CHAPTER 2

(Part 2)

FUNCTIONS

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Definition

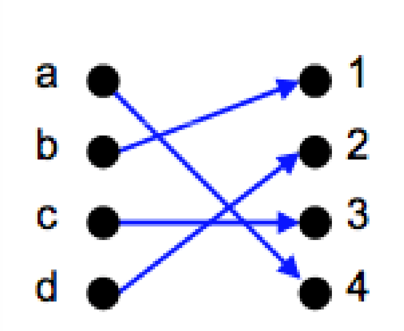
A **function** f from a set **A** to a set **B** is a relation with domain **A** and co-domain **B** that satisfies the following two properties:

i. For every element x in **A**, there is an element y in **B** such that $(x, y) \in f$.

ii. For all element x in **A** and y and z in **B**, $if\, (x, y) \in f$ and $(x, z) \in f$, then $y = z$

Relations vs Functions

- Not all relations are functions.
- But consider the following function:



- All functions are relations.

Relations vs Functions (cont'd)

When to use which?

- A **function** is used when you need to obtain a single result for any element in the domain.
  Example: sin, cos, tan

- A **relation** is when there are multiple mappings between the domain and the co-domain.
  Example: students enrolled in multiple courses.

Notation of Function: $f(x)$

If **A** and **B** are sets and f is a function from **A** to **B**, then given any element x in **A**, the unique element in **B** that is related to x by f is denoted $f(x)$, which is read " f to x ".

Example:

For the function, $f = \{(1,a), (2,b), (3,a)\}$

We may write:

$$f(1) = a,\ \ f(2) = b,\ \ f(3) = a$$

Example:

Defined: $f = \{(x, x^2) | x \text{ is a real number}\}$

- $f(x) = x^2$
- $f(2) = 4, \; f(-3.5) = 12.25, \; f(0) = 0$

Domain, Co-domain, Range

A function from a set **X** to a set **Y** is denoted, $f : \mathbf{X} \rightarrow \mathbf{Y}$

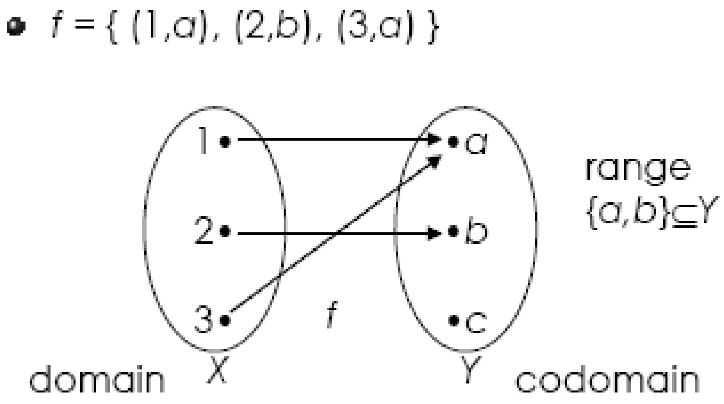
- The domain of f is the set **X**.
- The set **Y** is called the co-domain or target of f.
- The set $\{\, y \mid (x, y) \in f \,\}$ is called the range.

Example:

Given the relation, $f = \{(1,a), (2,b), (3,a)\}$ from $\mathbf{X} = \{1, 2, 3\}$ to $\mathbf{Y} = \{a, b, c\}$ is a function from **X** to **Y**. State the domain and range.

Solution:

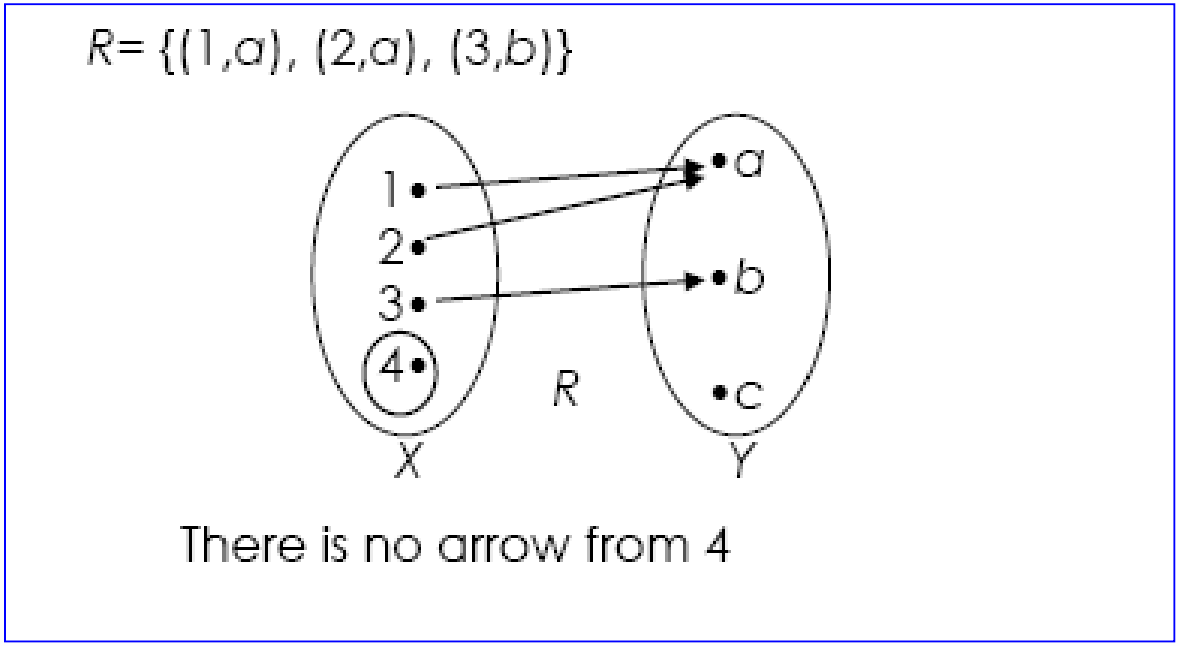
- ✓ The domain of f is **X**
- ✓ The range of f is $\{a, b\}$



Example:

The relation, $R = \{(1, a), (2, a), (3, b)\}$ from **X** = {1, 2, 3, 4} to **Y** = {*a*, *b*, *c*} is NOT a function from **X** to **Y**.

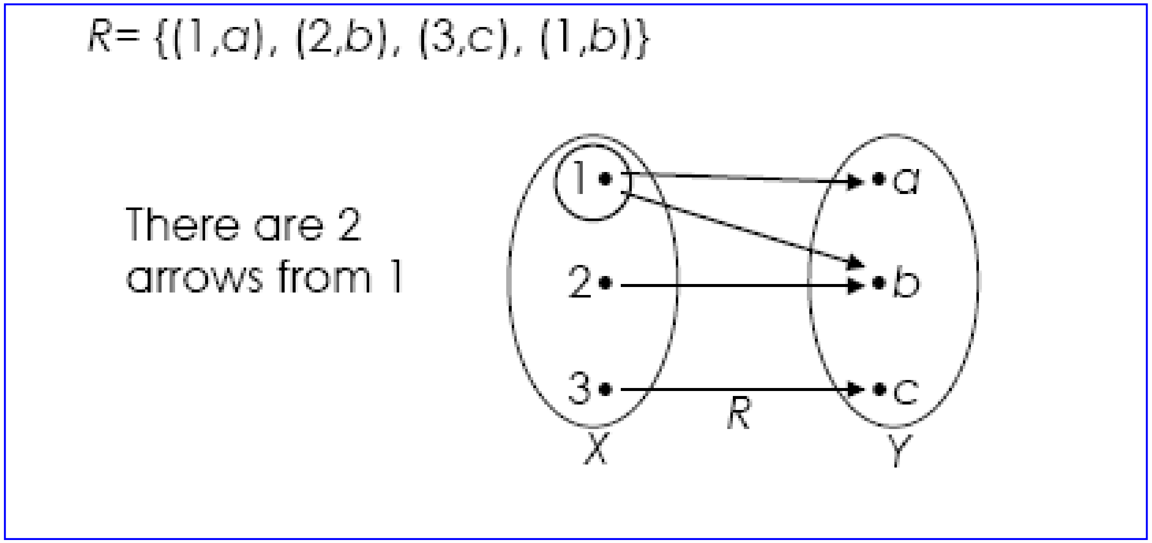
This is because the domain of *R*, {1, 2, 3} is not equal to **X**



Example:

The relation, $R = \{(1, a), (2, b), (3, c), (1, b)\}$ from **X** = {1, 2, 3} to **Y** = {a, b, c} is NOT a function from **X** to **Y**.

