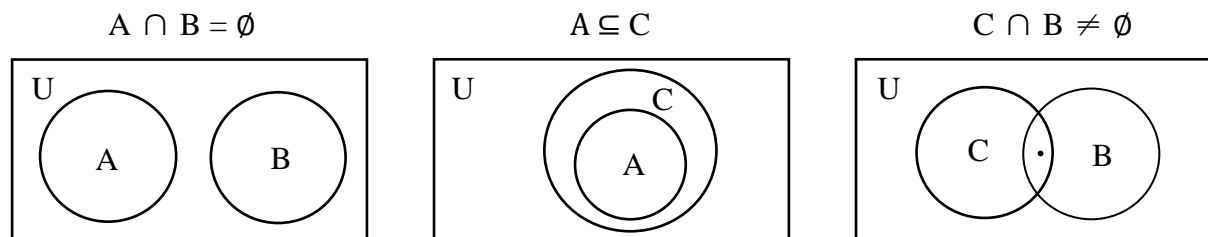
a) $A \cap B = \emptyset, A \subseteq C, C \cap B \neq \emptyset$

b) $A \subseteq B, C \subseteq B, A \cap C \neq \emptyset$

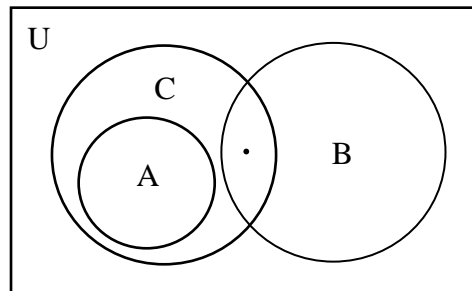
c) $A \cap B \neq \emptyset, B \cap C \neq \emptyset, A \cap C = \emptyset, A \not\subseteq B, C \not\subseteq B$

Solutions:

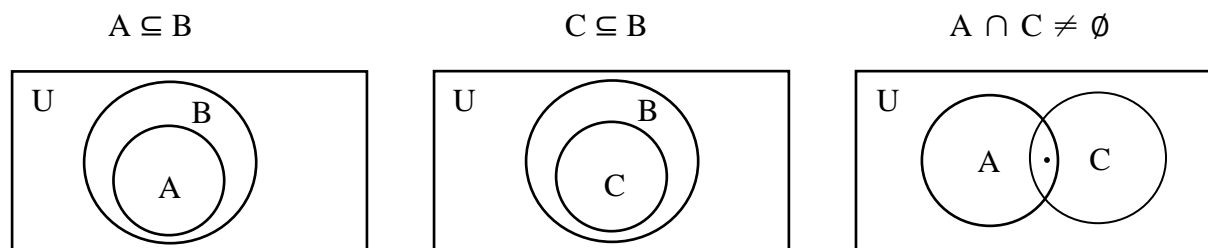
a) $A \cap B = \emptyset, A \subseteq C, C \cap B \neq \emptyset$



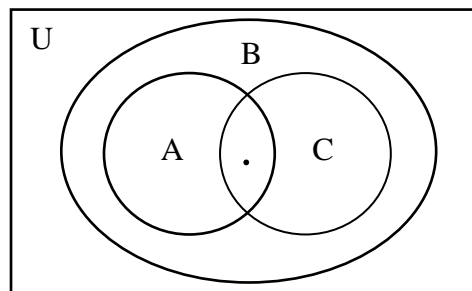
Thus, $A \cap B = \emptyset, A \subseteq C, C \cap B \neq \emptyset$



b) $A \subseteq B, C \subseteq B, A \cap C \neq \emptyset$

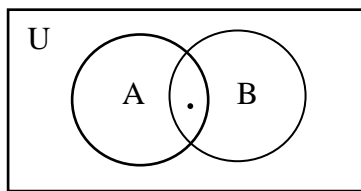


Thus, $A \subseteq B, C \subseteq B, A \cap C \neq \emptyset$

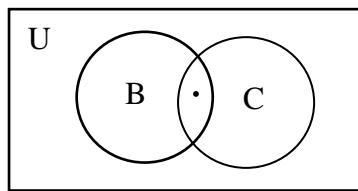


c) $A \cap B \neq \emptyset, B \cap C \neq \emptyset, A \cap C = \emptyset, A \not\subset B, C \not\subset B$

