CHAPTER 4

GRAPH THEORY

(Part 3)

Shortest Path Problem

Introduction

- Let ***G*** be a weighted graph.
- Let ***u*** and ***v*** be two vertices in ***G***, and let ***P*** be a path in ***G*** from ***u*** to ***v***.
- The length of path ***P***, written ***L*(*P*)**, is the sum of the weights of all the edges on path ***P***.
- A **shortest path** from a vertex to another vertex is a path with the shortest length between the vertices.

Shortest Path using Dijkstra's Algorithm

1. $S := \emptyset$
2. $N := V$
3. For all vertices, $u \in V$, $u \neq a$, $L(u) := \infty$
4. $L(a) := 0$
5. While $z \notin S$ do,

   5.a :Let $v \in N$ be such that

   $L(v) = \min\{L(u) | u \in N\}$

   5.b : $S := S \in \{v\}$

   5.c : $N := N - \{v\}$

   5.d : For all $w \in N$ such that there is an edge from v to w

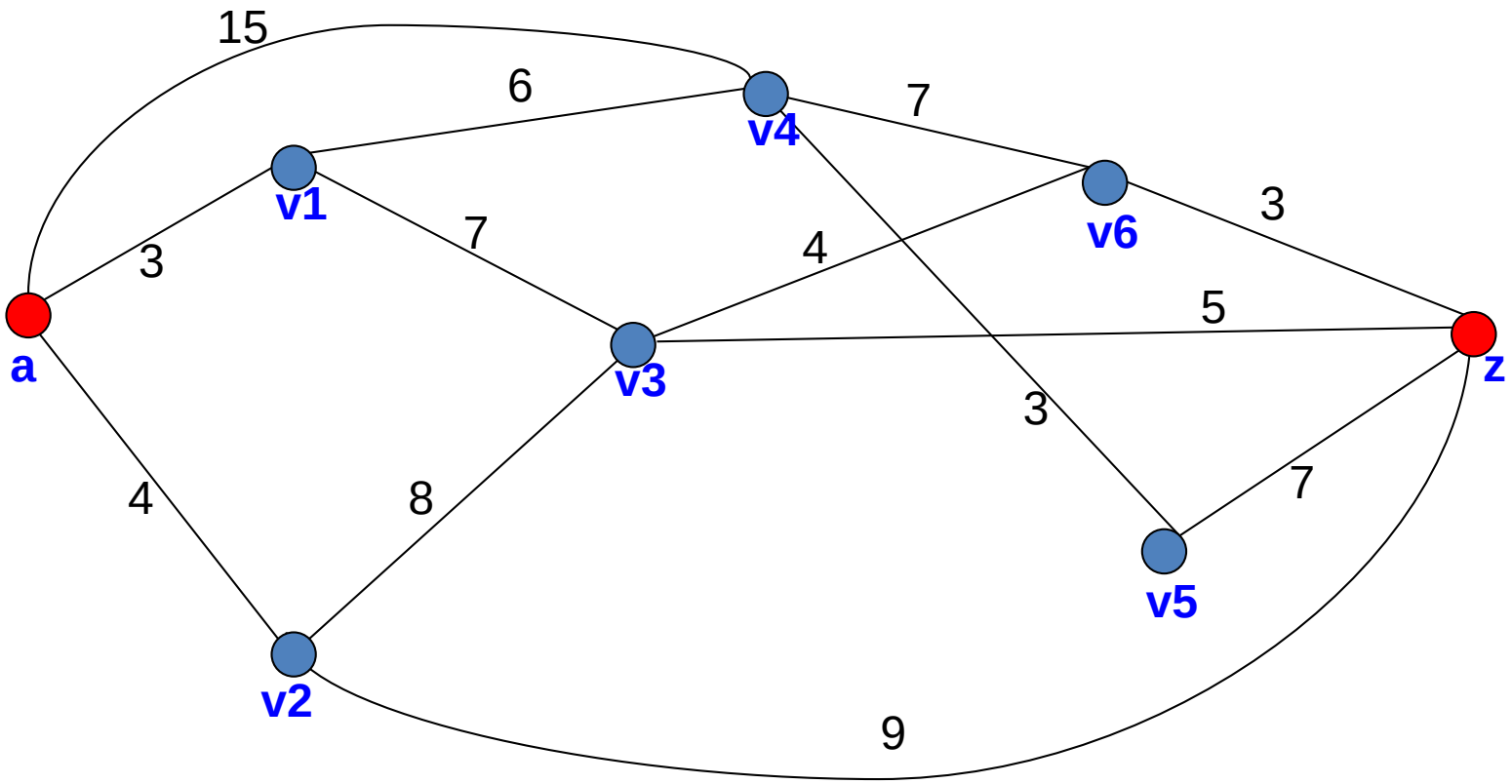
   5.d.(i): If $L(v) + W[v,w] < L(w)$ then

   $L(w) = L(v) + W[v,w]$

http://en.wikipedia.org/wiki/Dijkstra%27s_algorithm

Example

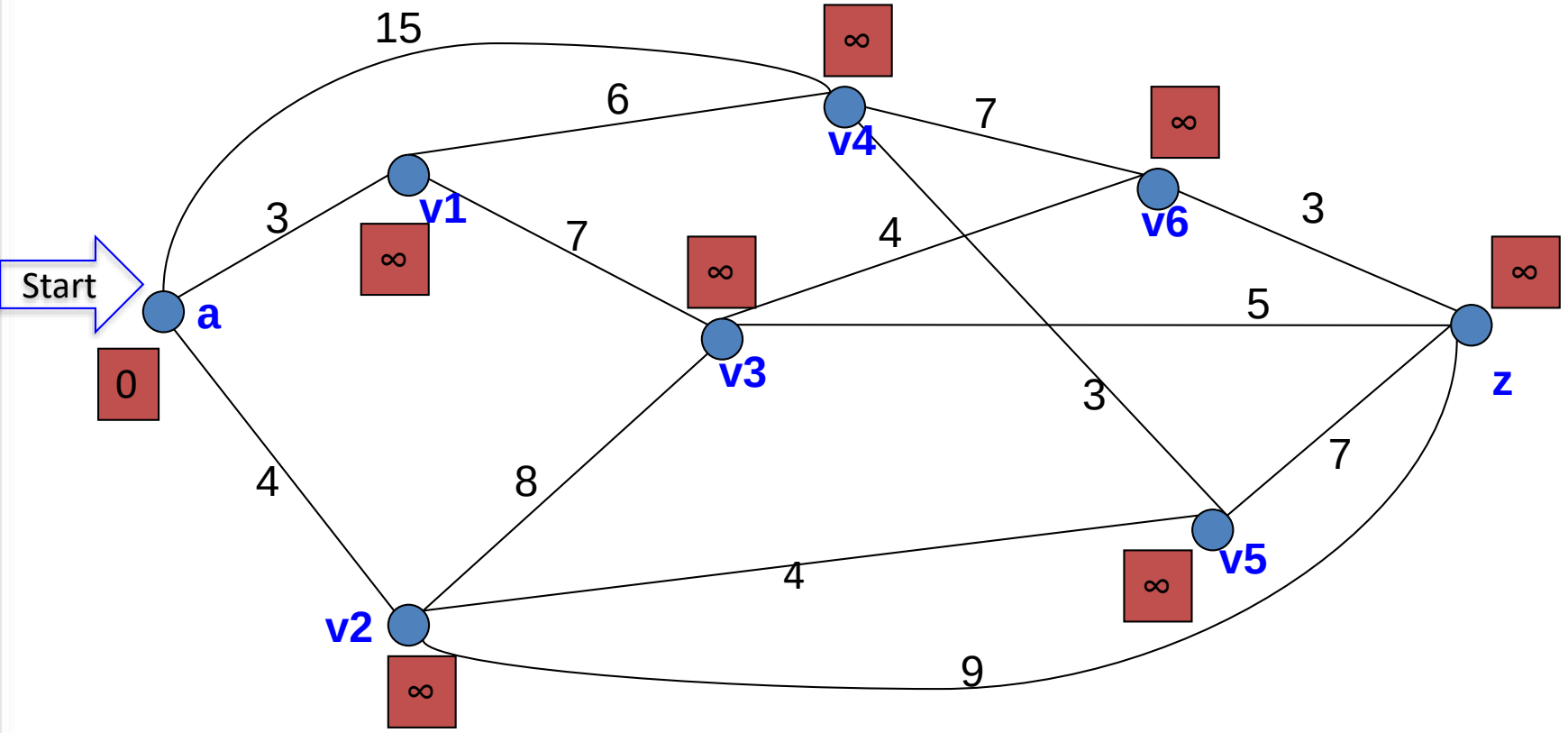
What is the shortest path from **a** to **z**?