This is because (1, a) and (1, b) in R but $a \neq b$



One-to-One Function

Let f be a function from a set **X** to a set **Y**. f is **one-to-one** (or injective) if, and only if, for all elements x_1 and x_2 in **X**,

$$\text{if } f(x_1) = f(x_2), \text{ then } x_1 = x_2$$

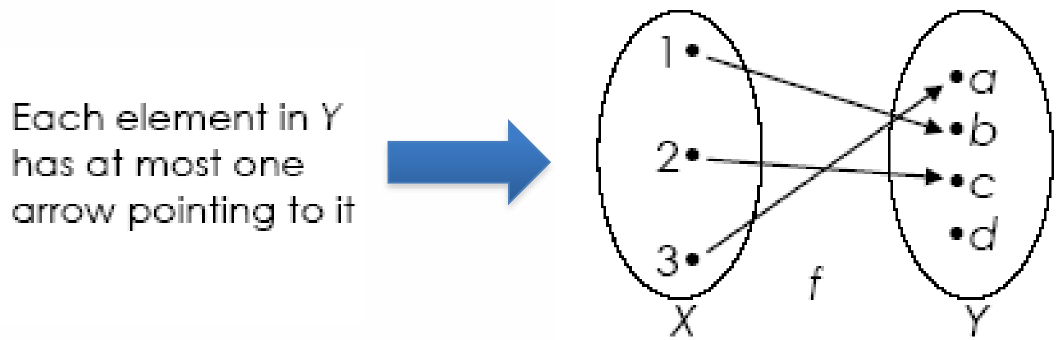
Or, equivalently, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

Symbolically,

$$f : X \rightarrow Y \text{ is one-to-one } \Leftrightarrow \text{"}\, x_1, x_2 \in X, \text{ if } f(x_1) = f(x_2) \text{ then } x_1 = x_2.$$

Example:

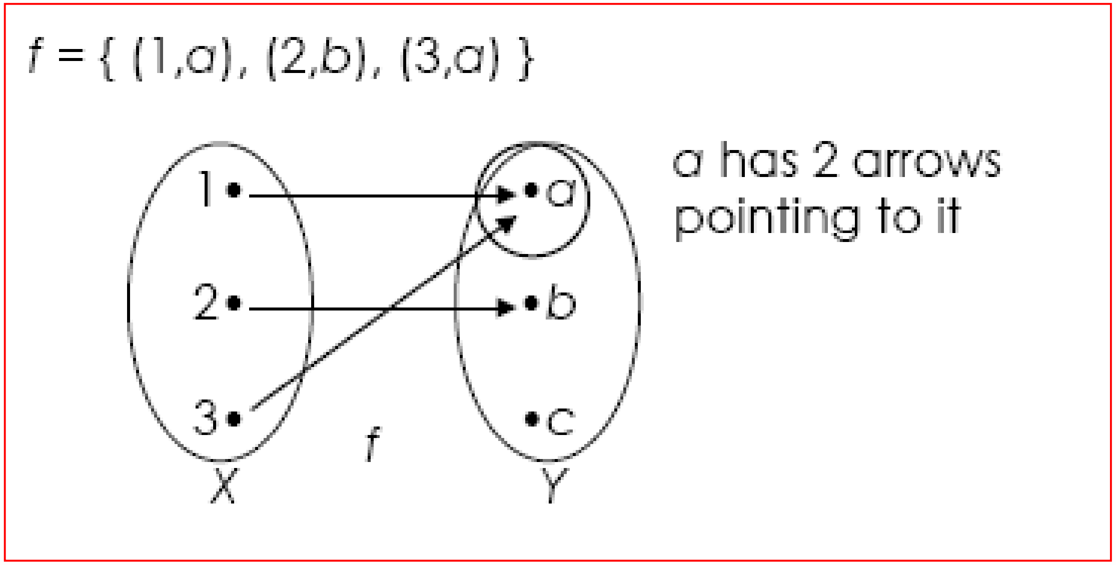
The function, $f = \{(1,b), (3,a), (2,c)\}$ from $\mathbf{X} = \{ 1, 2, 3 \}$ to $\mathbf{Y} = \{a, b, c, d\}$ is one-to-one.



In terms of arrow diagrams, a one-to-one function can be thought of as a function that separates points. That is, it takes distinct points of the domain to distinct points of the co-domain.

