CHAPTER 2

(Part 3)

SEQUENCE, RECURRENCE RELATIONS & RECURSIVE ALGORITHM

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SEQUENCE
&
RECURRENCE RELATIONS

Sequence

- Definition:

  A sequence is a function whose domain is either all the integers between two given integers or all the integers greater than or equal to a given integer.

- In the sequence denoted

$$a_m, a_{m+1}, a_{m+2}, \ldots a_n$$

- Note:

  Each individual element a_k is called ***term***; k = index/subscript

  a_m = initial term; a_n = final term

- An explicit formula or general formula for a sequence is a rule that shows how the values of a_k depend on k.

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Example:

- Consider the sequence $\{a_n\}$, where

$$a_n = \frac{1}{n}$$

- The list of the terms of this sequence, beginning with a_1, namely

$$a_1, a_2, a_3, a_4, \ldots.$$

start with

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots.$$

Recurrance Relations

- A recurrence relations (or recurrence) for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of sequence, namely, a_0, a_1, $a_2,\ldots,a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

- A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.

- The information called the initial condition(s) for the recurrence must be provided to give enough information about the equation to get started.

Simple Recurrence Relation

- The simplest form of a **recurrence relation** is the case where the next term depends only on the immediately previous term.

- For example, given an initial condition, $a_1 = 3$, the list of terms $a_1, a_2, a_3, \ldots,$ begin with 3, 8, 13, 18, 23, . . ., is generated from a recurrence relation defined by

$$a_n = a_{n-1} + 5, \qquad n \geq 2$$

n-th Term of a Sequence

Recurrence relation can be used to compute any n-th term of the sequence.

Example:

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1, 2, 3, \ldots,$ and suppose that $a_0 = 2$. What are a_1, a_2, and a_3?

Solution:

We see from the recurrence relation that,

$a_1 = a_0 + 3 = 2+3 = 5;$
$a_2 = a_1 + 3 = 5+3 = 8;$
$a_3 = a_2 + 3 = 8+3 = 11$

Example:

Consider the following sequence:

$$3, 9, 27, 81, 243, \ldots$$

The above sequence shows a pattern:

$$3^1,\ 3^2,\ 3^3,\ 3^4,\ 3^5, \ldots$$
$$a_1,\ a_2,\ a_3,\ a_4,\ a_5, \ldots$$

Recurrence relation is defined by:

$$a_n = 3^n, n \geq 1$$

Example:

Given initial condition, $a_0 = 1$ and recurrence relation:

$$\boldsymbol{a_n = 1 + 2a_{n-1} \quad , n \geq 1}$$

First few sequence are:

$$a_1 = 1 + 2\,(1) = 3$$
$$a_2 = 1 + 2(3) = 7$$
$$a_3 = 1 + 2(7) = 15$$

1, 3, 7, 15, 31, 63, …

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Example:

Given initial conditions, $a_0 = 1$, $a_1 = 2$ and recurrence relation:

$$\boldsymbol{a_n = 3(a_{n-1} + a_{n-2}),\ n \geq 2}$$

First few sequence are:

$$a_2 = 3(2 + 1) = 9$$
$$a_3 = 3(9 + 2) = 33$$
$$a_4 = 3(33 + 9) = 126$$

1, 2, 9, 33, 126, 477, 1809, 6858, 26001,…

Example:

For a photo shoot, the staff at a company have been arranged such that there are 10 people in the front row and each row has 7 more people than in the row in front of it. Find the recurrence relation and compute number of staff in the first 5 rows.

Solution:

Notice that the difference between the number of people in successive rows is a constant amount. This means that the *n-th* term of this sequence can be found using:

$$\boldsymbol{a_n} = \boldsymbol{a_{n-1}} + 7,\ \ n \geq 2 \text{ with } \boldsymbol{a_1 = 10}$$

Number of staff in the first 5 rows:

$\boldsymbol{a_1 = 10}$,

$\boldsymbol{a_2 = a_1} + 7 \Rightarrow\ 10 + 7 = 17,$

$\boldsymbol{a_3 = a_2} + 7 \Rightarrow\ 17 + 7 = 24,$

$\boldsymbol{a_4 = a_3} + 7 \Rightarrow\ 24 + 7 = 31,$

$\boldsymbol{a_5 = a_4} + 7 \Rightarrow\ 31 + 7 = 38.$

10, 17, 24, 31, 38

Example:

Find a recurrence relation and initial condition for

1, 5, 17, 53, 161, 485, …

Solution:

Look at the differences between terms:

4, 12, 36, 108, …

the difference in the sequence is growing by a factor of 3.

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Solution (cont'd):

However the original sequence is not.

$$1(3)=\mathbf{3},\ 5(3)=\mathbf{15},17(3)=\mathbf{51},\ldots$$

1, 5, 17, 53, 161, 485, …

It appears that we always end up with 2 less than the next term.

