CHAPTER 2

[Part 1]

RELATIONS

nzah@utm.my

Relations on Sets

- The most basic relation is " = " (e.g. $x = y$)
- **Binary relation** is a subset of the Cartesian product of two sets.

Example: **A** and **B** are two sets. Define a relation R from **A** to **B**.

$$(x,y) \in \mathbf{A}\times\mathbf{B} \text{ and } R \subseteq \mathbf{A}\times\mathbf{B},$$

$$x\,R\,y \leftrightarrow (x,y) \in R$$

- If $(x,y) \in R$, where $x \in$ **A** and $y \in$ **B**, can be written as: $x\,R\,y$ => x is related to y.
- If $(x,y) \notin R$, where $x \in$ **A** and $y \in$ **B**, can be written as: $x \not{R}\, y$ => x is not related to y.

nzah@utm.my

Relations on Sets (cont'd)

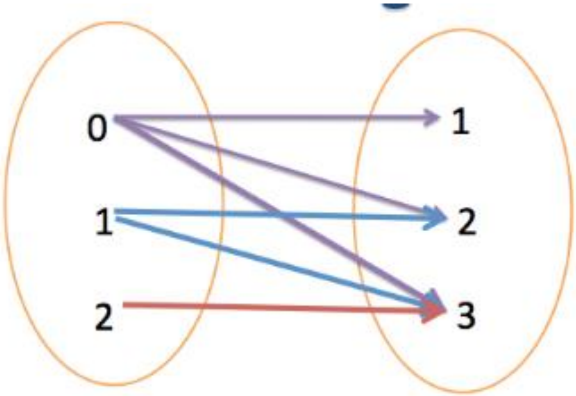
To show the relations, we can use one of the options:

i) Use traditional notation:

$0 < 1, 0 < 2, 0 < 3, 1 < 2, 1 < 3, 2 < 3$

ii) Use set notation: $R = \{(0,1),(0,2),(0,3), (1,2),(1,3),(2,3)\}$

iii) Use arrow diagrams:



$R=\{(0,1),(0,2),(0,3), (1,2),(1,3),(2,3)\}$

Example:

Suppose that **A** and **B** be sets,

A = {Ipoh, Kangar, Johor Bahru, Kuala Terengganu, Kuantan, Dungun, Muar};

B = {Terengganu, Johor, Perak, Perlis, Pahang}.

If $a \in$ **A**, $b \in$ **B**, and the relation between *a* and *b* is defined as "*a* is the city in the state *b*". Then,

R = {(Ipoh, Perak), (Kangar, Perlis), (Johor Bahru, Johor), (Kuantan, Pahang), (Kuala Terengganu, Terengganu), (Dungun, Terengganu), (Muar, Johor)}.

Example:

A and **B** are sets,
A = {1, 2, 3, 4}; **B** = {p, q, r}.

Define a relation S from **A** to **B** as follows:
$S = \{(1, q), (2, r), (3, q), (4, p)\}$, and $S \subseteq \mathbf{A} \times \mathbf{B}$

i) Is 1 S q ? Yes
ii) Is 3 S p ? No

Example:

Define a relation L from **R** to **R** as follows: For all real numbers x and y,

$$x \, L \, y \leftrightarrow x < y$$

i) Is 57 L 53 ?
ii) Is (-17) < -14 ?
iii) Is 143 L 143?
iv) Is (-35) L 1?

Solution:

i) No, 57 > 53.
ii) Yes.
iii) No, 143 = 143.
iv) Yes.

Example:

Define a relation R from $\mathbf{Z}$ to $\mathbf{Z}$ as follows: For all integer number m and n, $(m,n) \in \mathbf{Z}\times\mathbf{Z}$,

$$m\ R\ n \leftrightarrow m - n \text{ is even}$$

i) Is 4 R 0?

ii) Is 2 R 6?

iii) Is 3 R (-3)?

iv) Is 5 R 2?

v) List 5 integers that are related by R to 1.

Example - Solution:

i) Yes, 4 – 0 = 4 (even).

ii) Yes, 2 – 6 = -4 (even).

iii) Yes, 3 – (-3) = 6 (even).

iv) No, 5 – 2 = 3 (odd).

v) There are many such lists. One is

a) $1\ R\ 1 \Rightarrow 1-1 = 0$
b) $3\ R\ 1 \Rightarrow 3-1 = 2$
c) $5\ R\ 1 \Rightarrow 5-1 = 4$
d) $-3\ R\ 1 \Rightarrow -3-1 = -4$

.....

Domain & Range

Let R, a relation from **A** to **B**.

The **domain** of relations R is the set of all **first** elements in ordered pairs (a,b) which is the element of R,

$$\{a \in \mathbf{A} | (a, b) \in R \text{ for any } b \in \mathbf{B}\}$$

The **range** of relations R is the set of all **second** elements in ordered pairs (a,b) which is the element of R,

$$\{b \in \mathbf{B} | (a, b) \in R \text{ for any } a \in \mathbf{A}\}$$

Example:

Let R be a relation on $\mathbf{X} = \{1, 2, 3, 4\}$ defined by $(x,y) \in R$ if $x \leq y$, and $x,y \in \mathbf{X}$.

Then,

$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3),$
$(2,4), (3,3), (3,4), (4,4)\}$

The **domain and range** of $\boldsymbol{R}$ are both equal to $\mathbf{X}$.