Example:

The function, $f = \{(1,a), (2,b), (3,a)\}$ from $\mathbf{X} = \{1, 2, 3\}$ to $\mathbf{Y} = \{a, b, c\}$ is NOT one-to-one.



Example

Show that the function,

$$f(n) = 2n + 1$$

on the set of positive integers is one-to-one.

Solution:

For all positive integer, n_1 and n_2 if $f(n_1) = f(n_2)$, then $n_1 = n_2$.

Let $f(n_1) = f(n_2)$, $f(n) = 2n + 1$, then

$$2n_1 + 1 = 2n_2 + 1 \qquad \ldots(-1)$$

$$2n_1 = 2n_2 \qquad \ldots(\div 2)$$

$$n_1 = n_2$$

This shows that f is one-to-one.

Example:

Show that the function,

$$f(n) = 2^n - n^2$$

on the set of positive integers is NOT one-to-one.

Solution:

Need to find 2 positive integers, n_1 and n_2,

$$n_1 \neq n_2 \; with \; f(n_1) = f(n_2).$$

Trial and error, $f(2) = f(4)$ →

$$f(n) = 2^n - n^2$$

$$n = 2 \Rightarrow 2^2 - 2^2 = 0$$

$$n = 4 \Rightarrow 2^4 - 4^2 = 0$$

This shows that f is not one-to-one.

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Onto Function

Let f be a function from a set **X** to a set **Y**. f is **onto** (or **surjective**) if, and only if, given any element y in **Y**, it is possible to find an element x in **X** with the property that $y = f(x)$.

Symbolically,

$$f : X \rightarrow Y \text{ is onto } \Leftrightarrow \text{"}\ y \in Y, \$x \in X \text{ such that } f(x) = y.$$

Example:

Let **X** = {1, 2, 3, 4} and **Y** = {*a*, *b*, *c*}. Define *h*: **X** → **Y** as follows:

$$h(1) = c,\ h(2) = a,\ h(3) = c,\ h(4) = b.$$

Define *k*: **X** → **Y** as follows:

$$k(1) = c,\ k(2) = b,\ k(3) = b,\ k(4) = c.$$

Is either *h* or *k* onto?

h is onto because each of the three elements of the co-domain of *h* is the image of some element of the domain of *h*.

k is not onto because $a \neq k(x)$ for any x in $\{1, 2, 3, 4\}$

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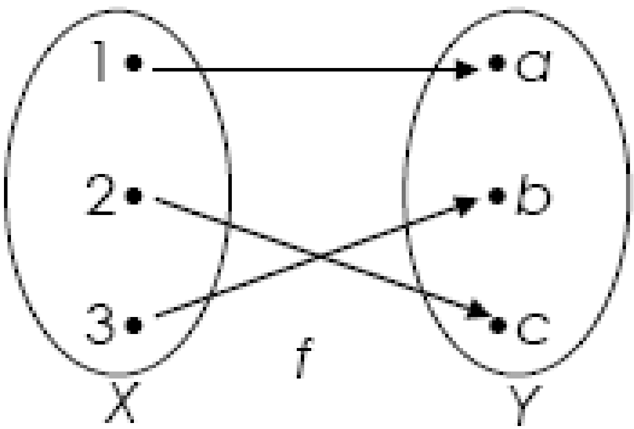
Example

The function, $f = \{(1,a), (2,c), (3,b)\}$ from $\mathbf{X} = \{1, 2, 3\}$ to $\mathbf{Y} = \{a, b, c\}$ is one-to-one and onto $\mathbf{Y}$.