Example -Solution

$S = \emptyset$

$N = \{a, v1, v2, v3, v4, v5, v6, z\}$



Example –Solution (cont'd)

Dijkstra's Algorithm: Iterations Table

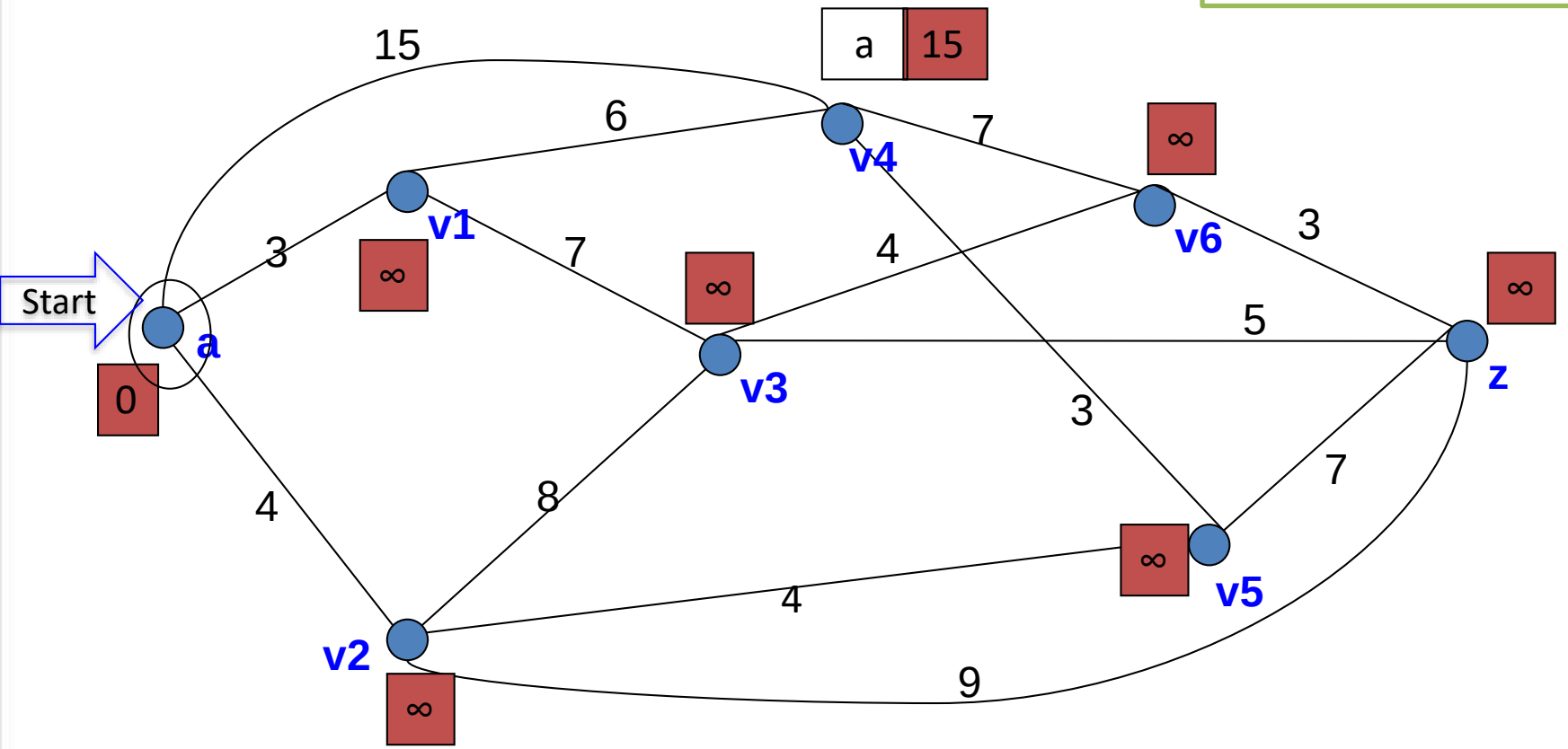
Iteration	S	N	L(a)	L(v1)	L(v2)	L(v3)	L(v4)	L(v5)	L(v6)	L(z)
0	{ }	{a, v1, v2, v3, v4, v5, v6, z}	0	∞	∞	∞	∞	∞	∞	∞

Example –Solution (cont'd)

$S = \{a\}$

$N = \{v1, v2, v3, v4, v5, v6, z\}$

$L(a)+W[a,v4] < L(v4)$
$0+15=15 < \infty$
$L(v4)=15$

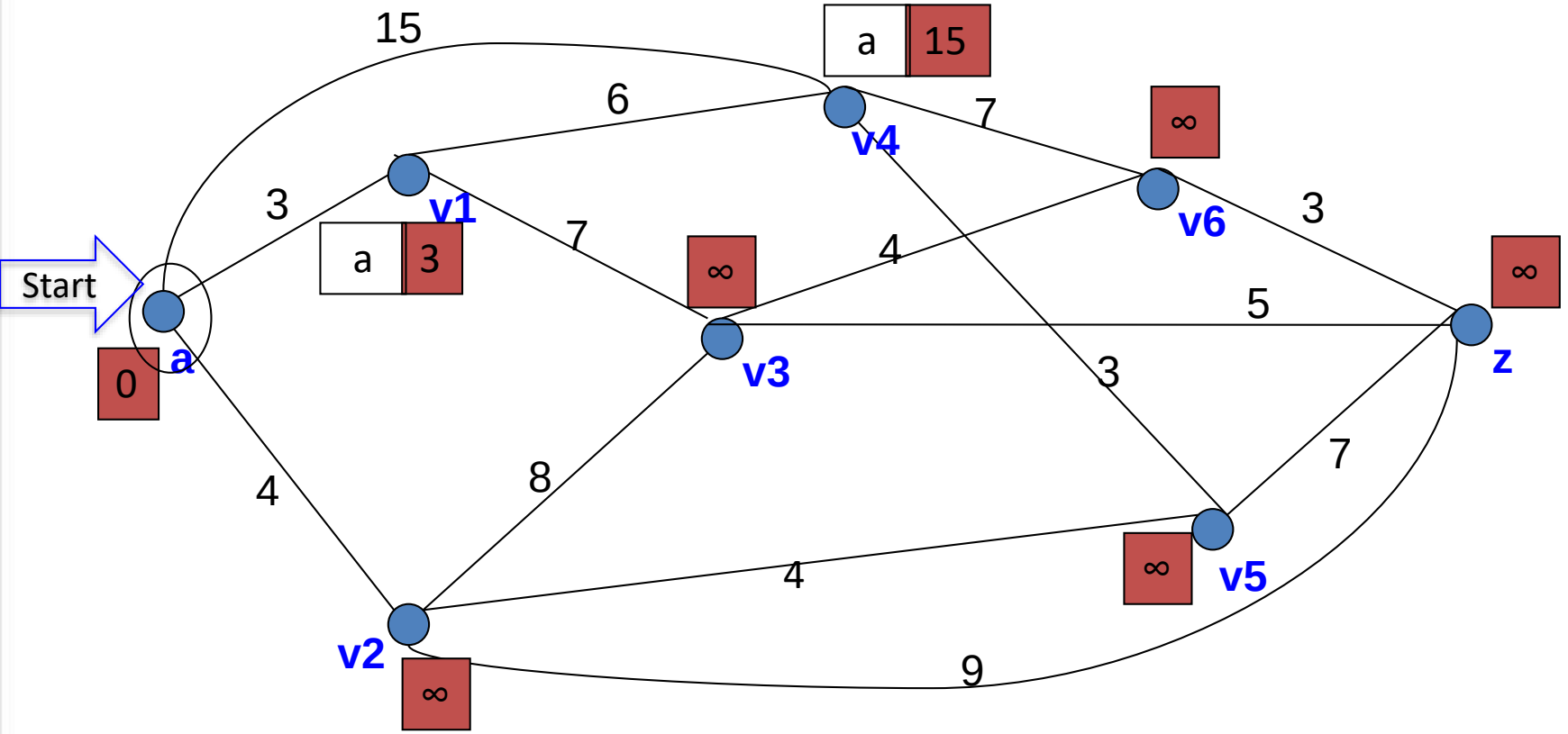


Example –Solution (cont'd)

$S = \{a\}$

$N = \{v1, v2, v3, v4, v5, v6, z\}$

$L(a)+W[a,v1] < L(v1)$
$0+3 = 3 < \infty$
$L(v1)=3$



Example –Solution (cont'd)

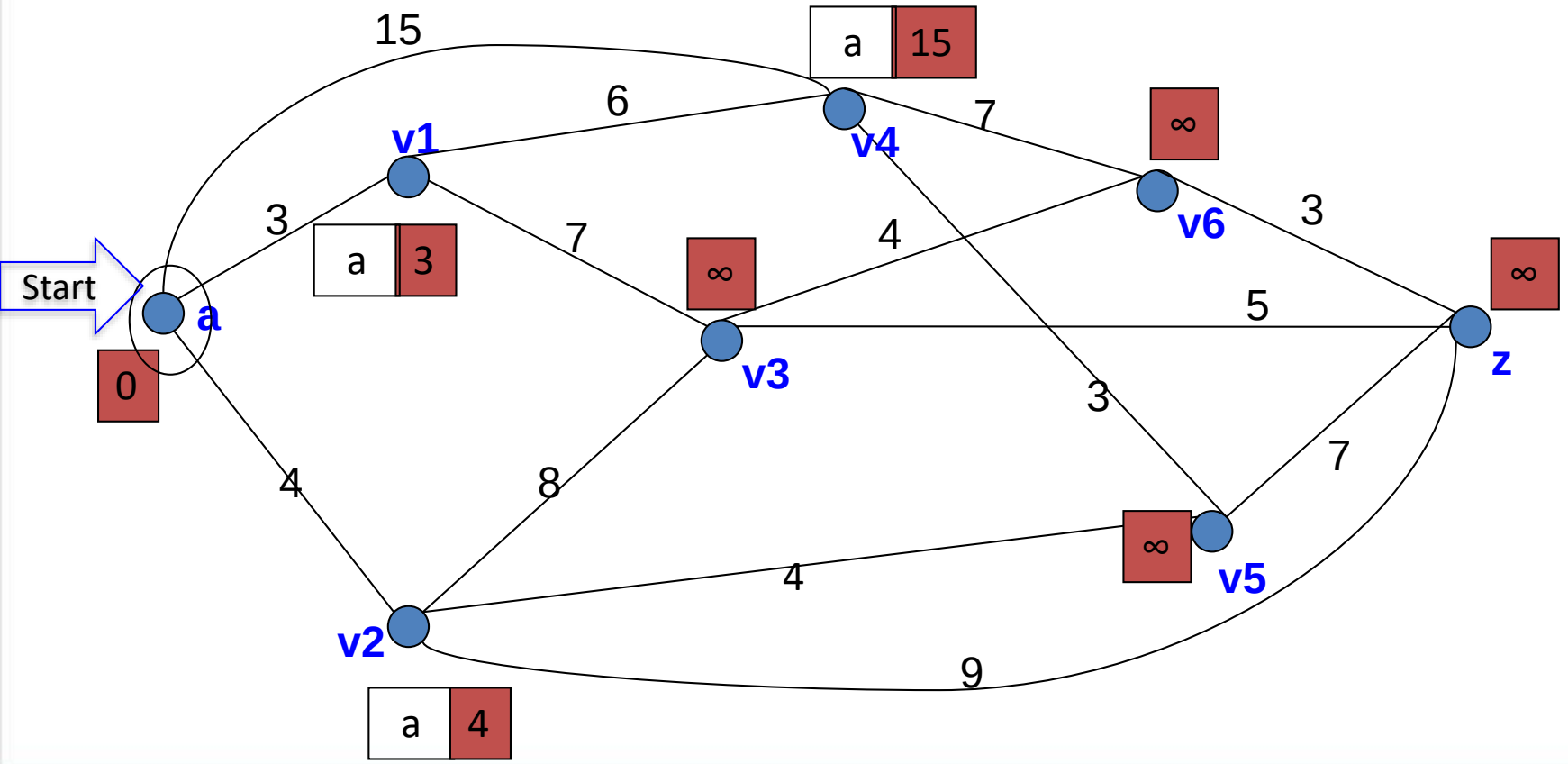
$S = \{a\}$

$N = \{v1, v2, v3, v4, v5, v6, z\}$

$L(a)+W[a,v2] < L(v2)$

$0+4 = 4 < \infty$

$L(v2)=4$



Example –Solution (cont'd)

