

CHAPTER 1

SET THEORY

[Part 1: Set & Subset]

Session 2020/2021 – Semester 1

Sets

- The concept of set is basic to all of mathematics and mathematical applications.
- A set is a **well-defined collection of distinct objects**.
- These objects are called **members** or **elements** of the set.

Sets (cont'd)

- Well-defined means that we can tell for certain whether an object is a member of the collection or not.
- If a set is finite and not too large, we can describe it by listing the elements in it.

Example

i) **A** is a set of all positive integers less than 10:

$$\mathbf{A} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

ii) **B** is a set of first 5 positive odd integers:

$$\mathbf{B} = \{1, 3, 5, 7, 9\}$$

iii) **C** is a set of vowels:

$$\mathbf{C} = \{a, e, i, o, u\}$$

Defining Sets

This can be done by:

- ✓ Listing ALL elements of the set within braces.
- ✓ Listing enough elements to show the pattern then an ellipsis.
- ✓ Use set builder notation to define “rules” for determining membership in the set.

Example

1. Listing ALL elements. $A = \{1, 2, 3, 4\}$
2. Demonstrating a pattern. $\mathbb{N} = \{1, 2, 3, \dots\}$
3. Using set builder notation. $P = \{x \mid x \in \mathbb{R} \text{ and } x \notin \mathbb{C}\}$

Note:

$\mathbb{N}$ = Natural numbers

$\mathbb{R}$ = Real numbers

$\mathbb{C}$ = Complex numbers

<https://www.mathsisfun.com/sets/number-types.html>

Defining Sets (cont'd)

A set is **determined by its elements** and **not by any particular order** in which the element might be listed.

Example:

For set $A = \{1, 2, 3, 4\}$, this set might just as well be specified as,

$\{2, 3, 4, 1\}$ or $\{4, 1, 3, 2\}$

Defining Sets (cont'd)

The elements making up a set are assumed to be **distinct**, we may have duplicates in our list, only one occurrence of each element is in the set.

Example:

$$\{a, b, c, a, c\} \longrightarrow \{a, b, c\}$$

$$\{1, 3, 3, 5, 1\} \longrightarrow \{1, 3, 5\}$$

Defining Sets (cont'd)

- Use uppercase letters $A, B, C \dots$ to denote sets, lowercase denote the elements of set.
- The symbol $\in$ stands for ‘belongs to’
- The symbol $\notin$ stands for ‘does not belong to’

Example

- $X = \{ a, b, c, d, e \}$, $b \in X$ and $m \notin X$
- $A = \{ \{1\}, \{2\}, 3, 4 \}$, $\{1\} \in A$ and $1 \notin A$

Defining Sets (cont'd)

- If a set is a large finite set or an infinite set, we can describe it by listing a property necessary for memberships.

- Let **S** be a set, the notation,

$$A = \{x \mid x \in S, P(x)\} \text{ or } A = \{x \in S \mid P(x)\}$$

means that **A** is the set of all x in **S** such that P of x .

Example

- Let $A = \{1, 2, 3, 4, 5, 6\}$, we can also write A as,
$$A = \{x \mid x \in \mathbf{Z}, 0 < x < 7\},$$
if $\mathbf{Z}$ denotes the set of integers.
- Let $B = \{x \mid x \in \mathbf{Z}, x > 0\} \rightarrow B = \{1, 2, 3, 4, \dots\}$

Example – Standard definition

The set of natural numbers: $\mathbb{N} = \{1, 2, 3, \dots\}$

The set of integers: $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

The set of positive integers: $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$

The set of Rational Numbers (fractions): $\frac{1}{2}, \frac{2}{3}, \frac{5}{7}, \text{etc} \in \mathbb{Q}$

More formally: $\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{R}, b \neq 0 \right\}$

The set of Irrational Numbers: $\sqrt{2}, \pi$, or e are irrational

The Real numbers = $\mathbb{R}$ = the union of the rational numbers
with the irrational numbers

Symbols Used With Set Builder Notation

The standard form of notation for this is called "set builder notation".

For instance, $\{x \mid x \text{ is an odd positive integer}\}$ represents the set $\{1, 3, 5, 7, 9, \dots\}$

$\{x \mid x \text{ is an odd positive integer}\}$ is read as

"the set consisting of all x such that x is an odd positive integer".

The vertical bar, " $\mid$ ", stands for "such that"

Other "short-hand" notation used in working with sets

" $\forall$ " stands for "for every"

" $\exists$ " stands for "there exists"

" $\cup$ " stands for "union"

" $\cap$ " stands for "intersection"

" $\subseteq$ " stands for "is a subset of"

" $\subset$ " stands for "is a (proper) subset of"

" $\not\subset$ " stands for "is not a (proper) subset of"

" $\emptyset$ " stands for the "empty set"

" $\in$ " stands for "is an element of"

" $\notin$ " stands for "is not an element of"

" $\times$ " stands for "cartesian cross product"

" $=$ " stands for "is equal to"

Subset [$\subseteq$]

- If every element of **A** is an element of **B**, we say that **A** is a **subset** of **B** and write $A \subseteq B$.