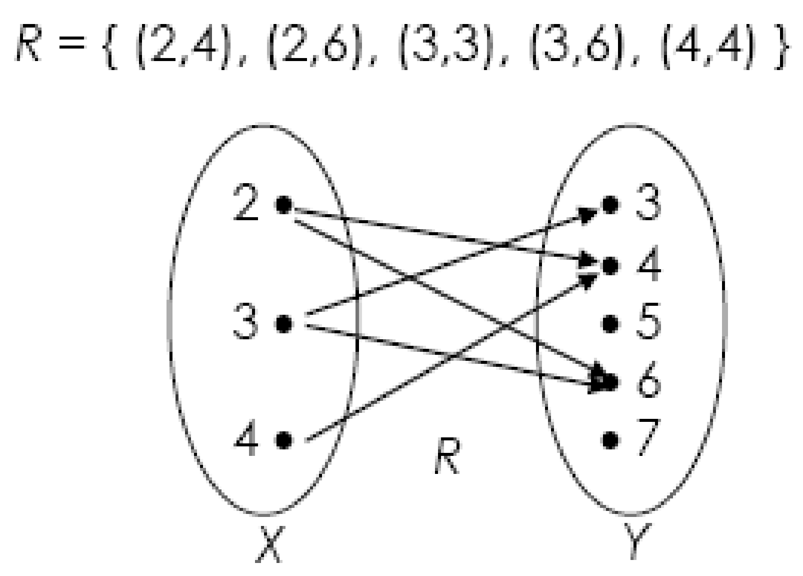
Example:

Let **X** = {2, 3, 4} and **Y** = {3, 4, 5, 6, 7}
If we define a relation *R* from **X** to **Y** by,

$$(x, y) \in R \text{ if } \frac{y}{x} \text{ (with zero remainder)}$$

We obtain,
R = {(2,4), (2,6), (3,3), (3,6), (4,4)}

The **domain** of *R* is {2,3,4}
The **range** of *R* is {3,4,6}



Arrow diagram

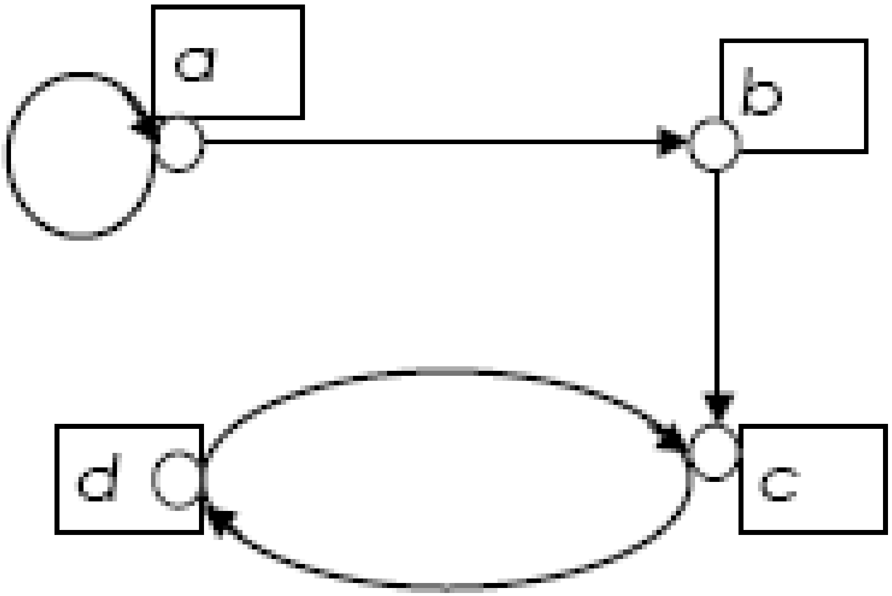
Directed Graph

- An informative way to **picture a relation** on a set is to draw its directed graph (digraph)
- The steps are:
  - Let R be a relation on a finite set **A**.
  - Draw dots (vertices) to represent the elements of **A**.
  - If the element $(a,b) \in R$, draw an arrow (called a directed edge) from a to b.

Example:

The relation R on $A = \{a, b, c, d\}$,

$R = \{(a, a), (a, b), (c, d), (d, c), (b,c)\}$

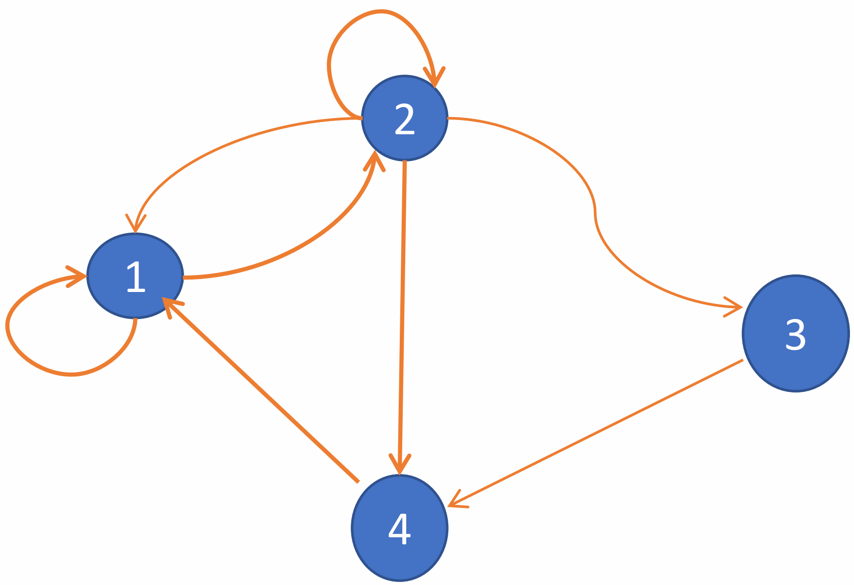


Example:

Let the relation R on **A** = {1,2,3,4},

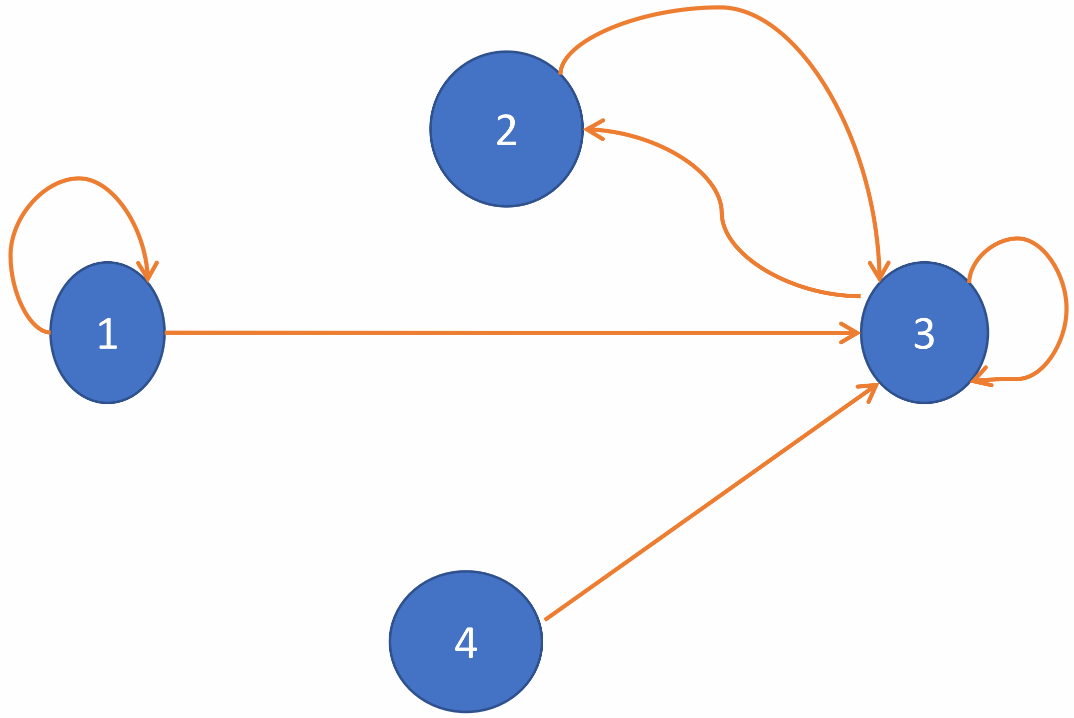
R = {(1,1), (1,2), (2,1), (2,2), (2,3), (2,4), (3,4) , (4,1)}

Draw the digraph of ***R***.



Example:

Find the relation determined by digraph below.



Answer: $R = \{(1,1), (1,3), (2,3), (3,2), (3,3), (4,3)\}$

Representing Relations Using Matrices

A relation is defined to be a set of pairs, but it can also be represented in other ways, e.g., by **matrix**.

- A matrix is simply a rectangular array of numbers.
- If a matrix has *n* rows and *m* columns, we call it an *n* x *m* matrix.
- In a matrix M, the number at the intersection of its *i*-th row and *j*-th column is written as M_{ij}.

Representing Relations Using Matrices (cont'd)

A matrix is a convenient way to represent a relation *R* from **A** to **B**.

- Label the rows with the elements of **A** (in some arbitrary order).
- Label the columns with the elements of **B** (in some arbitrary order).

Representing Relations Using Matrices (cont'd)

Let $A=\{a_1, a_2,\ldots, a_n\}$ and $B=\{b_1, b_2,\ldots, b_p\}$ be finite nonempty sets.

Let R be a relation from A into B.

Let $M_R = [m_{ij}]_{nxp}$ be the Boolean nxp matrix, where

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{otherwise} \end{cases}$$

Representing Relations Using Matrices (cont'd)

$$M_R = \begin{bmatrix} m_{11} & m_{12} & .. & .. & m_{1p} \\ m_{21} & m_{22} & .. & .. & m_{2p} \\ \vdots & \vdots & .. & .. & \vdots \\ \vdots & \vdots & .. & .. & \vdots \\ m_{n1} & m_{n2} & .. & .. & m_{np} \end{bmatrix}$$

UTM UNIVERSITI TEKNOLOGI MALAYSIA

Example:

Let **A** = {1, 3, 5} and **B** = {1,2}.
Let R be a relation from **A** to **B**, where
R = {(1,1), (3,2), (5,1)}.
Then the matrix represent R is:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Example:

- The relation,
  $R = \{ (1,b),(1,d),(2,c),(3,c),(3,b),(4,a) \}$
  from, $X = \{ 1, 2, 3, 4 \}$ to $Y = \{ a, b, c, d \}$

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array} \qquad \begin{array}{c} \\ 2 \\ 3 \\ 4 \\ 1 \end{array} \begin{array}{c} \begin{array}{cccc} d & b & a & c \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

Example:

The matrix of the relation R from $\{ 2, 3, 4 \}$ to $\{ 5, 6, 7, 8 \}$ defined by

$$x \; R \; y \text{ if } x \text{ divides } y$$

$$\begin{array}{c c} & \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{array}{c} 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Example:

Let A={ a, b, c, d }

Let R be a relation on A.

R = { (a,a),(b,b),(c,c),(d,d),(b,c),(c,b) }

$$
\begin{array}{c}
\begin{array}{cccc} a & b & c & d \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \end{array}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\end{array}
$$

Example:

Let **A** = {1, 2, 3,4} and R is a relation on **A**.
Suppose R = {(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)}

i) What is R (represent)? $(x,y) \in R$ if $y > x$

ii) What is the matrix for R?

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Example:

An airline services the five cities c_1, c_2, c_3, c_4 and c_5.
Table below gives the cost (in dollars) of going from c_i to c_j.
Thus the cost of going from c_1 to c_3 is \$100, while the cost of going from c_4 to c_2 is \$200

To / From	c_1	c_2	c_3	c_4	c_5
c_1		140	100	150	200
c_2	190		200	160	220
c_3	110	180		190	250
c_4	190	200	120		150
c_5	200	100	200	150	

Example (cont'd):

If the relation R on the set of cities $\mathbf{A} = \{c_1, c_2, c_3, c_4, c_5\}$: $c_i \, R \, c_j$ if and only if the cost of going from c_i to c_j is defined and less than or equal to \$180. Find R.

Answer:

$$R = \{(c_1,c_2), (c_1,c_3), (c_1,c_4), (c_2,c_4),(c_3, c_1), (c_3,c_2), (c_4,c_3), (c_4,c_5), (c_4,c_5), (c_5,c_2), (c_5,c_4)\}$$

In Degree and Out Degree

If R is a relation on a set **A** and $a \in$ **A**, then the in-degree of a (relative to relation R) is the number of $b \in$ **A** such that $(b, a) \in R$.

The out degree of a is the number of $b \in$ **A** such that $(a, b) \in R$.

In terms of the digraph of R, is that the in-degree of a vertex is **the number of edges terminating at the vertex.**
The out-degree of a vertex is **the number of edges leaving the vertex.**

Example:

Let **A** = {*a*, *b*, *c*, *d*}, and let *R* be the relation on **A** that has the matrix (given below). Construct the digraph of *R*, and list in-degrees and out-degrees of all vertices.