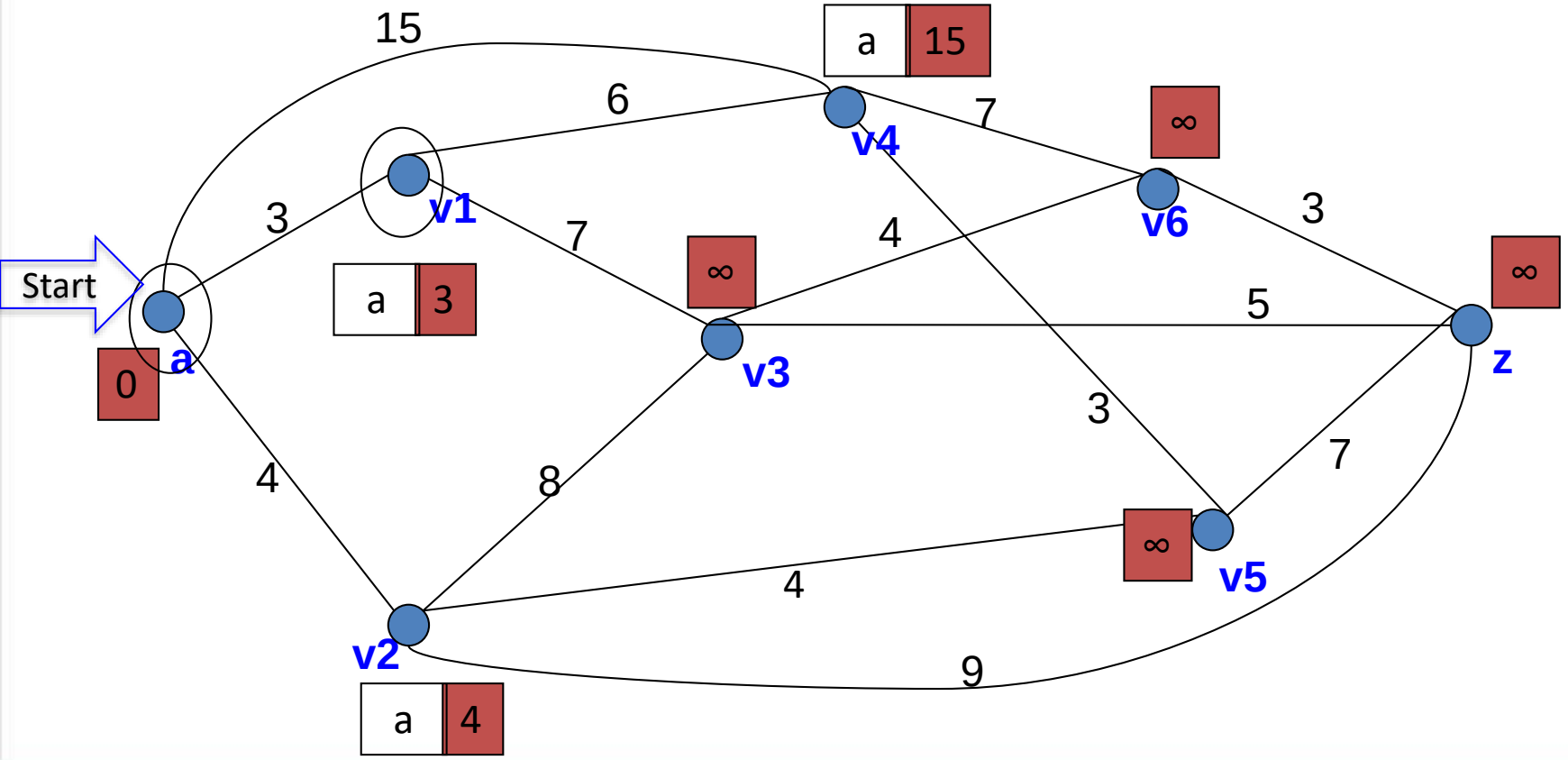
Iteration	*S*	*N*	*L*(a)	*L*(v1)	*L*(v2)	*L*(v3)	*L*(v4)	*L*(v5)	*L*(v6)	*L*(z)
0	{ }	{a, v1, v2, v3, v4, v5, v6, z}	0	∞	∞	∞	∞	∞	∞	∞
1	{a }	{v1, v2, v3, v4, v5, v6, z}	0	3	4	∞	15	∞	∞	∞

Example –Solution (cont'd)

$S = \{a\}$

$N = \{v1, v2, v3, v4, v5, v6, z\}$

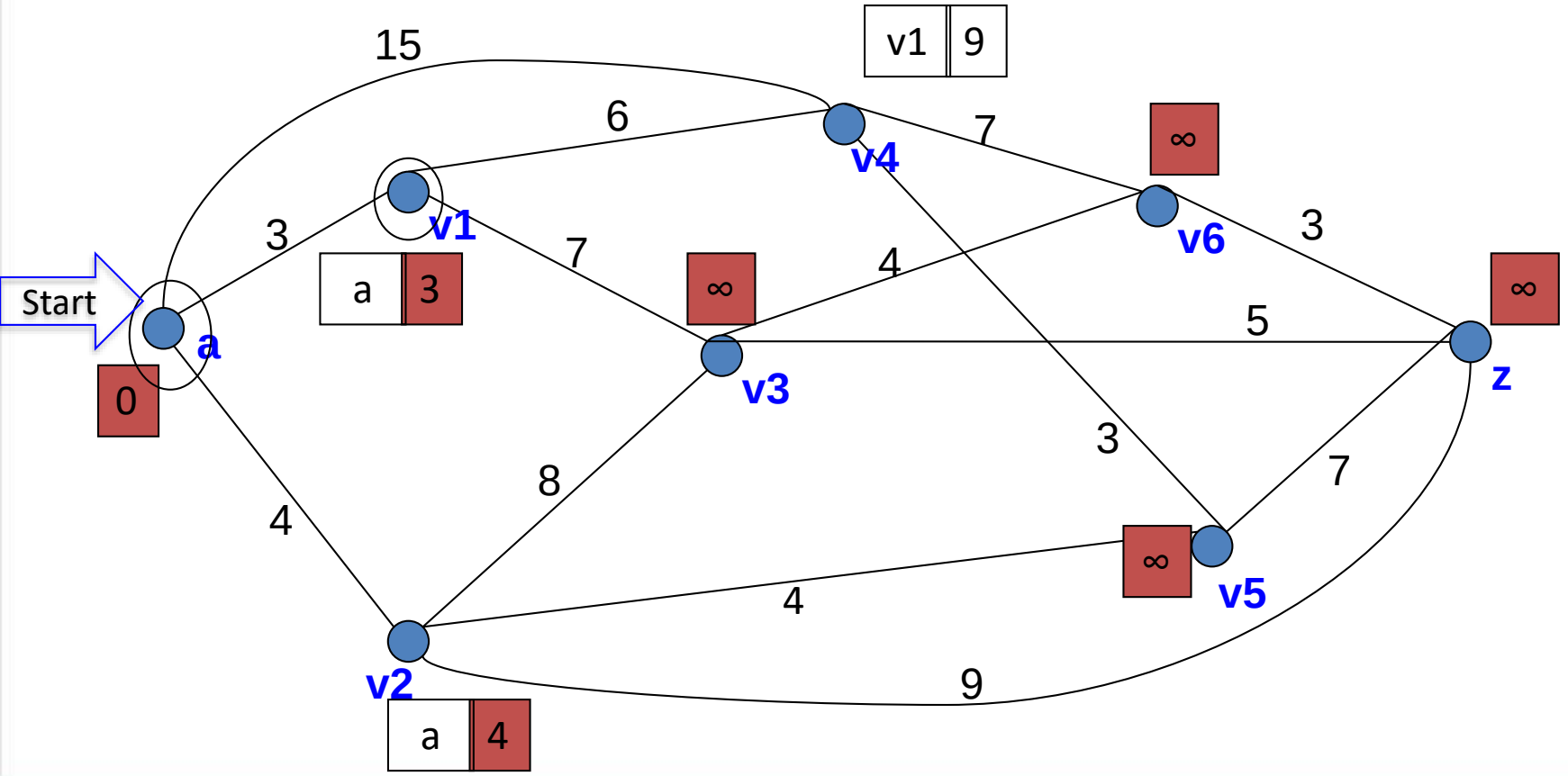
choose $v1$ because
$L(v1) = 3 = \min\{L(u) | u \in N\}$



$S = \{a, v1\}$

$N = \{v2, v3, v4, v5, v6, z\}$

$L(v1) + W[v1,v4] < L(v4)$
$3+6 = 9 < 15$
$L(v4) = 9$



Example –Solution (cont'd)

