

# CHAPTER 2

(Part 3)

## SEQUENCE, RECURRENCE RELATIONS & RECURSIVE ALGORITHM

# SEQUENCE & RECURRENCE RELATIONS

# Sequence

- Definition:

A sequence is a function whose domain is either all the integers between two given integers or all the integers greater than or equal to a given integer.

- In the sequence denoted

$$a_m, a_{m+1}, a_{m+2}, \dots a_n$$

- Note:

Each individual element  $a_k$  is called **term**;  $k$  = index/subscript

$a_m$  = initial term;  $a_n$  = final term

- An **explicit formula** or **general formula** for a sequence is a rule that shows how the values of  $a_k$  depend on  $k$ .

# Example:

- Consider the sequence  $\{a_n\}$ , where

$$a_n = \frac{1}{n}$$

- The list of the terms of this sequence, beginning with  $a_1$ , namely

$$a_1, a_2, a_3, a_4, \dots$$

start with

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$$

# Recurrence Relations

- A recurrence relations (or recurrence) for the **sequence**  $\{a_n\}$  is an equation that expresses  $a_n$  in terms of one or more of the previous **terms** of sequence, namely,  $a_0, a_1, a_2, \dots, a_{n-1}$ , for all integers  $n$  with  $n \geq n_0$ , where  $n_0$  is a nonnegative integer.
- A sequence is called a **solution** of a recurrence relation if its terms satisfy the recurrence relation.
- The information called the **initial condition(s)** for the recurrence must be provided to give enough information about the equation to get started.

# Simple Recurrence Relation

- The simplest form of a **recurrence relation** is the case where the next **term** depends only on the immediately previous term.
- For example, given an initial condition,  $a_1 = 3$ , the list of terms  $a_1, a_2, a_3, \dots$ , begin with 3, 8, 13, 18, 23,  $\dots$ , is generated from a recurrence relation defined by

$$a_n = a_{n-1} + 5, \quad n \geq 2$$

# $n$ -th Term of a Sequence

Recurrence relation can be used to compute any  $n$ -th term of the sequence.

## Example:

Let  $\{a_n\}$  be a sequence that satisfies the recurrence relation  $a_n = a_{n-1} + 3$  for  $n = 1, 2, 3, \dots$ , and suppose that  $a_0 = 2$ . What are  $a_1$ ,  $a_2$ , and  $a_3$ ?

### Solution:

We see from the recurrence relation that,

$$a_1 = a_0 + 3 = 2 + 3 = 5;$$

$$a_2 = a_1 + 3 = 5 + 3 = 8;$$

$$a_3 = a_2 + 3 = 8 + 3 = 11$$

## Example:

Consider the following sequence:

$$3, 9, 27, 81, 243, \dots$$

The above sequence shows a pattern:

$$3^1, 3^2, 3^3, 3^4, 3^5, \dots$$

$$a_1, a_2, a_3, a_4, a_5, \dots$$

Recurrence relation is defined by:

$$a_n = 3^n, n \geq 1$$

# Example:

Given initial condition,  $a_0 = 1$  and recurrence relation:

$$a_n = 1 + 2a_{n-1} , n \geq 1$$

First few sequence are:

$$a_1 = 1 + 2(1) = 3$$

$$a_2 = 1 + 2(3) = 7$$

$$a_3 = 1 + 2(7) = 15$$

**1, 3, 7, 15, 31, 63, ...**

## Example:

Given initial conditions,  $a_0 = 1$ ,  $a_1 = 2$  and recurrence relation:

$$a_n = 3( a_{n-1} + a_{n-2} ), \quad n \geq 2$$

First few sequence are:

$$a_2 = 3( 2 + 1 ) = 9$$

$$a_3 = 3( 9 + 2 ) = 33$$

$$a_4 = 3( 33 + 9 ) = 126$$

**1, 2, 9, 33, 126, 477, 1809, 6858, 26001,...**

# Example:

For a photo shoot, the staff at a company have been arranged such that there are 10 people in the front row and each row has 7 more people than in the row in front of it. Find the recurrence relation and compute number of staff in the first 5 rows.

## Solution:

Notice that the difference between the number of people in successive rows is a constant amount. This means that the  $n$ -th term of this sequence can be found using:

$$a_n = a_{n-1} + 7, \quad n \geq 2 \text{ with } a_1 = 10$$

Number of staff in the first 5 rows:

$$a_1 = 10,$$

$$a_2 = a_1 + 7 \Rightarrow 10 + 7 = 17,$$

$$a_3 = a_2 + 7 \Rightarrow 17 + 7 = 24,$$

$$a_4 = a_3 + 7 \Rightarrow 24 + 7 = 31,$$

$$a_5 = a_4 + 7 \Rightarrow 31 + 7 = 38.$$

**10, 17, 24, 31, 38**

## Example:

Find a recurrence relation and initial condition for

**1, 5, 17, 53, 161, 485, ...**

### Solution:

Look at the differences between terms:

**4, 12, 36, 108, ...**

the difference in the sequence is growing by a factor of 3.

## Solution (cont'd):

However the original sequence is not.

$$1(3)=\mathbf{3}, 5(3)=\mathbf{15}, 17(3)=\mathbf{51}, \dots$$

$$1, 5, 17, \mathbf{53}, \mathbf{161}, \mathbf{485}, \dots$$

It appears that we always end up with 2 less than the next term.

So, the recurrence relation is defined by:

$$a_n = 3(a_{n-1}) + 2, \quad n \geq 1, \quad \text{with initial condition, } a_0 = 1$$

## Example:

A depositor deposits RM10,000 in a savings account at a bank yielding 5% per year with interest compounded annually. How much money will be in the account after 30 years? Let  $P_n$  denote the amount in the account after  $n$  years.