In notation: $A = B$, if $A \subseteq B$ and $B \subseteq A$

- The **empty set** ($\emptyset$) is a subset of every set.

Example

Let, set **A** = {1, 2, 3}

The **subset** of **A** are

$\emptyset$, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}

Note: **A** is a subset of **A**

Proper Subsets [$\subset$]

- If $A \subseteq B$ and B contains an element that is **not** in A (i.e., A doesn't equal B), then we say “ A is a **proper subset** of B ”: $A \subset B$ or $B \supset A$
- In notation: $A \subseteq B$ and $A \neq B$ ($B \not\subseteq A$)
- Formally: $A \subseteq B$ means $\forall x [x \in A \rightarrow x \in B]$
- For all sets: $A \subseteq A$

Example

- $A = \{1, 2, 3\}$
- The **proper subset** of A :
 $\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}$

Note:

A proper subset of a set A is a subset of A that is not equal to A : $\{1,2,3\} \subsetneq A$

Example

Lets, set **B** = {1, 2, 3, 4, 5, 6} and **A** = {1, 2, 3}

$\therefore$ **A** is a proper subset of **B**.

Example

Given a set as follows:

$$\mathbf{A} = \{a, b, c, d, e, f, g, h\}$$

$$\mathbf{B} = \{b, d, e\}$$

$$\mathbf{C} = \{a, b, c, d, e\}$$

$$\mathbf{D} = \{r, s, d, e\}$$

Which subset are proper subset of **A**?

Ans: **B** and **C**

Empty Sets [$\emptyset$ or $\{\}$]

- The **empty set** $\emptyset$ or $\{\}$ **but not** $\{\emptyset\}$ is the set without elements.
- Note:
 - ✓ Empty set has no elements.
 - ✓ Empty set is a subset of any set.
 - ✓ There is exactly one empty set.
 - ✓ Properties of empty set:

$$A \cup \emptyset = A; A \cap \emptyset = \emptyset$$

$$A \cap A' = \emptyset; A \cup A' = U$$

$$U' = \emptyset; \emptyset' = U$$

Example

i) $\emptyset = \{x \mid x \text{ is a real number and } x^2 = -3\}$

ii) $\emptyset = \{x \mid x \text{ is positive integer and } x^3 < 0\}$

Equal Sets [=]

- The set **A** and **B** are **equal** ($\mathbf{A} = \mathbf{B}$) if and only if each element of **A** is an element of **B** and vice versa.
- Formally: $\mathbf{A} = \mathbf{B}$ means $\forall x [x \in \mathbf{A} \leftrightarrow x \in \mathbf{B}]$

Example

i) $\mathbf{A} = \{a, b, c\}$; $\mathbf{B} = \{b, c, a\}$

$$\therefore \mathbf{A} = \mathbf{B}$$

ii) $\mathbf{C} = \{0, 2, 4, 6, 8\}$;

$$\mathbf{D} = \{x \mid x \text{ is a positive integer and } 2x < 10\}$$

$$\therefore \mathbf{C} = \mathbf{D}$$

Equivalent Set

- Two sets, **A** and **B** are **equivalent** if there exists a one-to-one correspondence between them.
- An equivalent set is simply a set with an **equal number of elements**.
- **Example:** Lets **A** and **B** be sets.

$$\mathbf{A} = \{a, b, c, d, e\} ; \mathbf{B} = \{1, 2, 3, 4, 5\}$$

$\therefore$ **A** and **B** are **equivalent**

Finite Set

- A set **A** is **finite** if it is empty or if there exists a nonnegative integer n such that **A** has n elements.
- Example: **A** = $\{1, 2, 3, \dots, n\}$; **A** is called a finite set with n elements).

Example

Let, **A** and **B** be sets.

$$\mathbf{A} = \{1, 2, 3, 4\};$$

$$\mathbf{B} = \{x \mid x \text{ is an integer, } 1 \leq x \leq 4\}$$

$\therefore$ **A** and **B** are **finite set**.

Infinite Sets

- A set **A** is **infinite** if there is not exists a natural number n such that **A** is equivalent to $\{1, 2, 3, \dots, n\}$.
- Infinite set is **uncountable**.
- Are all infinite sets equivalent?
 - A set is infinite if it is equivalent to a proper subset of itself!