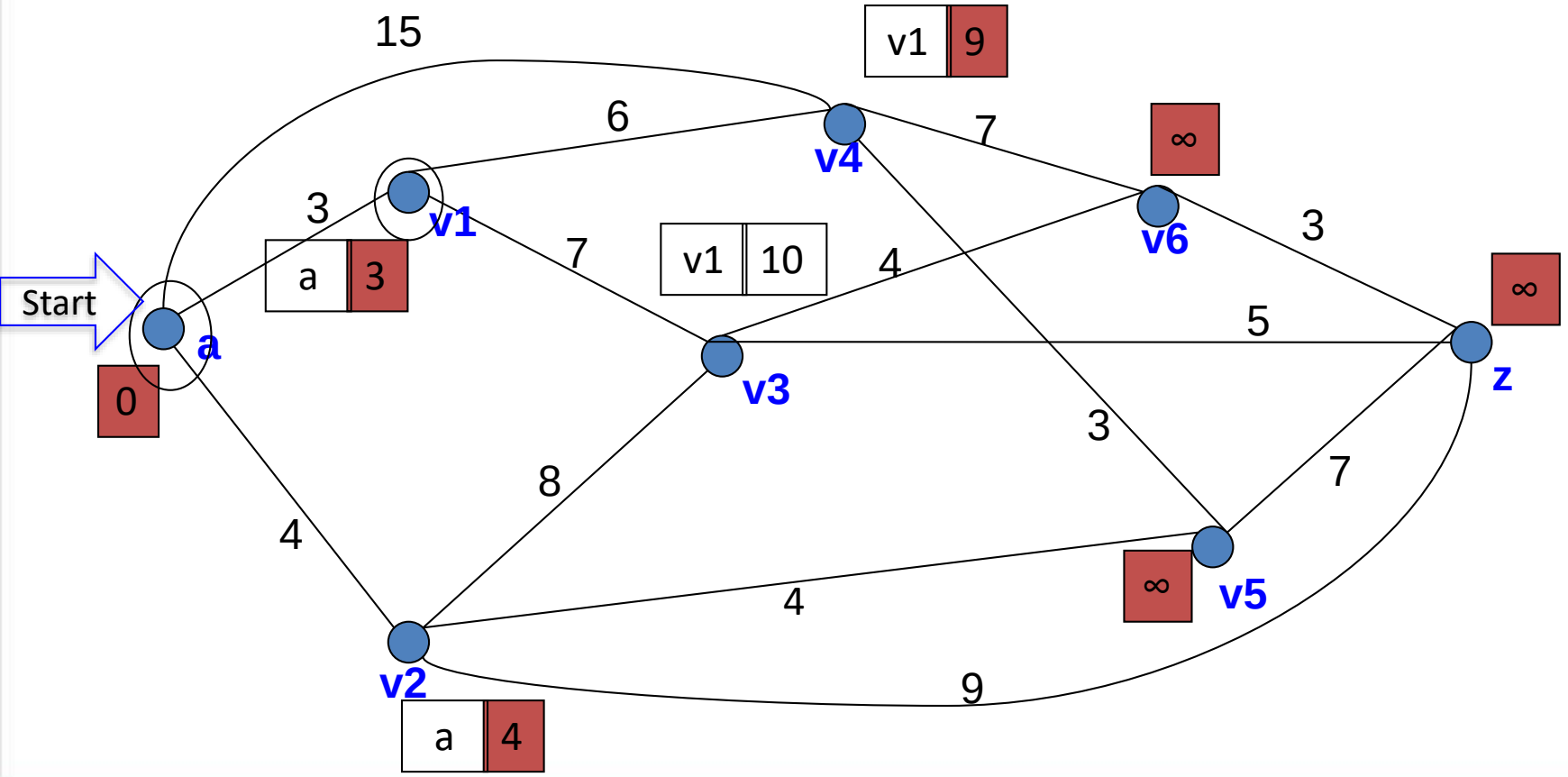
$S = \{a, v1\}$

$N = \{v2, v3, v4, v5, v6, z\}$

$L(v1) + W[v1,v3] < L(v3)$

$3+7 = 10 < \infty$

$L(v4) = 10$



Example –Solution (cont'd)

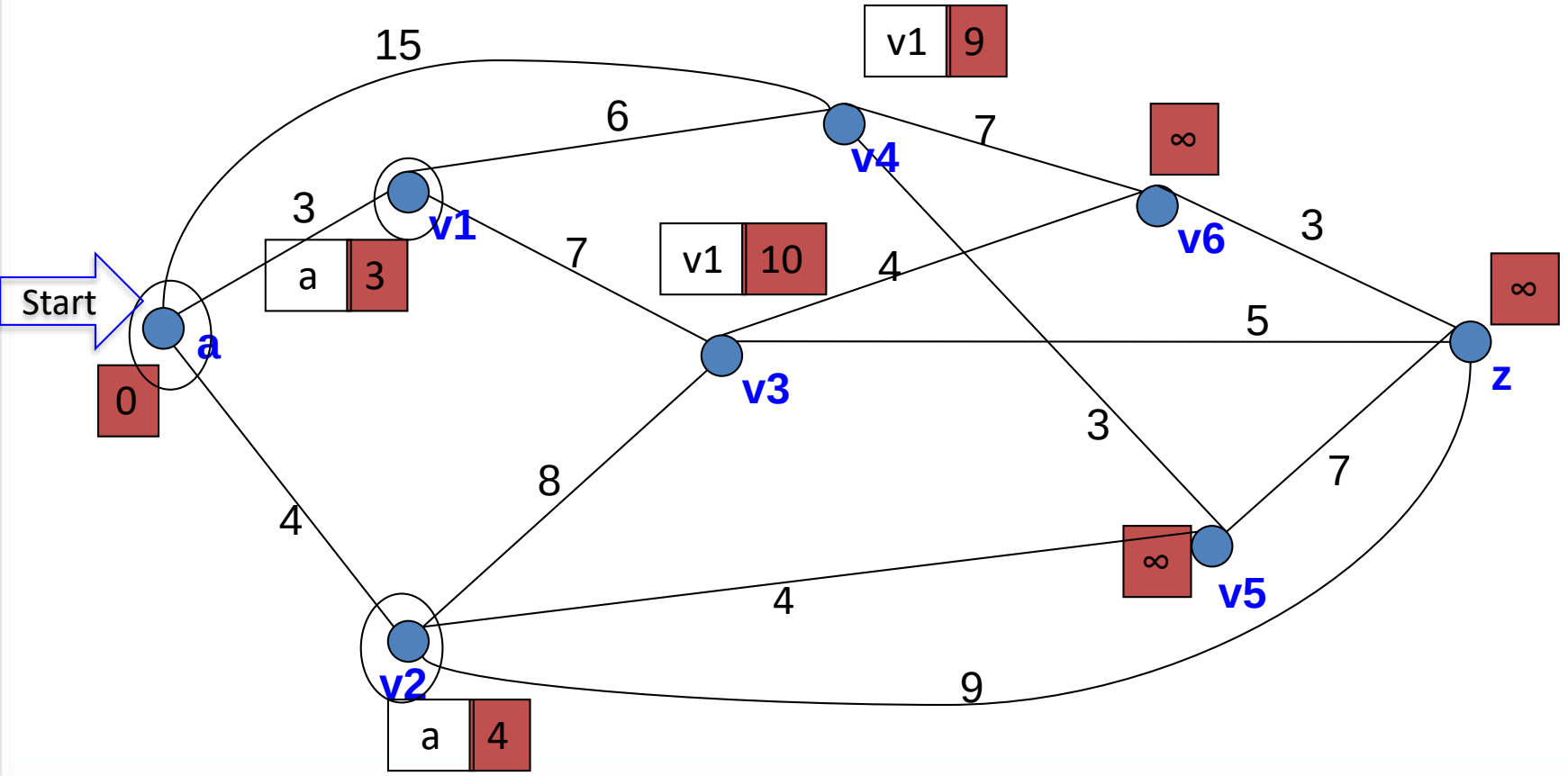
Iteration	S	N	L(a)	L(v1)	L(v2)	L(v3)	L(v4)	L(v5)	L(v6)	L(z)
0	{ }	{a, v1, v2, v3, v4, v5, v6, z}	0	∞	∞	∞	∞	∞	∞	∞
1	{a }	{v1, v2, v3, v4, v5, v6, z}	0	3	4	∞	15	∞	∞	∞
2	{a , v1}	{v2, v3, v4, v5, v6, z}	0	3	4	10	9	∞	∞	∞

Example –Solution (cont'd)

