UTM
UNIVERSITI TEKNOLOGI MALAYSIA

CHAPTER 1

SET THEORY
[Part 1: Set & Subset]

Session 2020/2021 – Semester 1

Sets

- The concept of set is basic to all of mathematics and mathematical applications.
- A set is a well-defined collection of distinct objects.
- These objects are called members or elements of the set.

Sets (cont'd)

- Well-defined means that we can tell for certain whether an object is a member of the collection or not.
- If a set is finite and not too large, we can describe it by listing the elements in it.

Example

i) **A** is a set of all positive integers less than 10:

$\mathbf{A} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

ii) **B** is a set of first 5 positive odd integers:

$\mathbf{B} = \{1, 3, 5, 7, 9\}$

iii) **C** is a set of vowels:

$\mathbf{C} = \{a, e, i, o, u\}$

Defining Sets

This can be done by:

- ✓ Listing ALL elements of the set within braces.
- ✓ Listing enough elements to show the pattern then an ellipsis.
- ✓ Use set builder notation to define "rules" for determining membership in the set.

Example

1. Listing ALL elements. $A = \{1,\ 2,\ 3,\ 4\}$
2. Demonstrating a pattern. $\mathbb{N} = \{1,\ 2,\ 3,\ \ldots\}$
3. Using set builder notation. $P = \{x \mid x \in \mathbb{R} \text{ and } x \notin \mathbb{C}\}$

Defining Sets (cont'd)

A set is **determined by its elements** and **not by any particular order** in which the element might be listed.

Example:

For set **A** = {1, 2, 3, 4}, this set might just as well be specified as,

{2, 3, 4, 1} or {4, 1, 3, 2}

Defining Sets (cont'd)

The elements making up a set are assumed to be distinct, we may have duplicates in our list, only one occurrence of each element is in the set.

Example:

$\{a, b, c, a, c\} \rightarrow \{a, b, c\}$

$\{1, 3, 3, 5, 1\} \rightarrow \{1, 3, 5\}$

Defining Sets (cont'd)

- Use uppercase letters *A*, *B*, *C* … to denote sets, lowercase denote the elements of set.
- The symbol $\in$ stands for 'belongs to'
- The symbol $\notin$ stands for 'does not belong to'

Example

- $\mathbf{X}=\{ a, b, c, d, e \}$, $b \in \mathbf{X}$ and $m \notin \mathbf{X}$

- $\mathbf{A}=\{\{1\}, \{2\}, 3, 4\}$, $\{1\} \in \mathbf{A}$ and $1 \notin \mathbf{A}$

Defining Sets (cont'd)

- If a set is a large finite set or an infinite set, we can describe it by listing a property necessary for memberships.

- Let **S** be a set, the notation,

  $A = \{x \mid x \in S, P(x)\}$ or $A = \{x \in S \mid P(x)\}$

  means that **A** is the set of all x in **S** such that P of x.

Example

- Let **A**={1, 2, 3, 4, 5, 6}, we can also write **A** as,

  $$\mathbf{A} = \{x \mid x \in \mathbf{Z},\ 0 < x < 7\},$$

  if **Z** denotes the set of integers.

- Let $\mathbf{B} = \{x \mid x \in \mathbf{Z}, x > 0\}$ → **B** = {1, 2, 3, 4, ...}

Example – Standard definition

The set of natural numbers: $\mathbb{N} = \{1, 2, 3, \ldots\}$

The set of integers: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

The set of positive integers: $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$

The set of Rational Numbers (fractions): $\frac{1}{2}, \frac{2}{3}, \frac{5}{7}, etc \in \mathbb{Q}$

More formally: $\mathbb{Q} = \left\{ \frac{a}{b} \middle| a, b \in \mathbb{R}, b \neq 0 \right\}$

The set of Irrational Numbers: $\sqrt{2}, \pi$, or e are irrational

The Real numbers = $\mathbb{R} =$ the union of the rational numbers with the irrational numbers

Symbols Used With Set Builder Notation

The standard form of notation for this is called "set builder notation".

For instance, $\{x|x \text{ is an odd positive integer}\}$ represents the set $\{1, 3, 5, 7, 9, \ldots\}$

$\{x|x \text{ is an odd positive integer}\}$ is read as

"the set consisting of all x such that x is an odd positive integer".