Example

Lets, **B**, **C**, **D**, **Z** and **S** be sets.

$$\mathbf{C} = \{5, 6, 7, 8, 9, 10\}$$

$$\mathbf{B} = \{x \mid x \text{ is an integer, } 10 < x < 20\}$$

$$\mathbf{D} = \{x \mid x \text{ is an integer, } x > 0\}$$

$$\mathbf{Z} = \{x \mid x \text{ is an integer}\} \text{ or } \mathbf{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

$$\mathbf{S} = \{x \mid x \text{ is a real number and } 1 \leq x \leq 4\}$$

*$\therefore$ **C** and **B** are finite sets; **D**, **Z** and **S** are infinite sets.*

Universal Set

- Sometimes we are dealing with sets all of which are subsets of a set **U**.
- This set **U** is called a **universal** set or a universe.
- The set **U** must be explicitly given or inferred from the context.

Universal Set (cont'd)

- Typically we consider a set **A** as a part of a universal set, **U**, which consist of all possible elements.
- To be entirely correct we should say,

$\forall x \in \mathbf{U}[x \in \mathbf{A} \leftrightarrow x \in \mathbf{B}]$ **instead of** $\forall x[x \in \mathbf{A} \leftrightarrow x \in \mathbf{B}]$ for $\mathbf{A} = \mathbf{B}$

Note:

$\{x|0 < x < 5\}$ is can be ambiguous, compare to $\{x|0 < x < 5, x \in \mathbf{N}\}$ with $\{x|0 < x < 5, x \in \mathbf{Q}\}$

Example

- The sets **A** = {1,2,3}, **B** = {2,4,6,8} and **C** = {5,7}
- One may choose **U** = {1,2,3,4,5,6,7,8} as a universal set.
- Any **superset** of **U** can also be considered a universal set for these sets **A**, **B**, and **C**.
For example, **U** = { x | x is a positive integer}

Cardinality of Set

- Let **S** be a finite set with n distinct elements, where $n \geq 0$.
- Then we write $|\mathbf{S}| = n$ and say that the **cardinality** (or **the number of elements**) of **S** is n .
- **Example:** **A** and **B** be sets.

$$\mathbf{A} = \{1, 2, 3\} \rightarrow |\mathbf{A}| = 3$$

$$\mathbf{B} = \{a, b, c, d, e, f, g\} \rightarrow |\mathbf{B}| = 7$$

Power Set

- The set of all subsets of a set **A**, denoted $P(\mathbf{A})$, is called the **power set of A**.

$$P(\mathbf{A}) = \{x \mid x \subseteq \mathbf{A}\}$$

- If $|\mathbf{A}| = n$, then $|P(\mathbf{A})| = 2^n$

Example

- Lets, a set $\mathbf{A} = \{1,2,3\}$
- The power set of $\mathbf{A}$,
$$P(\mathbf{A}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$
- Notice that $|\mathbf{A}| = 3$, and $|P(\mathbf{A})| = 2^3 = 8$

Summary

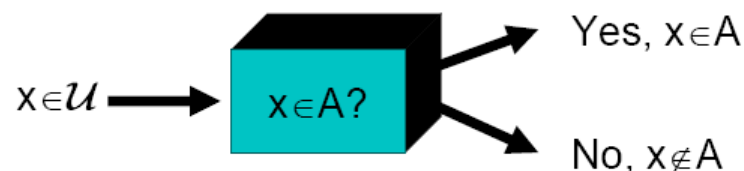
How to Think of Sets

The elements of a set do not have an ordering,
hence $\{a,b,c\} = \{b,c,a\}$

The elements of a set do not have multitudes,
hence $\{a,a,a\} = \{a,a\} = \{a\}$

All that matters is: “Is x an element of A or not?”

The size of A is thus the number of *different* elements



Exercise # 1

1. Determine whether each of these pairs of sets are equal.

(b) $\emptyset, \{\emptyset\}$

(c) $\{1, 2, \{2\}\}, \{1, 2\}$

2. For each of the following sets, determine whether 2 is an element of that set.

(a) $\{x \in \mathbf{R} \mid x \text{ is an integer greater than } 1\}$

(b) $\{x \in \mathbf{R} \mid x \text{ is the square of an integer}\}$

(c) $\{2, \{2\}\}$

(d) $\{\{2\}, \{2, \{2\}\}\}$

Exercise # 2

Determine whether each pair of sets is equal.

a) $\{1, 2, 2, 3\} ; \{1, 3, 2\}$

b) $\{x \mid x^2 + x = 2\} ; \{1, -2\}$

c) $\{x \mid x \text{ is a real number and } 0 < x \leq 2\}; \{1, 2\}$

Exercise # 3

- Let $X = \{0, 2, a\}$
- Find:
 - a) $|X|$
 - b) Proper subset of X
 - c) Power set of X