$S = \{a, v1\}$

$N = \{v2, v3, v4, v5, v6, z\}$

choose $v2$ because
$L(v2) = 4 = \min\{L(u) | u \in N\}$

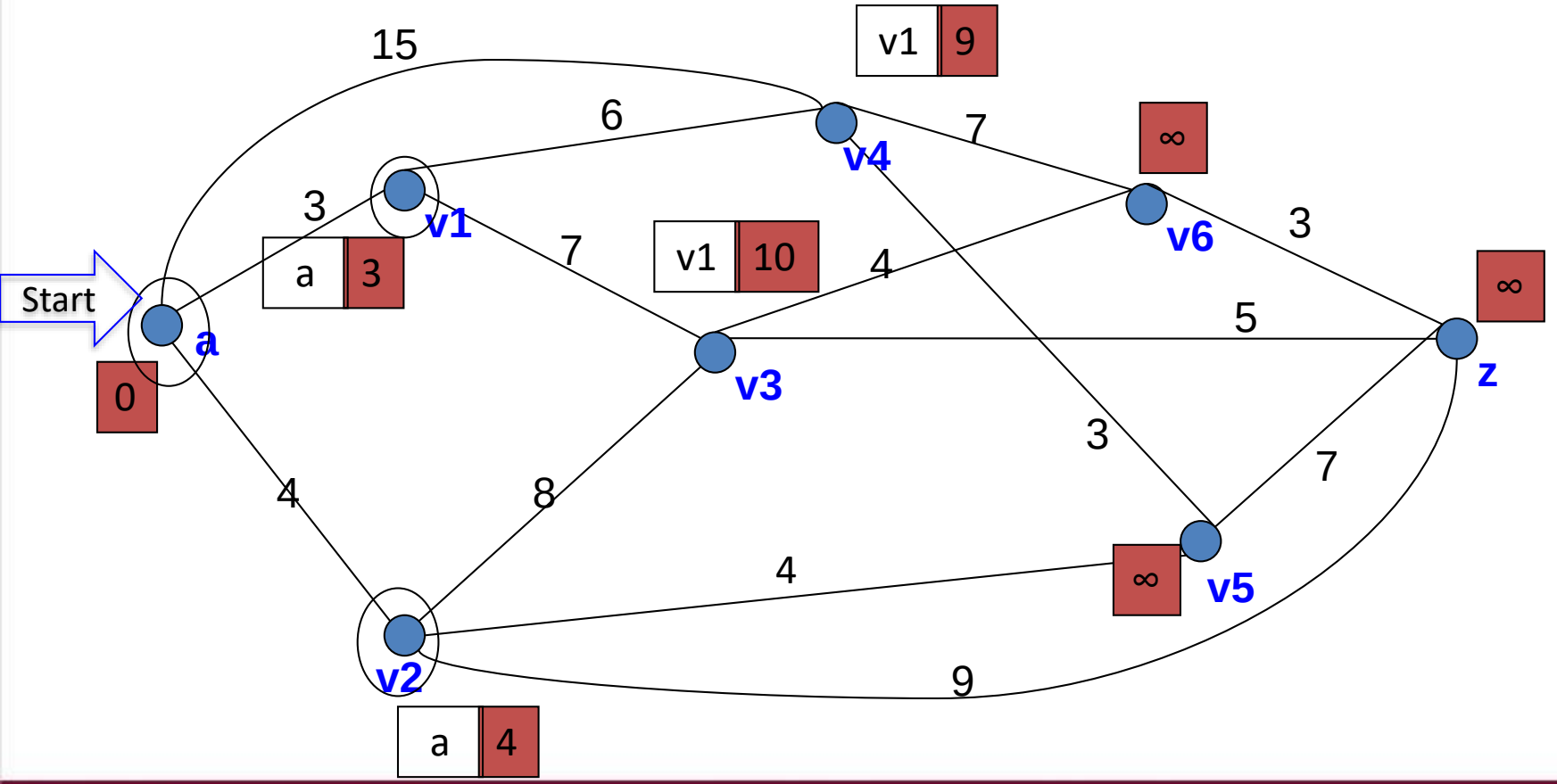


Example –Solution (cont'd)

$S=\{a, v1, v2\}$

$N=\{v3, v4, v5, v6, z\}$

$L(v2) + W[v2,v3] < L(v3)$
$4+8 = 12 > 10$
$L(v3)$ remains the same.



Example –Solution (cont'd)

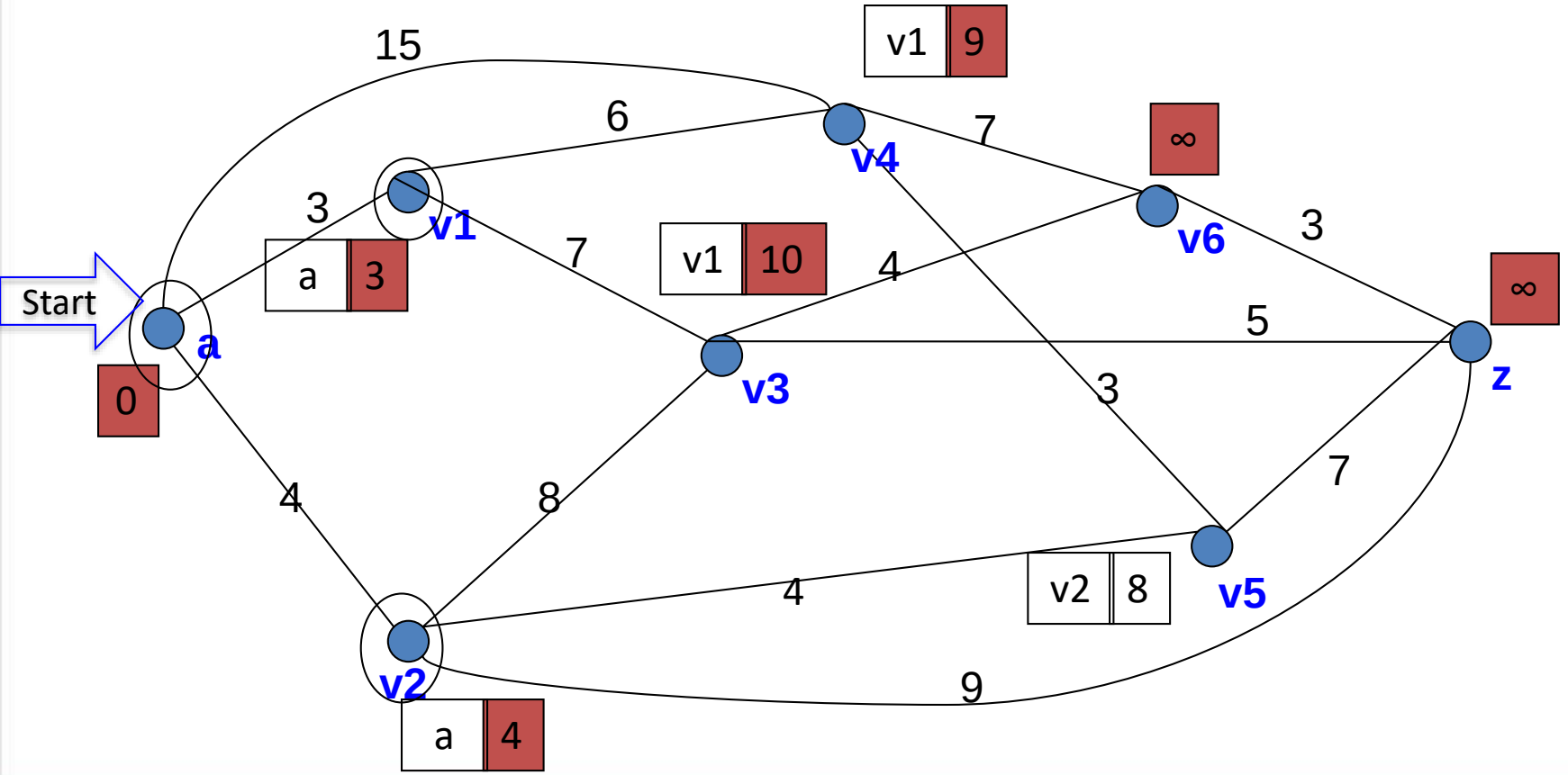
$S = \{a, v1, v2\}$

$N = \{v3, v4, v5, v6, z\}$

$L(v2) + W[v2,v5] < L(v5)$

$4+4 = 8 < \infty$

$L(v5) = 8$



Example –Solution (cont'd)

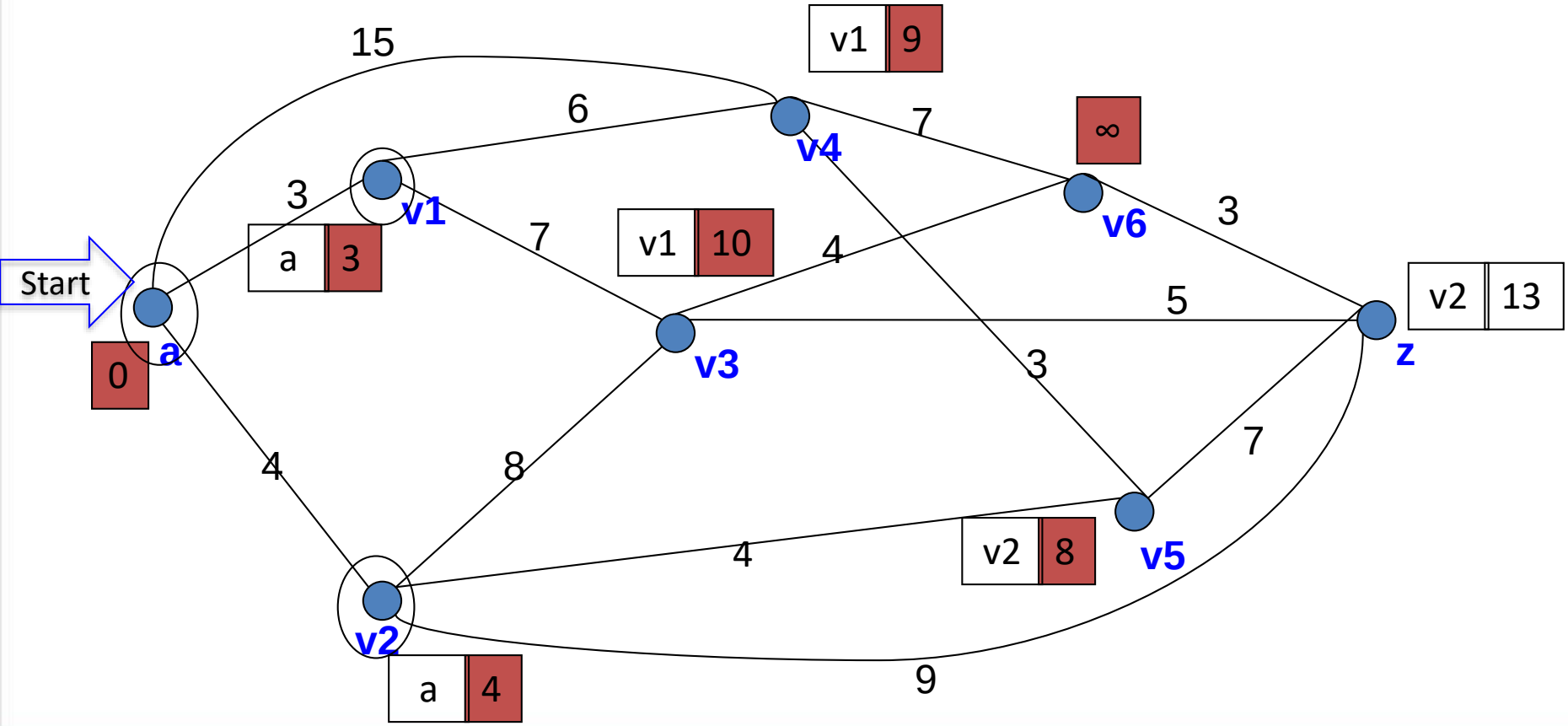
$S = \{a, v1, v2\}$

$N = \{v3, v4, v5, v6, z\}$

$L(v2) + W[v2, z] < L(z)$

$4+9 = 13 < \infty$

$L(z) = 13$



Example –Solution (cont'd)