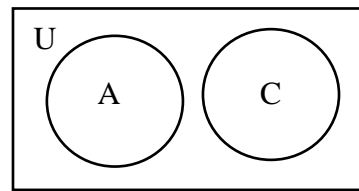
$A \cap B \neq \emptyset$



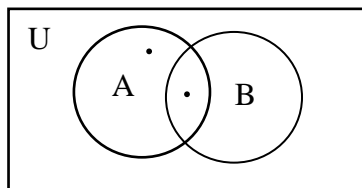
$B \cap C \neq \emptyset$



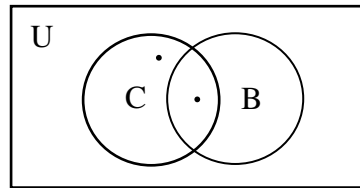
$A \cap C = \emptyset$



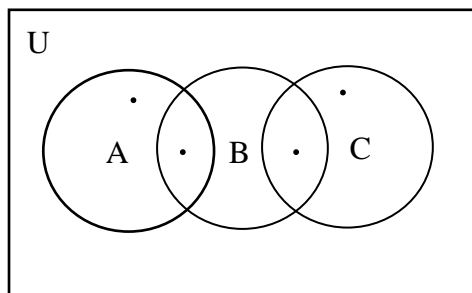
$A \not\subset B$



$C \not\subset B$



Thus, $A \cap B \neq \emptyset, B \cap C \neq \emptyset, A \cap C = \emptyset, A \not\subset B, C \not\subset B$



3. Given two relations S and T from A to B,

$$S \cap T = \{(x,y) \in A \times B \mid (x,y) \in S \text{ and } (x,y) \in T\}$$

$$S \cup T = \{(x,y) \in A \times B \mid (x,y) \in S \text{ or } (x,y) \in T\}$$

Let $A = \{-1, 1, 2, 4\}$ and $B = \{1, 2\}$ and defined binary relations S and T from A to B as follows:

For all $(x,y) \in A \times B$, $x S y \leftrightarrow |x| = |y|$

For all $(x,y) \in A \times B$, $x T y \leftrightarrow x - y$ is even

State explicitly which ordered pairs are in $A \times B$, S, T, $S \cap T$, and $S \cup T$.

Solutions:

$$A \times B = \{(-1, 1), (-1, 2), (1, 1), (1, 2), (2, 1), (2, 2), (4, 1), (4, 2)\}$$

$$S = \{(-1, 1), (1, 1), (2, 2)\}$$

$$T = \{(-1, 1), (1, 1), (2, 2), (4, 2)\}$$

$$S \cap T = \{(-1, 1), (1, 1), (2, 2)\}$$

$$S \cup T = \{(-1, 1), (1, 1), (2, 2), (4, 2)\}$$

4. Show that $\neg((\neg p \wedge q) \vee (\neg p \wedge \neg q)) \vee (p \wedge q) \equiv p$. State carefully which of the laws are used at each stage.

$$\neg((\neg p \wedge q) \vee (\neg p \wedge \neg q)) \vee (p \wedge q)$$

$$= \neg(\neg p \wedge (q \vee \neg q)) \vee (p \wedge q) \quad (\text{Distributive law})$$

$$= \neg(\neg p) \vee (p \wedge q) \quad (\text{Double Negation law})$$

$$= p \vee (p \wedge q) \quad (\text{Absorption law})$$

$$= p$$

5. $R_1 = \{(x, y) \mid x + y \leq 6\}$; R_1 is from X to Y ; $R_2 = \{(y, z) \mid y > z\}$; R_2 is from Y to Z ; ordering of X , Y , and Z : 1, 2, 3, 4, 5.

Find:

- The matrix A_1 of the relation R_1 (relative to the given orderings)
- The matrix A_2 of the relation R_2 (relative to the given orderings)
- Is R_1 reflexive, symmetric, transitive, and/or an equivalence relation?
- Is R_2 reflexive, antisymmetric, transitive, and/or a partial order relation?

Solutions:

- The matrix A_1 of the relation R_1 (relative to the given orderings)

$$R_1 = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (4, 1), (4, 2), (5, 1)\}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- The matrix A_2 of the relation R_2 (relative to the given orderings)

$$R_2 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3), (5, 1), (5, 2), (5, 3), (5, 4)\}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

- Is R_1 reflexive, symmetric, transitive, and/or an equivalence relation?

Reflexive: not all $(a, a) \in R_1$; $(4, 4)$ and $(5, 5) \notin R_1$, thus R_1 is not reflexive relation.

Symmetric: $(a, b) \in R_1$ and $(b, a) \in R_1$; $(1, 2) \in R_1$ and $(2, 1) \in R_1$

Thus, R_1 is symmetric relation.

$$\text{Transitive: } \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Thus, R_1 is not transitive relation.