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \!\!\! \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

Answer:

	a	*b*	*c*	*d*
In-degree	2	3	1	1
Out-degree	1	1	3	2

Exercises

Exercise #1

Let **A** = {3,4,5} and **B** = {2,4,6,8,10} and *R* be the relation from **A** to **B** defined by,

$$(a,b) \in R, \text{ if } 2a \leq b+1$$

Write the ordered pair of *R*.

Exercise #2

Find range and domain for:

(i) The relation R ={(1,2), (2,1), (3,3), (1,1), (2,2)} on **X** ={1, 2, 3}

(ii) The relation R on {1, 2, 3, 4} defined by

$$(x, y) \in R \text{ if } x^2 \leq y$$

Exercise #3

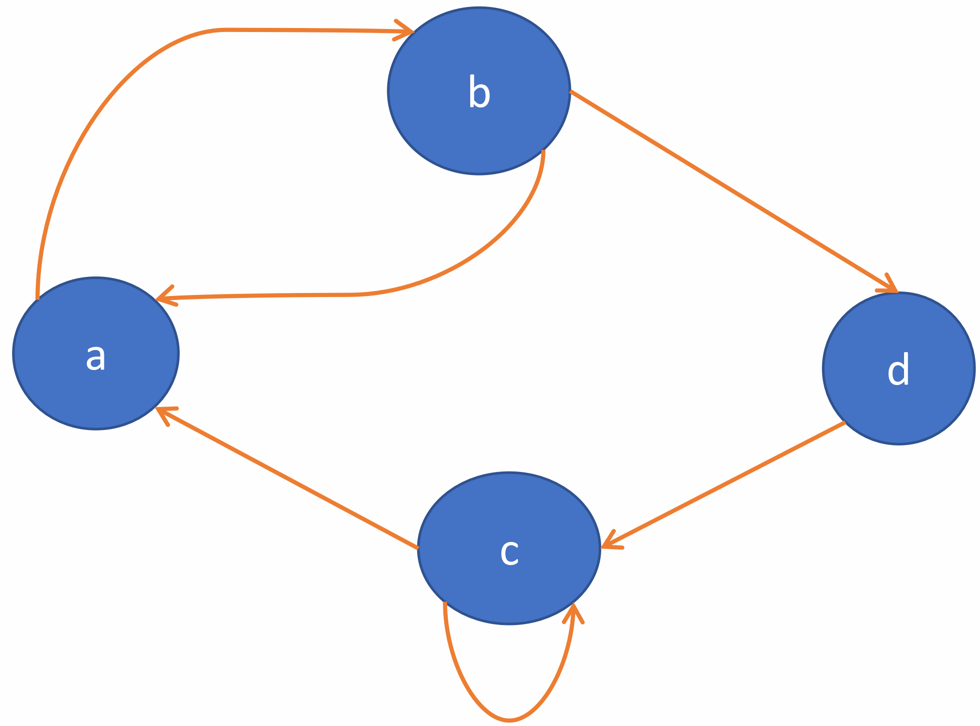
Draw the digraph of the following relation.

(i) $R = \{(a,6), (b,2), (a,1), (c,1)\}$

(ii) The relation $\boldsymbol{R}$ on {1, 2, 3, 4} defined by $(x,y) \in \boldsymbol{R}$ if $x^2 \geq y$

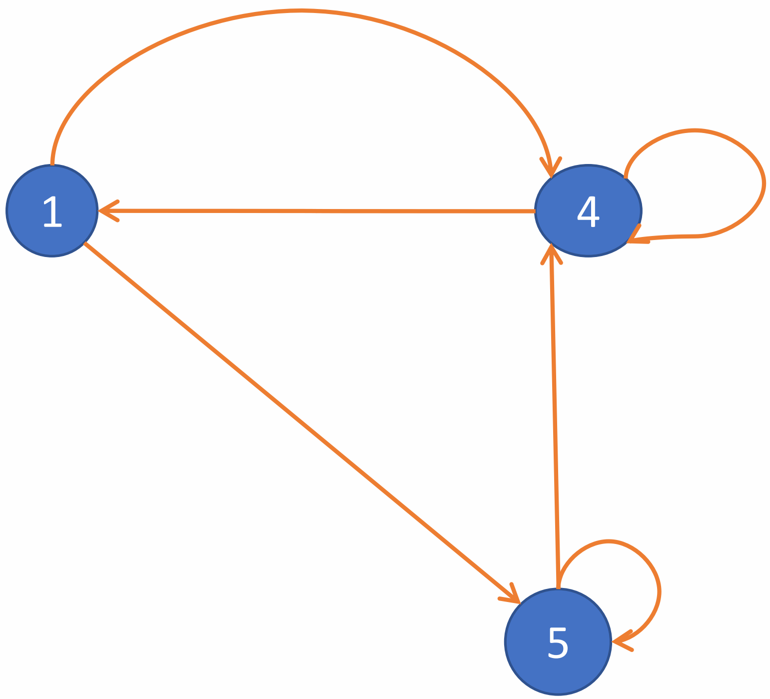
Exercise #3 (cont'd)

iii) Based on the digraph below, write the relation as a set of ordered pair.