- $f = \{ (1,a), (2,c), (3,b) \}$

One-to-one
Each element in Y has at most one arrow

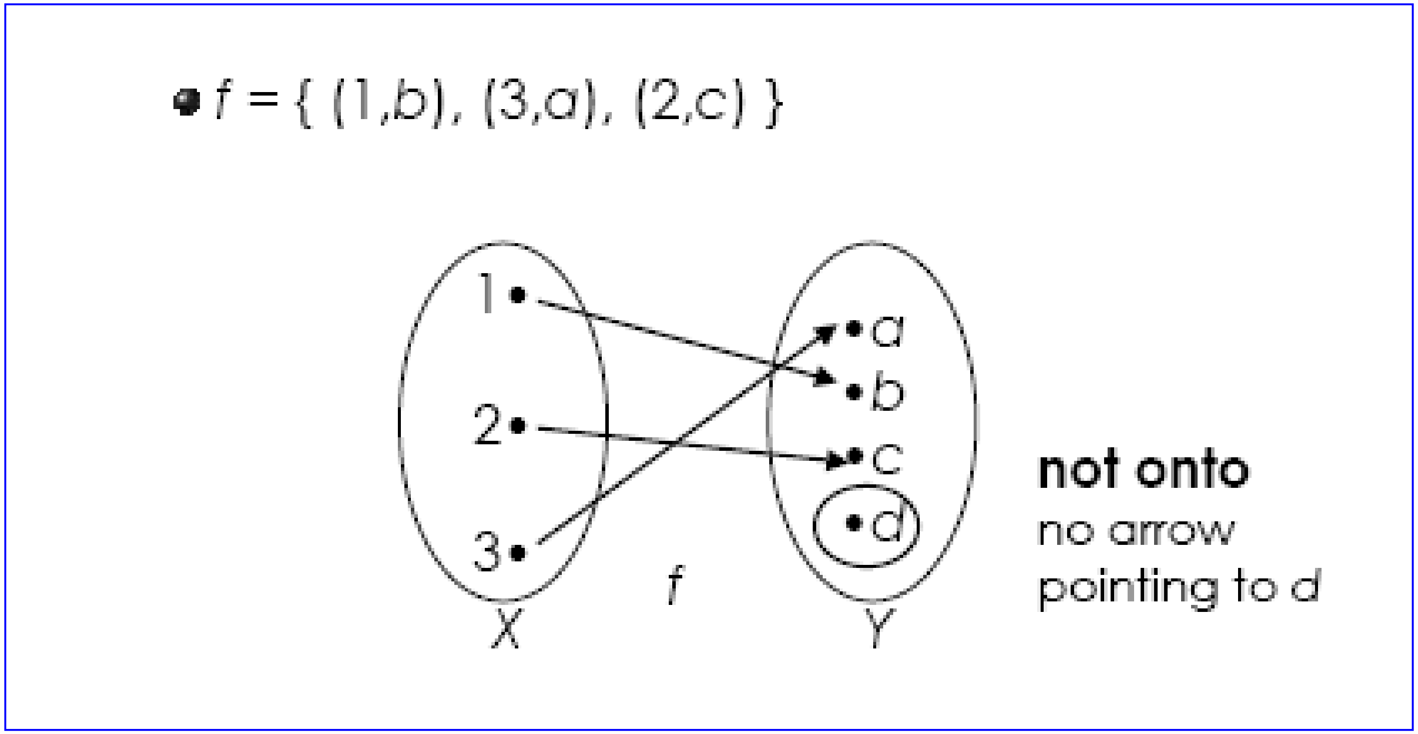


Onto
Each element in Y has at least one arrow pointing to it

Example:

The function, $f = \{(1,b), (3,a), (2,c)\}$ is not onto

$\mathbf{Y} = \{a, b, c, d\}$

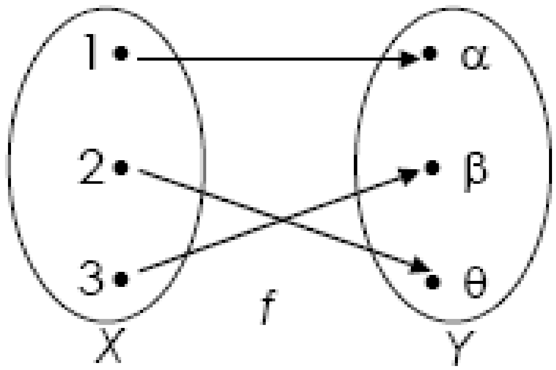


Bijection Function

A function, f is called one-to-one correspondence (or bijective/bijection) if f is both one-to-one and onto.