Equivalence: R_1 is symmetric but not reflexive and not transitive,

Thus, R_1 is not equivalence relation.

d) Is R2 reflexive, antisymmetric, transitive, and/or a partial order relation?

Reflexive: since for each $a \in Y$, $(a, a) \notin R2$, thus R2 is irreflexive relation.

Antisymmetric: $(2, 1) \in R2$ but $(1, 2) \notin R2$, thus R2 is antisymmetric relation.

$$\text{Transitive: } \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Thus, R2 is not transitive relation.

Partial order relation: R2 is antisymmetric but not reflexive and not transitive,
so R2 is not partial order relation.

6. Suppose that the matrix of relation R1 on {1, 2, 3} is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

relative to the ordering 1, 2, 3, and that the matrix of relation R2 on {1, 2, 3} is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

relative to the ordering 1, 2, 3. Find:

a) The matrix of relation $R1 \cup R2$

b) The matrix of relation $R1 \cap R2$

Solutions:

a) The matrix of relation $R1 \cup R2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

b) The matrix of relation $R1 \cap R2$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

7. If $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ are both one-to-one, is $f + g$ also one-to-one? Justify your answer.

Solutions:

Assume: $f(x) = x$, $g(x) = -x$ (x is for all real numbers)

$$f + g = (f + g)(x)$$

$$= f(x) + g(x)$$

$$= (x) + (-x)$$

$$= 0 \text{ (it is a constant function)}$$

For example, $(f + g)(1) = 0$ and $(f + g)(2) = 0$, but $1 \neq 2$

Thus, $f + g$ is not one-to-one.

8. With each step you take when climbing a staircase, you can move up either one stair or two stairs. As a result, you can climb the entire staircase taking one stair at a time, taking two at a time, or taking a combination of one- or two-stair increments. For each integer $n \geq 1$, if the staircase consists of n stairs, let C_n be the number of different ways to climb the staircase. Find a recurrence relation for c_1, c_2, \dots, C_n .

Solutions:

When one is climbing a staircase consisting of n stairs, the last step taken is either a single stair or two stairs together.

Numbers of ways to climb the staircase = C_n

The number of ways to climb the staircase and have the final step be a single stair is C_{n-1} .

The number of ways to climb the staircase and have the final step be two stairs is C_{n-2} .

When $n \geq 3$, the staircase contains more than 2 stairs and so will need to use a combination of single stair and two stairs.

Therefore, $C_n = C_{n-1} + C_{n-2}$, $n \geq 3$. Mention that $C_1 = 1$ and $C_2 = 2$.

9. The Tribonacci sequence (t_n) is defined by the equations,

$$t_0 = 0, t_1 = t_2 = 1, t_n = t_{n-1} + t_{n-2} + t_{n-3} \text{ for all } n \geq 3.$$

a) Find t_7 .

b) Write a recursive algorithm to compute t_n , $n \geq 3$.

Solutions:

a) Find t_7 .

$$t_n = t_{n-1} + t_{n-2} + t_{n-3}$$

$$\begin{aligned} t_3 &= t_{3-1} + t_{3-2} + t_{3-3} \\ &= t_2 + t_1 + t_0 \\ &= 1 + 1 + 0 \\ &= 2 \end{aligned}$$

$$\begin{aligned} t_4 &= t_{4-1} + t_{4-2} + t_{4-3} \\ &= t_3 + t_2 + t_1 \\ &= 2 + 1 + 1 \\ &= 4 \end{aligned}$$

$$\begin{aligned} t_5 &= t_{5-1} + t_{5-2} + t_{5-3} \\ &= t_4 + t_3 + t_2 \\ &= 4 + 2 + 1 \\ &= 7 \end{aligned}$$

$$\begin{aligned} t_6 &= t_{6-1} + t_{6-2} + t_{6-3} \\ &= t_5 + t_4 + t_3 \\ &= 7 + 4 + 2 \\ &= 13 \end{aligned}$$

$$\begin{aligned} t_7 &= t_{7-1} + t_{7-2} + t_{7-3} \\ &= t_6 + t_5 + t_4 \\ &= 13 + 7 + 4 \\ &= 24 \end{aligned}$$

b) Write a recursive algorithm to compute t_n , $n \geq 3$.

Input: n

Output: $t(n)$

```
t(n) {  
    if (n=1 or n=2)  
        return 1  
    return  $t_{n-1} + t_{n-2} + t_{n-3}$   
}
```