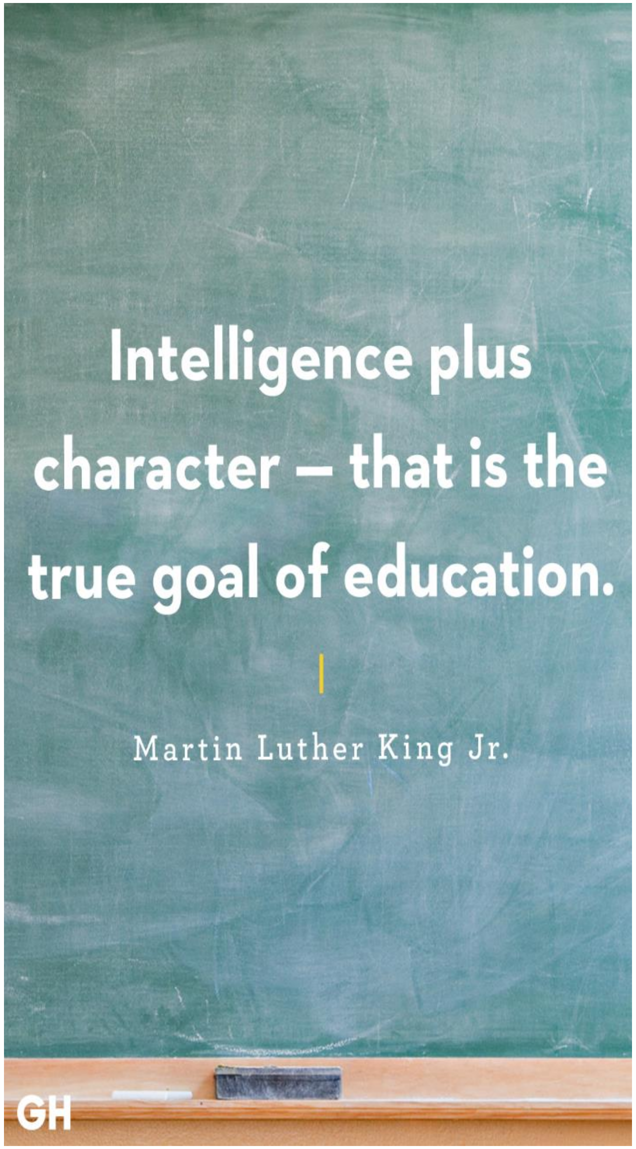


Exercise #4

Let **A** = {1, 4, 5} and let R be given by the digraph shown below. Find M_R and R. Then, list the in-degree and the out-degree.



Next..


Intelligence plus character – that is the true goal of education.
Martin Luther King Jr.
GH

Properties of Relations

Reflexive Relations

Let R be a relation on a set **A**. Then R is called **reflexive** if $(x,x) \in R$ for every $x \in$ **A**. That is,

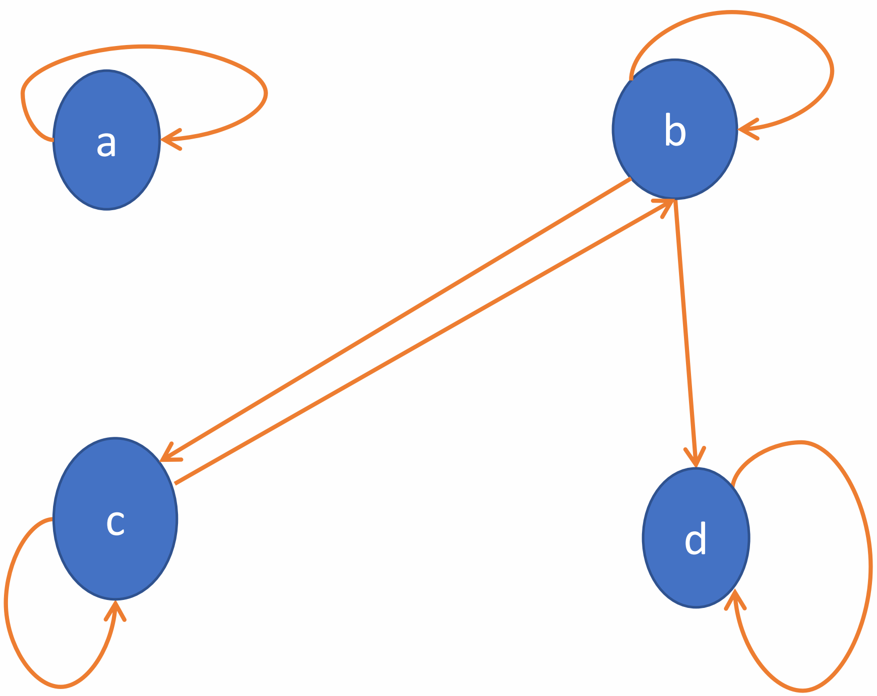
$x \; R \; x$ for all $x \in$ **A**

The matrix of reflexive relation must have the value **1** on its diagonal.

Digraph R will have a loop at every vertex.

Reflexive Relations (cont'd)

Digraph



Matrix

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 0 & 0 & 0 \\ b & 0 & 1 & 1 & 0 \\ c & 0 & 1 & 1 & 0 \\ d & 0 & 0 & 0 & 1 \end{array}$$

Example:

The relation R on $\mathbf{X} = \{1, 2, 3, 4\}$ defined by $(x,y) \in R$ if $x \leq y$, $x,y \in \mathbf{X}$

Determine whether R is a reflexive relation.

Solution:

$R = \{(1,1), (2,2), (3,3), (4,4)\}$

Since for each $x \in \mathbf{X}$, $(x,x) \in R$, thus R is a reflexive relation.

Irreflexive Relations

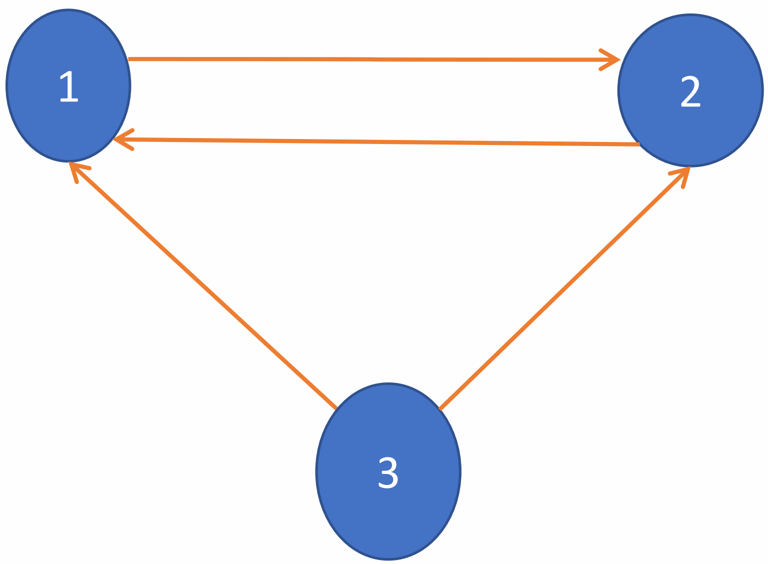
Let R be a relation on a set $\boldsymbol{A}$. Then R is called **irreflexive** if $(x,x) \notin R$ for every $x \in \mathbf{A}$.