Example:

- $f = \{ (1,\alpha), (2,\theta), (3,\beta) \}$



One-to-one and onto Y
-**bijection**

Exercise # 1

Determine which of the relations f are functions from the set **X** to the set **Y**. In case any of these relations are functions, determine if they are one-to-one, onto **Y**, and/or bijection.

a) **X** = {-2, -1, 0, 1, 2} , **Y** = {-3, 4, 5} and
f = {(-2,-3), (-1,-3), (0,4), (1,5), (2,-3)}

b) **X** = {-2, -1, 0, 1, 2} , **Y** = {-3, 4, 5} and
f = {(-2,-3), (1,4), (2,5)}

c) **X** = **Y**= {-3, -1, 0, 2} and
f = {(-3,-1), (-3,0), (-1,2), (0,2), (2,-1)}

Exercise # 2

Let **X** = {1, 2, 3 }, **Y** = { 1, 2, 3, 4} and **Z** = {1, 2}.

i) Define a function f : **X** → **Y** that is one-to-one but not onto.

ii) Define a function g: **X** → **Z** that is onto but not one-to-one.

iii) Define a function h :X → X that is neither one-to-one nor onto.

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Inverse Function

If f is a one-to-one correspondence from a set **X** to a set **Y**, then there is a function from **Y** to **X** that "undoes" the action of f (it sends each element of **Y** back to the element of **X** that it came from). This function is called the inverse function for f.

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Theorem

Suppose f: **X** $\rightarrow$ **Y** is one-to-one correspondence; that is, suppose f is one-to-one and onto. Then there is a function

f^{-1}:**Y** $\rightarrow$ **X** that is defined as follows:

Given any element y in **Y**, $f^{-1}(y)$ = that unique element x in **X** such that $f(x) = y$.

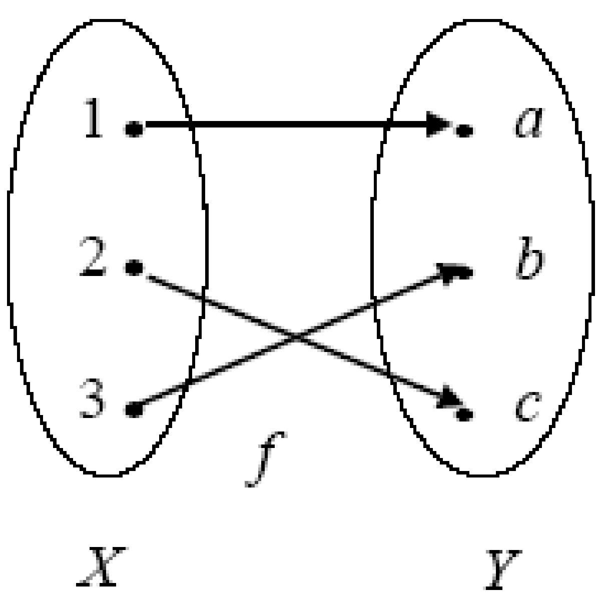
In other words,

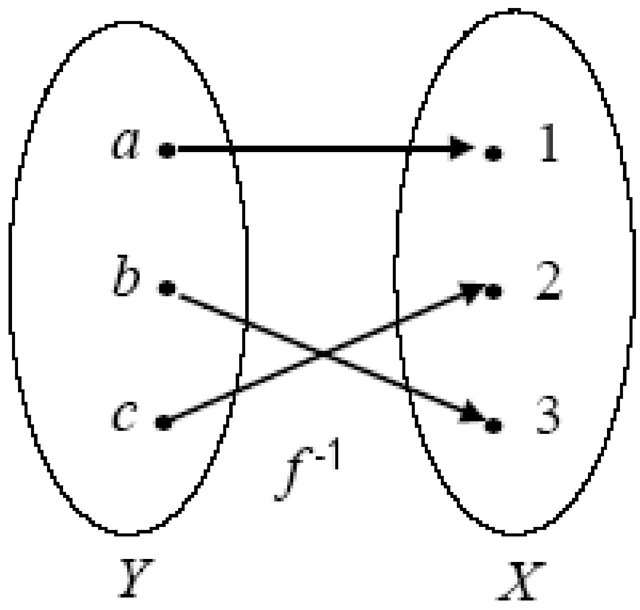
$$f^{-1}(y) = x \Leftrightarrow y = f(x)$$

Example:

$f = \{(1,a),(2,c),(3,b)\}$

$f^{-1} = \{(a,1),(c,2),(b,3)\}$





Example:

The function, f : **R** → **R** defined by the formula

$$f(x) = 4x - 1 \text{ for all } x \in \mathbf{R} \text{ (real number)}$$

This function is both one-to-one. Find the inverse function.