So, the recurrence relation is defined by:

$$\boldsymbol{a_n = 3(a_{n-1}) + 2,\ n \geq 1, \text{ with initial condition, } a_0 = 1}$$

Example:

A depositor deposits RM10,000 in a savings account at a bank yielding 5% per year with interest compounded annually. How much money will be in the account after 30 years? Let P_n denote the amount in the account after n years.

Solution:

Derive the following **recurrence relation**:

$$\boldsymbol{P_n = P_{n\text{-}1} + 0.05P_{n\text{-}1} = 1.05P_{n\text{-}1}}$$

Where, $\boldsymbol{P_n}$ = Current balance and $\boldsymbol{P_{n\text{-}1}}$ = Previous year balance and 0.05 is the compounding interest.

Solution (cont'd):

Initial condition, $P_0 = 10{,}000$. Then,

$$
\begin{aligned}
&P_1 = 1.05P_0 \\
&P_2 = 1.05P_1 = (1.05)^2P_0 \\
&P_3 = 1.05P_2 = (1.05)^3P_0 \\
&\ldots \\
&P_n = 1.05P_{n-1} = (1.05)^nP_0\ ,
\end{aligned}
$$

now we can use this formula to calculate *n-th* term without iteration.

Solution (cont'd):

Let us use this formula to find P_{30} under the initial condition $P_0 = 10{,}000$:

$$P_{30} = (1.05)^{30}(10{,}000) = 43{,}219.42$$

After 30 years, the account contains RM 43,219.42.

Exercise # 1

Consider the following sequence:

1, 5, 9, 13, 17

Find the recurrence relation that defines the above sequence.

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Exercise # 2

A basketball is dropped onto the ground from a height of 15 feet. On each bounce, the ball reaches a maximum height 55% of its previous maximum height.

a) Write a recursive formula, a_n , that completely defines the height reached on the n_{th} bounce, where the first term in the sequence is the height reached on the ball's first bounce.

b) How high does the basketball reach after the 4^{th} bounce? Give your answer to two decimal places.

RECURSIVE FUNCTION

Recursive Function

- Many functions are defined in such away that the value at one input is defined in terms of other values of the function.
- We call such function as recursive.
- A recursive function is a function that invokes/calls itself.
- A recursive algorithm is an algorithm that contains a recursive.
- Recursive is a way that can be used to solve a large class of problems.

Recursive Function (cont'd)

Factorial problem

$$n! = n\ (n - 1)\ (n - 2)\ldots.$$

Example:

$$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$$

If $n \geq 1$, $\quad n! = n(n-1)(n-2)\ \ldots .2 \times 1$

Notice that, if $n \geq 2$, n factorial can be written as,

$$n! = n(n\text{-}1)!$$

Factorial problem (cont'd)

- n=5
- 5!

$$5! = 5.\ 4.\ 3.\ 2.\ 1 = 5.\ 4!$$
$$4! = 4.\ 3!$$
$$3! = 3.\ 2!$$

Recursive Algorithm for Factorial

- Input: n, integer ≥ 0
- Output: $n!$
- Factorial (n) {

```
    if (n=0)
        return 1
    return n*factorial(n-1)
}
```

Recursive Function (cont'd)

Fibonacci Sequence

- Fibonacci sequence, f_n

$$f_1 = 1$$
$$f_2 = 1$$
$$f_n = f_{n-1} + f_{n-2}, \quad \text{for } n \geq 3$$

1, 1, 2, 3, 5, 8, 13, ...

Recursive Function (cont'd)

Recursive algorithm for Fibonacci Sequence :

- Input: n
- Output: $f(n)$
- $f(n)$ {

  if (n=1 or n=2)

  return 1

  return $f(n-1) + f(n-2)$

  }

Example:

Consider the following arithmetic sequence:

1, 3, 5, 7, 9, ………

Suppose a_n is the term sequence. The generating rule is

$$a_n = a_{n-1} + 2, \text{ for } n \geq 1$$

The relevant recursive algorithm can be written as

```
f (n)
  { if (n =1)
        return 1
    return f (n-1) + 2
  }
```

Use the above recursive algorithm to trace $n = 4$.

Example- Solution

Trace the output if n = 4

f(4)

n = 4
Because n≠1
return f(3) +2

f(3)

n = 3
Because n≠1
return f(2) +2

f(2)

n = 2
Because n≠1
return f(1) +2

f(1)

n = 1
Because n =1
return 1

f(4) =7

Return 5 +2

f(3) =5

Return 3 +2

f(2) =3

Return 1 +2

f(1) =1

Return 1

Answer = 7

Exercise # 3

Shira and Zul are purchasing a new house costing RM300,000 with a down payment of RM30,000 and a long-term mortgage. The unpaid balance is being financed at a monthly rate of 1.2% on the unpaid balance and a payment of RM2000 per month. Find a recurrence relation and an initial condition to determine the unpaid balance after n monthly payments.