The matrix of reflexive relation must have the value **0** on its diagonal.

Digraph R will do not have loop at any vertex.

Irreflexive Relations (cont'd)

Digraph



Matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

Not Reflexive Relations

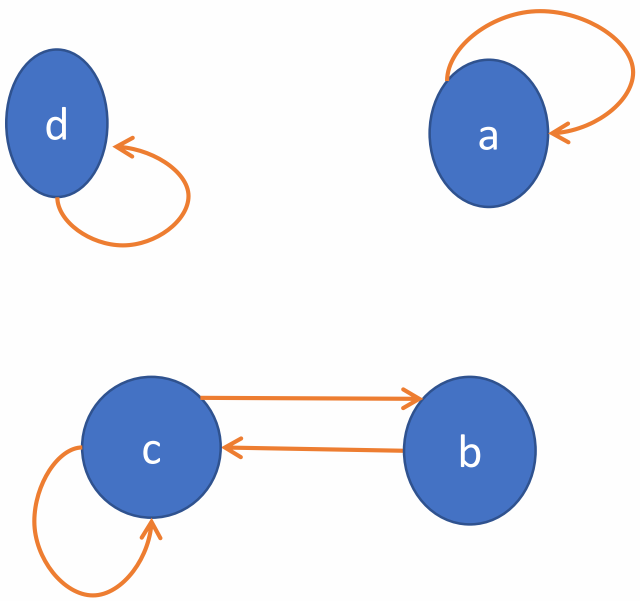
Let R be a relation on a set **A**. Then R is called **not reflexive** if there exist $(x,x) \in \cancel{R}$ for $x \in \mathbf{A}$.

The matrix of reflexive relation will have the value **0 and 1** on its diagonal.

Digraph R will have some loop at some vertices.

Not Reflexive Relations (cont'd)

Digraph



Matrix

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Example:

The relation $R=\{(a,a), (b,c), (c,b), (d,d)\}$ on $\mathbf{X} = \{a, b, c, d\}$ is not reflexive. Why?

Solution:

This is because $b \in \mathbf{X}$, but $(b,b) \notin R$.
Also $c \in \mathbf{X}$, but $(c,c) \notin R$.

Exercise #5

Consider the following relations on the set { 1, 2, 3}:

$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,1), (3,3)\}$

$R_2 = \{(1,1), (1,3), (2,2), (3,1)\}$

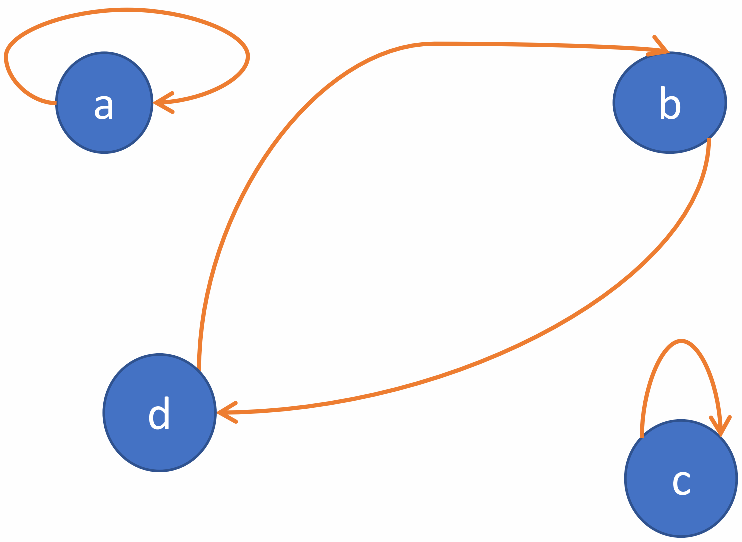
$R_3 = \{(2,3)\}$

$R_4 = \{(1,1)\}$

Which of them are reflexive?

Exercise #6

The relation R on $\mathbf{X}=\{a, b, c, d\}$ is shown on the digraph. Is R a reflexive relation? Justify your answer.



nzah@utm.my

Exercise #7

Let **A** = {1, 2, 3, 4}. Given R_i is a relation R on **A**. Construct a matrix for each R_i as follows. Then, determine whether the relation is reflexive, not reflexive or irreflexive.

(i) $R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,4)\}$.

(ii) $R_2 = \{(1,1), (1,2), (1,3), (3,1), (3,3), (4,4)\}$.

(iii) $R_3 = \{(1,1), (1,2), (1,3), (1,4), (3,1), (3,2), (3,3), (3,4), (4,2)\}$.

(iv) $R_4 = \{(1,2), (1,3), (1,4), (3,2), (3,4), (4,2)\}$.

Symmetric Relations

A relation R on a set $\mathbf{X}$ is called **symmetric** if for all $x,y \in \mathbf{X}$, if $(x,y) \in R$, then $(y,x) \in R$.

$$\forall x,y \in \mathbf{X},\ (x,y) \in R \rightarrow (y,x) \in R$$

Let M be the matrix of relation R. The relation R is symmetric if and only if for all i and j, the ij-th entry of M is equal to the ji-th entry of M.

Symmetric Relations (cont'd)

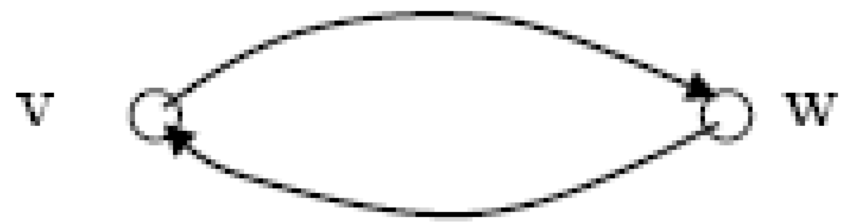
The matrix of relation M_R is symmetric if $M_R = M_R^T$

Example:

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} = M_R^T$$

Symmetric Relations (cont'd)

The digraph of a symmetric relation has the property that whenever there is a directed edge from v to w, there is also a directed edge from w to v.