Solution:

For any [*particular but arbitrarily chosen*] y in **R**, by definition of f^{-1},

$$f^{-1}(y) = \text{that unique real number } x \text{ such that } f(x) = y.$$

But,

$$\begin{aligned} & f(x) = y \\ \Leftrightarrow\ & 4x - 1 = y \\ \Leftrightarrow\ & x = \frac{y+1}{4} \end{aligned}$$

Hence $f^{-1}(y) = \frac{y+1}{4}$

Exercise # 3

Find each inverse function:

a) $f(x) = 4x + 2,\ \ x \in \mathbf{R}$

b) $f(x) = 3 + \frac{1}{x},\ \ \ x \in \mathbf{R}$

Composition

Suppose that g is a function from **X** to **Y** and f is a function from **Y** to **Z**.

The composition of f with g, $f \circ g$ is a function

$$(f \circ g)(x) = f\big(g(x)\big)$$

from **X** to **Z**.

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Composition (cont'd)

Composition sometimes allows us to decompose complicated functions into simpler functions.

Example:

$f(x) = \sqrt{\sin 2x}$; $g(x) = \sqrt{x}$; $h(x) = \sin x$; $w(x) = 2x$

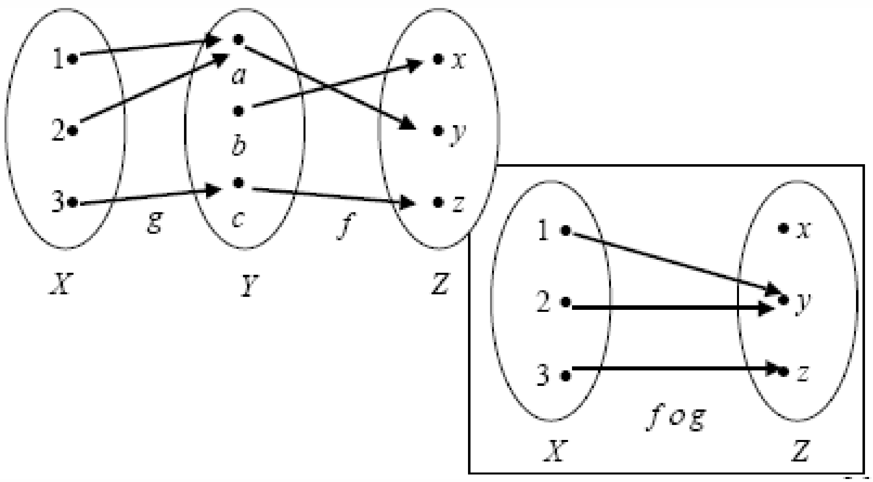
$$f(x) = g(h(w(x)))$$

Example:

Given, $g = \{(1,a), (2,a), (3,c)\}$ a function from **X** = $\{1, 2, 3\}$ to **Y** = $\{a, b, c\}$ and $f = \{(a,y), (b,x), (c,z)\}$ a function from **Y** to **Z** = $\{x, y, z\}$.

The composition function from **X** to **Z** is the function

$$f \circ g = \{(1, y), (2, y), (3, z)\}$$



Example:

Let, $f(x) = \log_3 x$, and $g(x) = x^4$. Find:

a) $f \circ g$

b) $g \circ f$

Solution:

a) $f \circ g = f(g(x)) = \log_3(x^4)$

b) $g \circ f = g(f(x)) = (\log_3 x)^4$

$\therefore Note : f \circ g \neq g \circ f$

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Example:

Define, $f : \mathbf{Z} \rightarrow \mathbf{Z}$ and $g: \mathbf{Z} \rightarrow \mathbf{Z}$ by the rules $f(a) = 7a$ and $g(a) = a \bmod 5$ for all integers a.

Find:

a) $(g \circ f)(0)$

b) $(g \circ f)(1)$

c) $(g \circ f)(2)$

d) $(g \circ f)(3)$

e) $(g \circ f)(4)$

Exercise # 4

Define, $f : \mathbf{Z} \rightarrow \mathbf{Z}$ and $g: \mathbf{Z} \rightarrow \mathbf{Z}$ by the rules $f(n) = n^3$, $g(n) = n$-1 for all integers n.

Find the compositions of the following:

a) $f \circ f$

b) $g \circ g$

c) $f \circ g$

d) $g \circ f$

e) Is $f \circ g = g \circ f$?