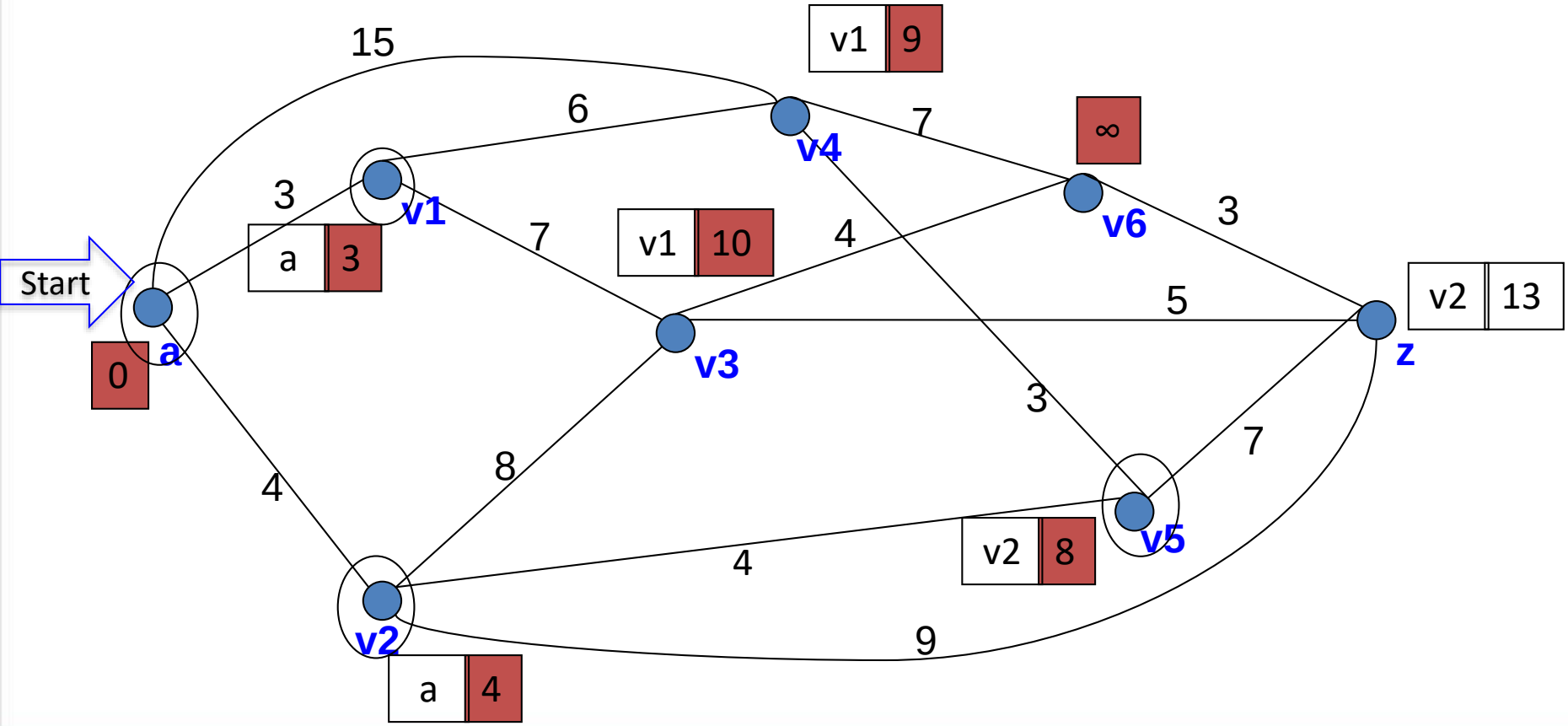
Iteration	S	N	L(a)	L(v1)	L(v2)	L(v3)	L(v4)	L(v5)	L(v6)	L(z)
0	{ }	{a, v1, v2, v3, v4, v5, v6, z}	0	∞	∞	∞	∞	∞	∞	∞
1	{a }	{v1, v2, v3, v4, v5, v6, z}	0	3	4	∞	15	∞	∞	∞
2	{a , v1}	{v2, v3, v4, v5, v6, z}	0	**3**	4	10	9	∞	∞	∞
3	{a , v1,v2}	{v3, v4, v5, v6, z}	0	**3**	**4**	10	9	8	∞	13

Example –Solution (cont'd)

$S = \{a, v1, v2\}$

$N = \{v3, v4, v5, v6, z\}$

choose $v5$ because
$L(v5) = 8 = \min\{L(u) | u \in N\}$

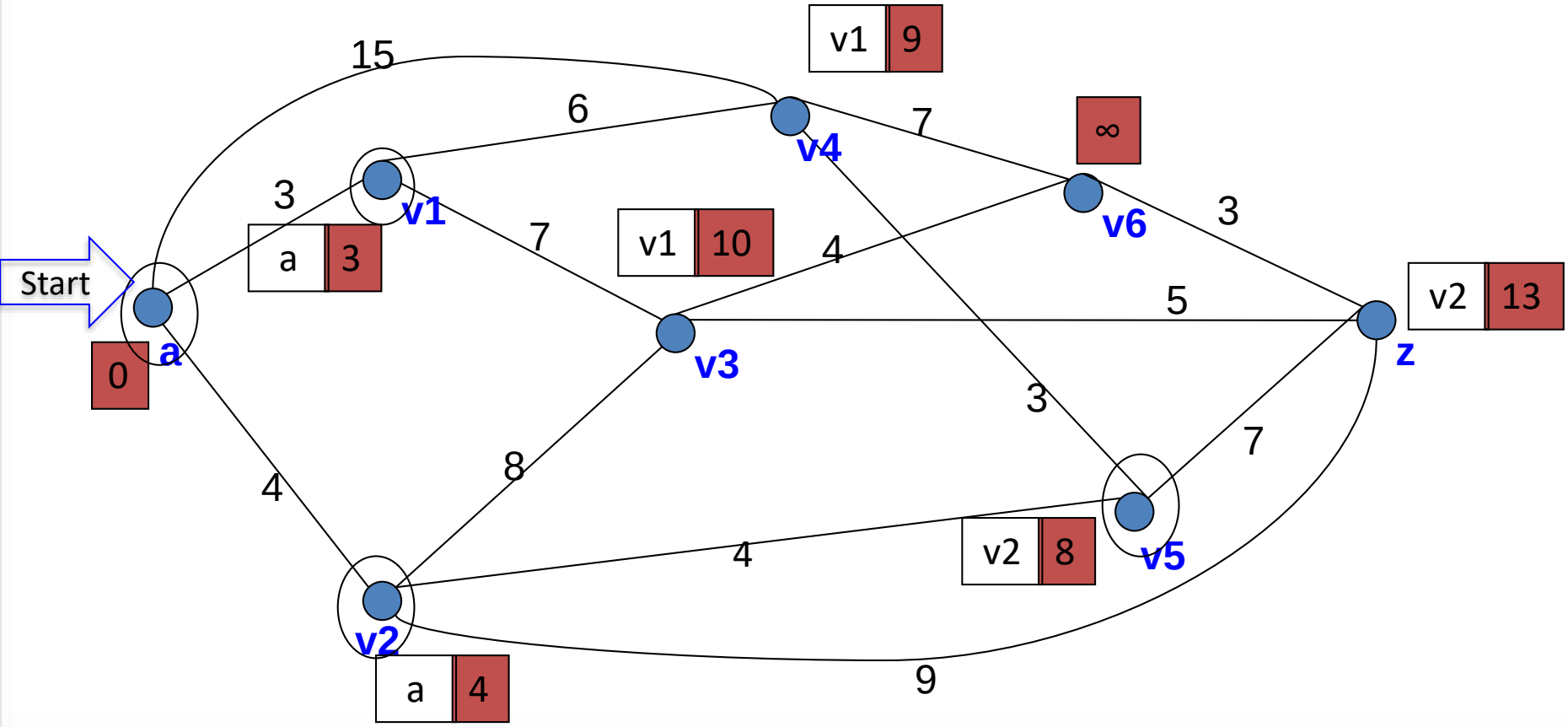


Example –Solution (cont'd)