Example:

The relation $R = \{(a,a),(b,c),(c,b),(d,d)\}$ on $\mathbf{X} = \{a,b,c,d\}$

$$\boxed{\begin{array}{l}(b,c) \in R \\ (c,b) \in R\end{array}}$$

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

symmetric

Antisymmetric

A relation R on set **A** is antisymmetric if $a \neq b$, whenever $a\ R\ b$, then $b\ R\ a$.

$$\forall a,b \in \mathbf{A},\ (a,b) \in R \wedge a \neq b \rightarrow (b, a) \notin R$$

or

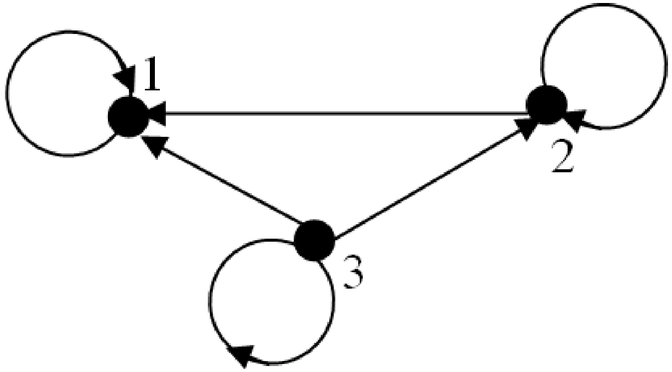
$$\forall a,b \in \mathbf{A},\ (a,b) \in R \wedge (b, a) \in R \rightarrow a = b$$

Antisymmetric (cont'd)

- Matrix $M_R = [M_{ij}]$ of an antisymmetric relation *R* satisfies the property that if $i \neq j$, then $m_{ij} = 0$ or $m_{ji} = 0$
- If *R* is antisymmetric relation, then for different vertices i and j there cannot be an edge from vertex i to vertex j and an edge from vertex j to vertex i, except $i=j$.
- In term of digraph, there is one directed relation and one way.

Example :

Let R be a relation on $\mathbf{A} = \{1, 2, 3\}$ defined as $(a, b) \in R$ if $a \geq b$; $a, b \in \mathbf{A}$



This is an antisymmetric relation because for all $a, b \in \mathbf{A}$, $(a, b) \in R$ and $a \neq b$, then $(b, a) \notin R$.

Here, $(3, 2) \in R$ but $(2, 3) \notin R$, and $(3, 3) \in R$ implies $a = b$

Example :

The relation R on $\mathbf{X} = \{1, 2, 3, 4\}$ defined by,

$$(x, y) \in R \text{ if } x \leq y;\ x, y \in \mathbf{X}$$

$$\boxed{\begin{array}{l} (1,2) \in R \\ (2,1) \notin R \end{array}}$$

$$\begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

antisymmetric

Example :

The relation

$R = \{ (a,a), (b,b), (c,c) \}$

on $X = \{ a, b, c \}$

R has no members of the form (x,y) with $x \neq y$, then R is antisymmetric

$$\begin{array}{c|ccc} & a & b & c \\ \hline a & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ c & 0 & 0 & 1 \end{array}$$

Asymmetric

- Let R be a relation on a set **A.** Then R is called **asymmetric** if $\forall a,b \in \mathbf{A}$, if $(a,b) \in R$, then $(b, a) \notin R$.

$$\boxed{\forall a,b \in \mathbf{A},\ (a,b) \in R \rightarrow (b,a) \notin R}$$

- In this sense, a relation is asymmetric if and only if it is both antisymmetric and irreflexive
- The matrix $M_R = [m_{ij}]$ of an asymmetric relation R satisfies the property that,
  - ✓ If $m_{ij} = 1$ then $m_{ji} = 0$
  - ✓ $m_{ii} = 0$ for all i (the main diagonal of matrix M_R consists entirely of 0's or otherwise)

Asymmetric (cont'd)

- If *R* is asymmetric relation, then the digraph of *R* cannot simultaneously have an edge from vertex *i* to vertex *j* and an edge from vertex *j* to vertex *i*.

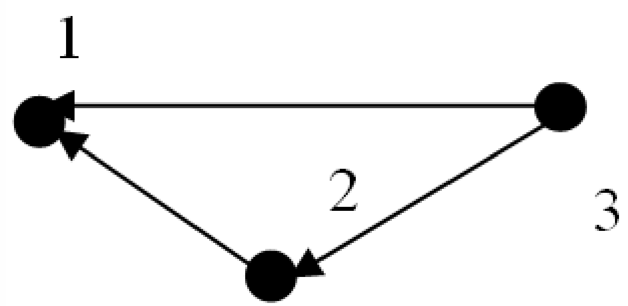
- All edges are one way and no loop.

- $M_R \neq M_R^T$

Example :

Let R be the relation on $\mathbf{A} = \{1, 2, 3\}$ defined by,

$$(a, b) \in R \text{ if } a > b;\ a,b \in \mathbf{A}$$



This is an asymmetric relation because,

$$(2, 1) \in R \text{ but } (1, 2) \notin R$$
$$(3, 1) \in R \text{ but } (1, 3) \notin R$$
$$(3, 2) \in R \text{ but } (2, 3) \notin R$$

Not Symmetric

Let R be a relation on a set **A**.