## Solution:

Derive the following **recurrence relation**:

$$P_n = P_{n-1} + 0.05P_{n-1} = 1.05P_{n-1}$$

Where,  $P_n$  = Current balance and  $P_{n-1}$  = Previous year balance and 0.05 is the compounding interest.

## Solution (cont'd):

Initial condition,  $P_0 = 10,000$ . Then,

$$P_1 = 1.05P_0$$

$$P_2 = 1.05P_1 = (1.05)^2P_0$$

$$P_3 = 1.05P_2 = (1.05)^3P_0$$

...

$$P_n = 1.05P_{n-1} = (1.05)^nP_0 ,$$

now we can use this formula to calculate  $n$ -th term without iteration.

## Solution (cont'd):

Let us use this formula to find  $P_{30}$  under the initial condition  $P_0 = 10,000$ :

$$P_{30} = (1.05)^{30}(10,000) = 43,219.42$$

After 30 years, the account contains RM 43,219.42.

# Exercise # 1

Consider the following sequence:

**1, 5, 9, 13, 17**

Find the recurrence relation that defines the above sequence.

## Exercise # 2

A basketball is dropped onto the ground from a height of 15 feet. On each bounce, the ball reaches a maximum height 55% of its previous maximum height.

- a) Write a recursive formula,  $a_n$ , that completely defines the height reached on the  $n_{\text{th}}$  bounce, where the first term in the sequence is the height reached on the ball's first bounce.
- b) How high does the basketball reach after the 4<sup>th</sup> bounce? Give your answer to two decimal places.

# RECURSIVE FUNCTION

# Recursive Function

- Many functions are defined in such away that the value at one input is defined in terms of other values of the function.
- We call such function as **recursive**.
- A recursive function is a function that invokes/calls itself.
- A recursive algorithm is an algorithm that contains a recursive.
- Recursive is a way that can be used to solve a large class of problems.

# Recursive Function (cont'd)

## Factorial problem

$$n! = n (n - 1) (n - 2) \dots$$

### Example:

$$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$$

$$\text{If } n \geq 1, \quad n! = n(n - 1)(n - 2) \dots 2 \times 1$$

Notice that, if  $n \geq 2$ ,  $n$  factorial can be written as,

$$n! = n(n-1)!$$

## Factorial problem (cont'd)

- $n=5$

- $5!$

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5 \cdot 4!$$

$$4! = 4 \cdot 3!$$

$$3! = 3 \cdot 2!$$

## Recursive Algorithm for Factorial

- Input:  $n$ , integer  $\geq 0$
- Output:  $n!$
- Factorial ( $n$ ) {  
    if ( $n=0$ )  
        return 1  
    return  $n * \text{factorial}(n-1)$   
}

# Recursive Function (cont'd)

## Fibonacci Sequence

- Fibonacci sequence,  $f_n$

$$f_1 = 1$$

$$f_2 = 1$$

for

$$f_n = f_{n-1} + f_{n-2}, \quad n \geq 3$$

1, 1, 2, 3, 5, 8, 13, ....

# Recursive Function (cont'd)

Recursive algorithm for Fibonacci Sequence :

- Input:  $n$
- Output:  $f(n)$
- $f(n)$  {  
    if ( $n=1$  or  $n=2$ )  
        return 1  
    return  $f(n-1) + f(n-2)$   
}

## Example:

Consider the following arithmetic sequence:

1, 3, 5, 7, 9, .....

Suppose  $a_n$  is the term sequence. The generating rule is

$$a_n = a_{n-1} + 2, \text{ for } n \geq 1$$

The relevant **recursive algorithm** can be written as

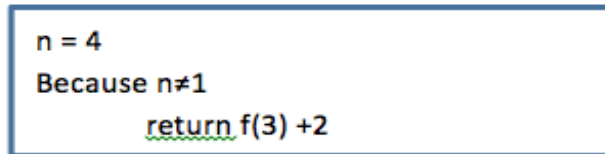
```
f(n)
{ if (n = 1)
    return 1
  return f(n-1) + 2
}
```

Use the above recursive algorithm to trace  $n = 4$ .

# Example- Solution

Trace the output if  $n = 4$

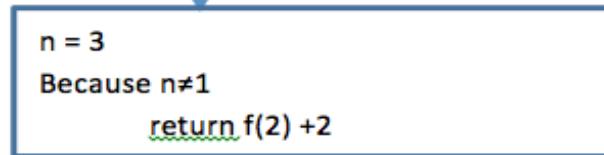
$f(4)$



$f(4) = 7$

Return  $5 + 2$

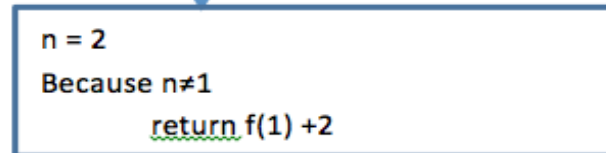
$f(3)$



$f(3) = 5$

Return  $3 + 2$

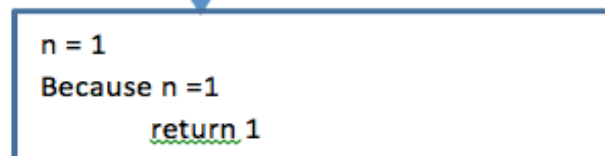
$f(2)$



$f(2) = 3$

Return  $1 + 2$

$f(1)$



$f(1) = 1$

Return  $1$

Answer = 7

## Exercise # 3

Shira and Zul are purchasing a new house costing RM300,000 with a down payment of RM30,000 and a long-term mortgage. The unpaid balance is being financed at a monthly rate of 1.2% on the unpaid balance and a payment of RM2000 per month. Find a recurrence relation and an initial condition to determine the unpaid balance after  $n$  monthly payments.