$S = \{a, v1, v2, v5\}$

$N = \{v3, v4, v6, z\}$

$L(v5) + W[v5, v4] < L(v4)$
$8+3 = 11 > 9$
$L(v4)$ remains the same

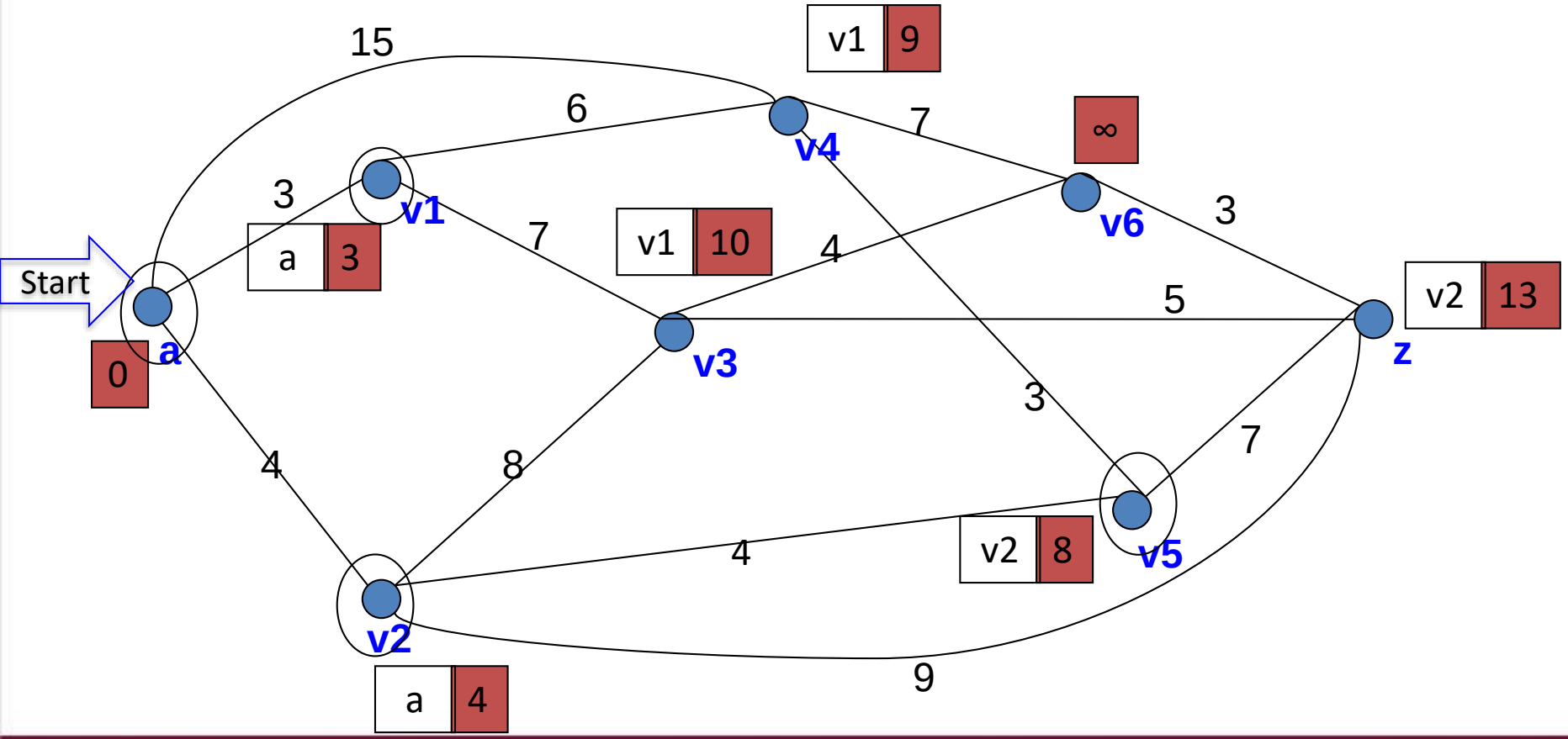


Example –Solution (cont'd)

$S = \{a, v1, v2, v5\}$

$N = \{v3, v4, v6, z\}$

$L(v5) + W[v5, z] < L(z)$
$8+7 = 15 > 13$
$L(z)$ remains the same



Example –Solution (cont'd)

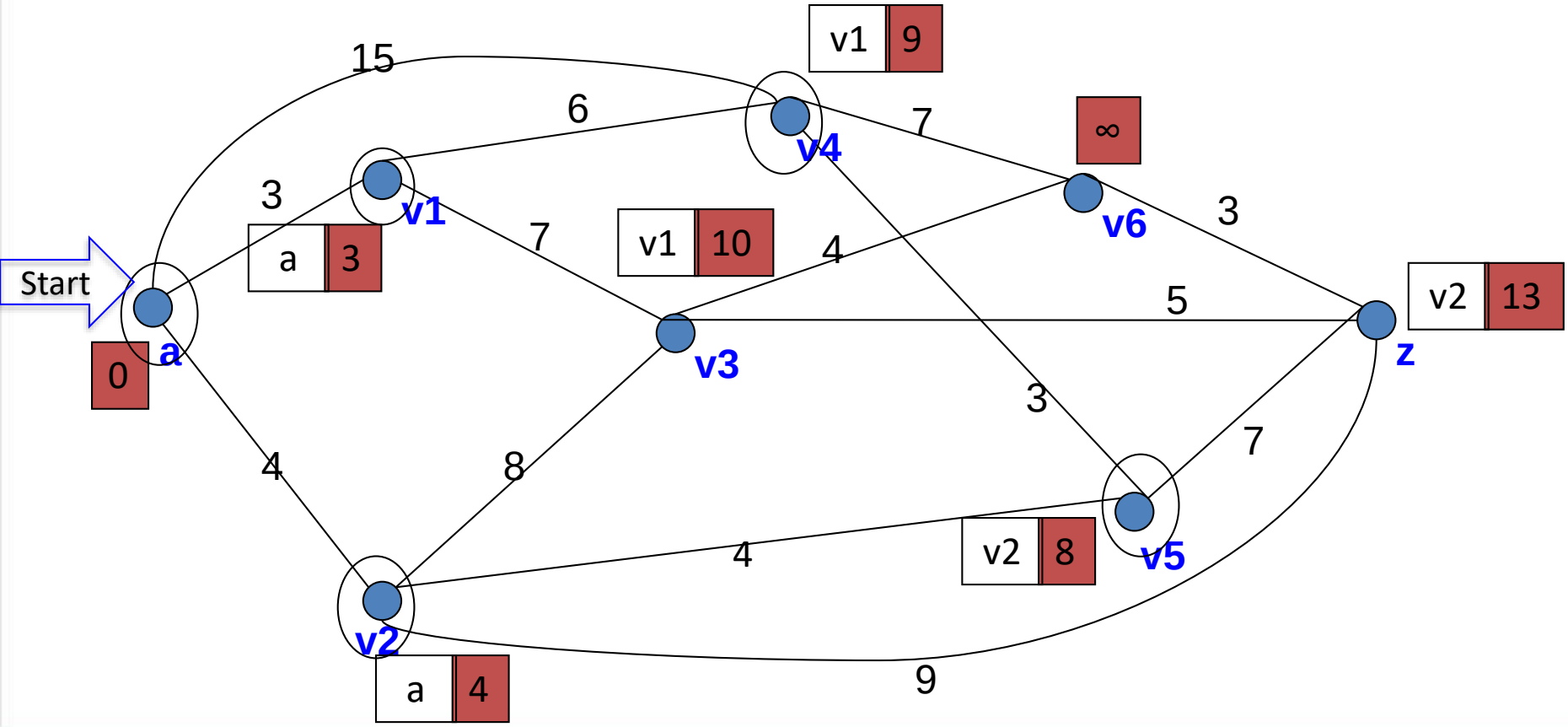
Iteration	S	N	L(a)	L(v1)	L(v2)	L(v3)	L(v4)	L(v5)	L(v6)	L(z)
0	{ }	{a, v1, v2, v3, v4, v5, v6, z}	0	∞	∞	∞	∞	∞	∞	∞
1	{a }	{v1, v2, v3, v4, v5, v6, z}	0	3	4	∞	15	∞	∞	∞
2	{a , v1}	{v2, v3, v4, v5, v6, z}	0	3	4	10	9	∞	∞	∞
3	{a , v1,v2}	{v3, v4, v5, v6, z}	0	3	4	10	9	8	∞	13
4	{a , v1,v2,v5}	{v3, v4, v6, z}	0	3	4	10	9	8	∞	13

Example –Solution (cont'd)

$S = \{a, v1, v2, v5\}$

$N = \{v3, v4, v6, z\}$

choose $v4$ because
$L(v4) = 9 = \min\{L(u) | u \in N\}$



Example –Solution (cont'd)