Then R is called **not symmetric**, if for all a, $b \in$ **A**, if $(a, b) \in R$, there exist $(b, a) \notin R$.

$$\exists a,b \in \mathbf{A}, (a, b) \in R \land (b, a) \notin R$$

Not Symmetric and Not Antisymmetric

Let R be a relation on a set **A**. Then R is called **not symmetric** and **not antisymmetric**, if and only if

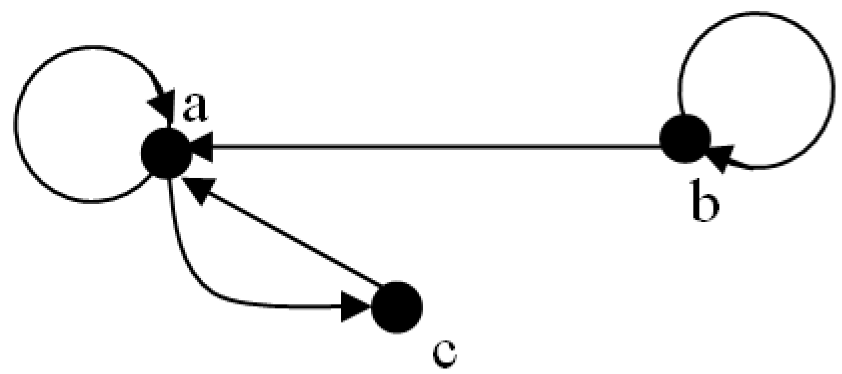
$$\exists a,b \in \mathbf{A},\ (a, b) \in R \wedge (b, a) \notin R$$

AND

$$\exists a,b \in \mathbf{A},\ (a, b) \in R \wedge a \neq b \wedge (b, a) \in R$$

Example:

Given a relation $R = \{(a, c), (b, b), (c, a), (b, a), (a, a)\}$ on $\mathbf{A} = \{a, b, c\}$.



This relation is not symmetric and not antisymmetric relation because there is $(a,c), (c,a) \in R$ and also $(b,a) \in R$ but $(a,b) \notin R$

Exercise #8

Let **A**={1,2,3,4} and let R = {(1,2), (2,2), (3,4), (4,1)} is a relation R on **A**.

Determine the properties of R is either symmetric, asymmetric or antisymmetric.

Exercise #9

Let **A**=**Z**, the set of integers and let,

$$R = \{(a,b) \in \mathbf{A} \times \mathbf{A} \mid a < b\}.$$

So that R is the relation "less than".

Is R symmetric, asymmetric or antisymmetric?

Exercise #10

Let **A** = {1,2,3,4}.

For each question below (i – iii), construct the matrix of relation of *R*. Then, determine whether the relation is symmetric, asymmetric, not symmetric or antisymmetric.

(i) $R = \{(1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,4)\}$

(ii) $R = \{(1,1), (1,2), (1,3), (3,1), (3,3), (4,4)\}$

(iii) $R =\{ (1,1), (1,2), (1,3), (1,4), (3,2), (3,3), (3,4), (4,2)\}$

Transitive Relations

A relation R on set **A** is **transitive** if for all $a,b,c \in$ **A**, if (a,b) and $(b,c) \in R$, then $(a,c) \in R$

$$\forall (a,b) \in \boldsymbol{A}, (a,b) \in \boldsymbol{R} \land (b,c) \in \boldsymbol{R} \rightarrow (a,c) \in \boldsymbol{R}$$

Normally, the matrix of the relation M_R is transitive if

$$M_R \otimes M_R = M_R$$

$\otimes$: product of Boolean

Product of Boolean

$$(1 \wedge 1) \vee (0 \wedge 0) \vee (1 \wedge 0) \vee (0 \wedge 0) = 1$$

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 0 & 1 & 0 \\ b & 0 & 1 & 0 & 0 \\ c & 0 & 1 & 1 & 0 \\ d & 0 & 0 & 0 & 1 \end{array} \otimes \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 0 & 1 & 0 \\ b & 0 & 1 & 0 & 0 \\ c & 0 & 1 & 1 & 0 \\ d & 0 & 0 & 0 & 1 \end{array} = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 1 & 1 & 0 \\ b & 0 & 1 & 0 & 0 \\ c & 0 & 1 & 1 & 0 \\ d & 0 & 0 & 0 & 1 \end{array}$$

$$(1 \wedge 0) \vee (0 \wedge 1) \vee (1 \wedge 1) \vee (0 \wedge 0) = 1$$

Example:

Consider the following relations on the set {1, 2, 3}:

$$R_1 = \{(1,1), (1,2), (2,3)\}$$
$$R_2 = \{(1,2), (2,3), (1,3)\}$$

Which of them is transitive?

Solution:

R_1 is not transitive because $(1, 2) \in R \wedge (2,3) \in R$, but $(1,3) \notin R$

R_2 is transitive because $(1, 2) \in R \wedge (2, 3) \in R \rightarrow (1,3) \in R$

Example:

The relation R on **A**=$\{a, b, c, d\}$ is $R = \{(a,a), (b,b), (c,c), (d,d), (a,c), (c,b)\}$ is not transitive. The matrix of relation M_R,

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$M_R \otimes M_R \neq M_R$$

The product of boolean,

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \otimes \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Note that,(a,c) and $(c,b) \in R$, $(a,b) \notin R$

Exercise #11

Let R be a relation on **A**=$\{1,2,3\}$ is defined by $(a,b) \in \boldsymbol{R}$ if $a \leq b$; $a,b \in \boldsymbol{A}$.

i) Find R.

ii) Is R a transitive relation?

Equivalence Relations

Relation R on set **A** is called an **equivalence relation** if it is a **reflexive, symmetric and transitive**.

Example:

Let R ={(1,1), (1,3), (2,2), (3,1), (3,3)} on {1,2,3}, the matrix of the relation M_R,

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

All the main diagonal matrix elements are 1, so it is **reflexive**.

Equivalence Relations (cont'd)

The transpose matrix M_R, M_R^T is equal to M_R, so R is **symmetric**.

$$M_R = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{array} \Rightarrow M_R^T = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{array}$$

The product of Boolean show that the matrix is **transitive**.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Partial Order Relations

Relation R on set **A** is called a **partial order relation** if it is a **reflexive, antisymmetric and transitive**.

Example:

Let R be a relation on a set **A**$=\{1,2,3\}$ defined by $(a,b) \in R$ if $a \leq b$, $a,b \in R$.

$$R = \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$$

Check each properties for partial order relations:

Reflexive: Since for each $x \in$ **A,** $(x,x) \in R$, thus R is a reflexive relation.

Antisymmetric: Since for all a, $b \in$ **A**, $(a, b) \in R$ and $a \neq b$, then $(b, a) \notin R$, thus R is antisymmetric.

Transitive: $(1, 2) \in R \wedge (2, 3) \in R \rightarrow (1,3) \in R$, thus R is transitive.

So, R is a partial order relation.