The vertical bar, " $|$ ", stands for "such that"

Other "short-hand" notation used in working with sets

"$\forall$" stands for "for every"

"$\exists$" stands for "there exists"

"$\bigcup$" stands for "union"

"$\bigcap$" stands for "intersection"

"$\subseteq$" stands for "is a subset of"

"$\subset$" stands for "is a (proper) subset of"

"$\not\subset$" stands for "is a not a (proper) subset of"

"$\varnothing$" stands for the "empty set"

"$\in$" stands for "is an element of"

"$\notin$" stands for "is not an element of"

"$\times$" stands for "cartesian cross product"

"$=$" stands for "is equal to"

Subset [$\subseteq$]

- If every element of **A** is an element of **B**, we say that **A** is a **subset** of **B** and write **A** $\subseteq$ **B**.

  In notation: **A** = **B**, if **A** $\subseteq$ **B** and **B** $\subseteq$ **A**

- The empty set ($\emptyset$) is a subset of every set.

Example

Let, set **A** = {1, 2, 3}

The subset of **A** are

$\emptyset$, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}

Note: **A** is a subset of **A**

UTM
UNIVERSITI TEKNOLOGI MALAYSIA

Proper Subsets [$\subset$]

- If **A** $\subseteq$ **B** and **B** contains an element that is not in **A** (i.e., A doesn't equal B), than we say "**A** is a **proper subset** of **B**": $\mathbf{A} \subset \mathbf{B}$ or $\mathbf{B} \supset \mathbf{A}$

.

- In notation: **A** $\subseteq$ **B** and **A** $\neq$ **B** (**B** $\not\subseteq$ **A**)

- Formally: $\mathbf{A} \subseteq \mathbf{B}$ means $\forall x\ [x \in \mathbf{A} \rightarrow x \in \mathbf{B}]$

- For all sets: $\mathbf{A} \subseteq \mathbf{A}$

Example

- **A**={1, 2, 3}

- The proper subset of **A**:

  $\varnothing$, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}

Note:

A proper subset of a set **A** is a subset of **A** that is not equal to **A** : {1,2,3} $\not\subset$ **A**

Example

Lets, set **B** = {1, 2, 3, 4, 5, 6} and **A** = {1, 2, 3}

∴ **A** is a proper subset of **B**.

Example

Given a set as follows:

A = $\{a, b, c, d, e, f, g, h\}$

B = $\{b, d, e\}$

C = $\{a, b, c, d, e\}$

D = $\{r, s, d, e\}$

Which subset are proper subset of **A**?

Empty Sets [∅ or { }]

- The **empty set** ∅ or **{ }** but not {∅} is the set without elements.
- Note:
  - ✓ Empty set has no elements.
  - ✓ Empty set is a subset of any set.
  - ✓ There is exactly one empty set.
  - ✓ Properties of empty set:

$$\mathbf{A} \cup \emptyset = \mathbf{A};\ \mathbf{A} \cap \emptyset = \emptyset$$
$$\mathbf{A} \cap \mathbf{A}' = \emptyset;\ \mathbf{A} \cup \mathbf{A}' = \mathbf{U}$$
$$\mathbf{U}' = \emptyset;\ \emptyset' = \mathbf{U}$$

Example

i) $\varnothing = \{x \mid x$ is a real number and $x^2 = -3\}$

ii) $\varnothing = \{x \mid x$ is positive integer and $x^3 < 0\}$

Equal Sets [=]

- The set **A** and **B** are **equal** (**A** = **B**) if and only if each element of **A** is an element of **B** and vice versa.

- Formally: **A** = **B** means $\forall x[x \in \mathbf{A} \leftrightarrow x \in \mathbf{B}]$

Example

i) **A** = $\{a, b, c\}$; **B** = $\{b, c, a\}$

$\therefore$ **A = B**

ii) **C** = {0, 2, 4, 6, 8} ;

D = $\{x \mid x$ is a positive integer and $2x < 10\}$

$\therefore$ $C = D$

Equivalent Set

- Two sets, **A** and **B** are **equivalent** if there exists a one-to-one correspondence between them.
- An equivalent set is simply a set with an **equal number of elements**.
- **Example:** Lets **A** and **B** be sets.

$$\mathbf{A} = \{a, b, c, d, e\} \,;\, \mathbf{B} = \{1, 2, 3, 4, 5\}$$

$\therefore$ **A** and **B** are equivalent

Finite Set

- A set **A** is finite if it is empty or if there exists a nonnegative integer n such that **A** has n elements.

- Example: **A** = {1, 2, 3,…, n} ; **A** is called a finite set with n elements).

Example

Let, **A** and **B** be sets.

A = {1, 2, 3, 4} ;

B = $\{x \mid x$ is an integer, $1 \leq x \leq 4\}$

$\therefore$ **A** and **B** are finite set.

Infinite Sets

- A set **A** is infinite if there is not exists a natural number n such that **A** is equivalent to $\{1, 2, 3, \ldots, n\}$.
- Infinite set is uncountable.
- Are all infinite sets equivalent?
  - A set is infinite if it is equivalent to a proper subset of itself!.

Example

Lets, **B**, **C** and **D** be sets.

C = {5, 6, 7, 8, 9, 10}

B = $\{x \mid x$ is an integer, $10 < x < 20\}$

D = $\{x \mid x$ is an integer, $x > 0\}$

Z = $\{x \mid x$ is an integer$\}$ or **Z** = {..., -3, -2, -1, 0, 1, 2, 3,...}

S = $\{x \mid x$ is a real number and $1 \leq x \leq 4\}$

$\therefore \mathbf{C}$ *and* $\mathbf{B}$ *are infinte sets*; $\mathbf{D}, \mathbf{Z}$ *and* $\mathbf{S}$ *are infinite sets.*

Universal Set

- Sometimes we are dealing with sets all of which are subsets of a set **U**.
- This set **U** is called a universal set or a universe.
- The set **U** must be explicitly given or inferred from the context.

Universal Set (cont'd)

- Typically we consider a set **A** as a part of a universal set, **U**, which consist of all possible elements.

- To be entirely correct we should say,

$\forall x \in \mathbf{U}[x \in \mathbf{A} \leftrightarrow x \in \mathbf{B}]$ instead of $\forall x[x \in \mathbf{A} \leftrightarrow x \in \mathbf{B}]$ for **A** = **B**

Note:
$\{x|0 < x < 5\}$ is can be ambiguous, compare to $\{x|0 < x < 5, x \in \mathbf{N}\}$ with $\{x|0 < x < 5, x \in \mathbf{Q}\}$

Example

- The sets **A** = {1,2,3}, **B** = {2,4,6,8} and **C** = {5,7}

- One may choose **U** = {1,2,3,4,5,6,7,8} as a universal set.

- Any superset of **U** can also be considered a universal set for these sets **A**, **B**, and **C**.

  For example, **U** = $\{x \mid x$ is a positive integer$\}$

Cardinality of Set

- Let **S** be a finite set with n distinct elements, where $n \geq 0$.
- Then we write |**S**|= n and say that the cardinality (or the number of elements) of **S** is n.
- **Example: A** and **B** be sets.

A = {1, 2, 3} → |**A**|=3

B = {*a*, *b*, *c*, *d*, *e*, *f*, *g*} → |**B**|=7

Power Set

- The set of all subsets of a set **A**, denoted $P(\mathbf{A})$, is called the power set of **A**.

$$P(\mathbf{A}) = \{x \mid x \subseteq \mathbf{A}\}$$

- If $|\mathbf{A}| = n$, then $|P(\mathbf{A})| = 2^n$

Example

- Lets, a set **A** = {1,2,3}

- The power set of **A**,

  $P(\mathbf{A}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

- Notice that $|\mathbf{A}| = 3$, and $|P(\mathbf{A})| = 2^3 = 8$

Summary

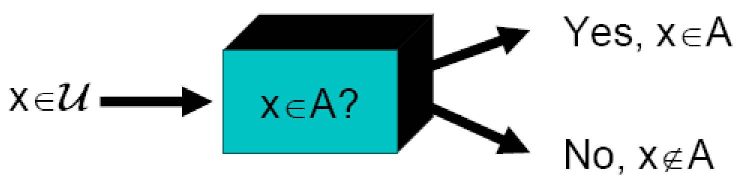
How to Think of Sets

The elements of a set do not have an ordering, hence {a,b,c} = {b,c,a}

The elements of a set do not have multitudes, hence {a,a,a} = {a,a} = {a}

All that matters is: “Is x an element of A or not?”

The size of A is thus the number of *different* elements



Exercise # 1

1. Determine whether each of these pairs of sets are equal.

   (b) $\varnothing, \{\varnothing\}$ ____________________

   (c) $\{1, 2, \{2\}\}, \{1, 2\}$ ____________________

2. For each of the following sets, determine whether 2 is an element of that set.

   (a) $\{x \in \mathbf{R} \mid x \text{ is an integer greater than } 1\}$ ____________________

   (b) $\{x \in \mathbf{R} \mid x \text{ is the square of an integer}\}$ ____________________

   (c) $\{2, \{2\}\}$ ____________________

   (d) $\{\{2\}, \{2, \{2\}\}\}$ ____________________

Exercise # 2

Determine whether each pair of sets is equal.

a) {1, 2, 2, 3} ; {1, 3, 2}

b) $\{ x \mid x^2 + x = 2\}$; {1, -2}

c) $\{x \mid x$ is a real number and $0 < x \leq 2\}$; {1,2}

Exercise # 3

- Let **X** = {0, 2, *a*}
- Find:
  a) |**X**|
  b) Proper subset of **X**
  c) Power set of **X**