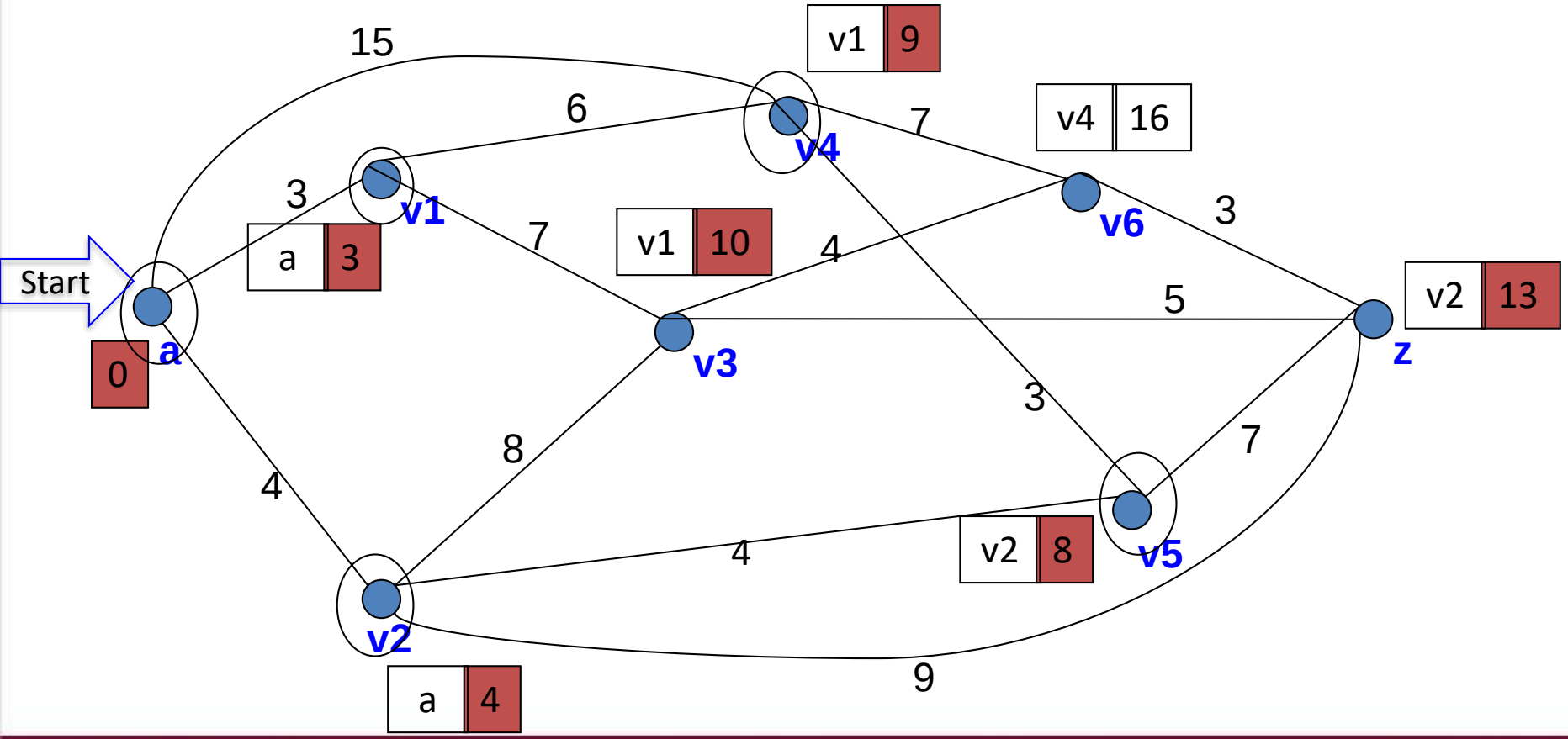
$S = \{a, v1, v2, v5, v4\}$

$N = \{v3, v6, z\}$

$L(v4) + W[v4, v6] < L(v6)$

$9+7 = 16 < \infty$

$L(v6) = 16$

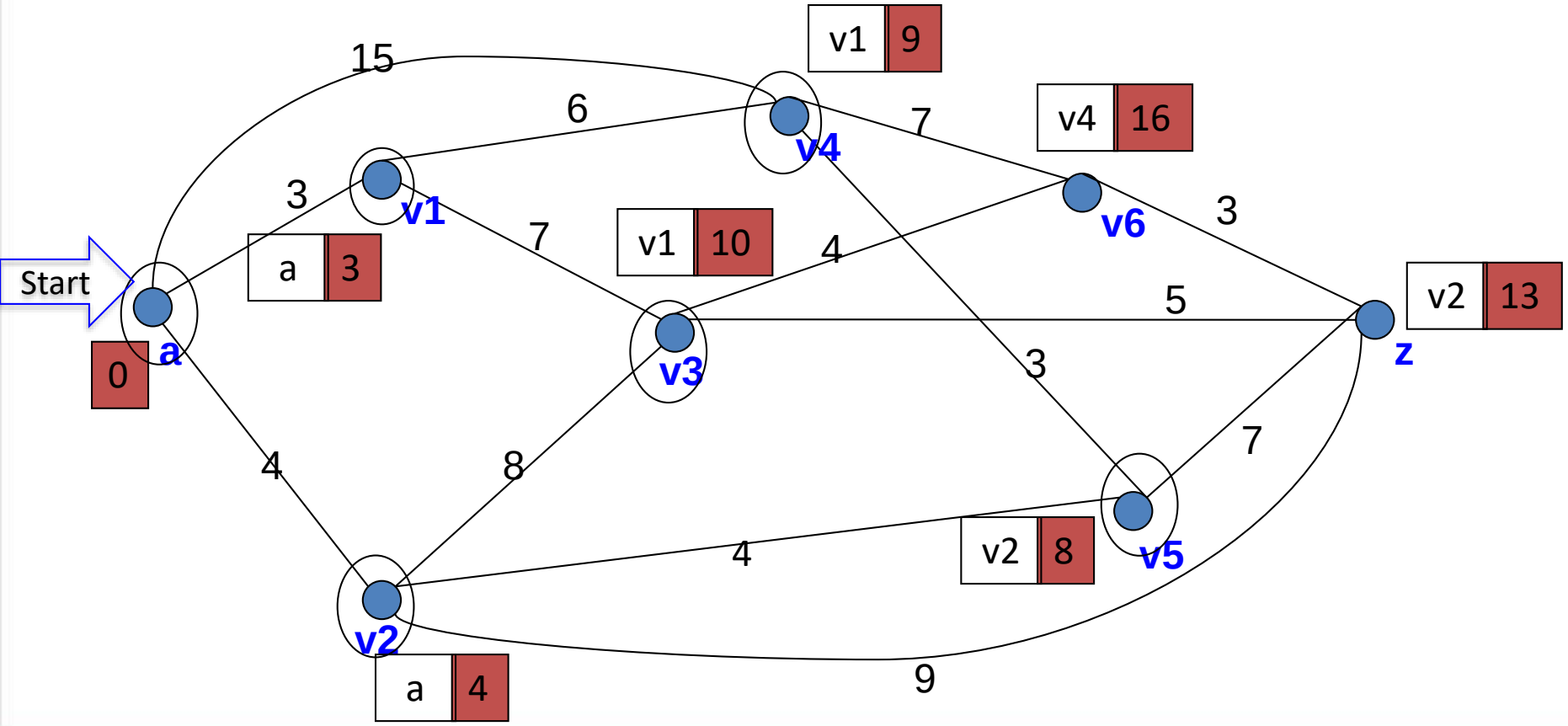


Example –Solution (cont'd)

$S = \{a, v1, v2, v5, v4\}$

$N = \{v3, v6, z\}$

choose $v3$ because
$L(v3) = 10 = \min\{L(u) | u \in N\}$



Example –Solution (cont'd)

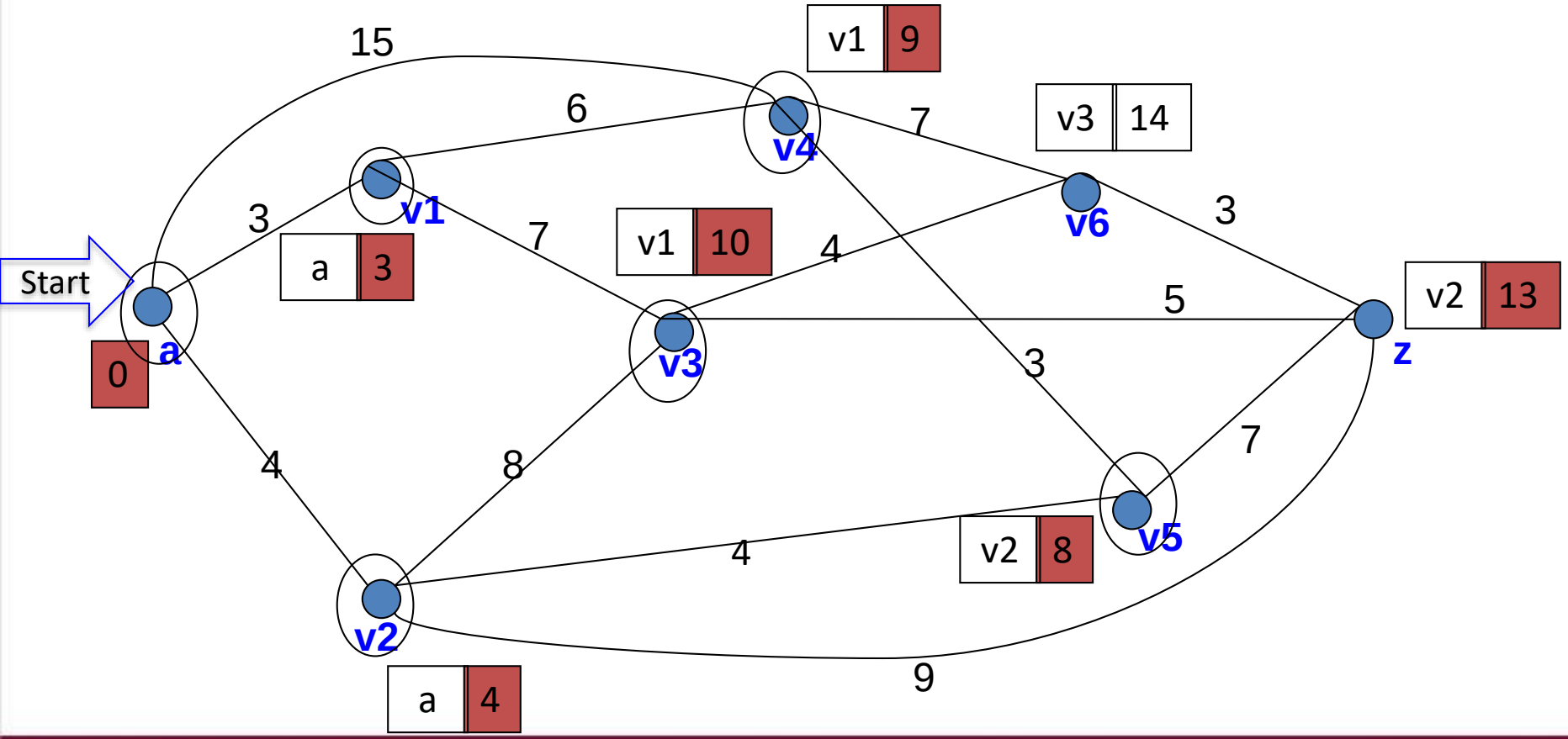
Iteration	S	N	L(a)	L(v1)	L(v2)	L(v3)	L(v4)	L(v5)	L(v6)	L(z)
0	{ }	{a, v1, v2, v3, v4, v5, v6, z}	0	∞	∞	∞	∞	∞	∞	∞
1	{a }	{v1, v2, v3, v4, v5, v6, z}	0	3	4	∞	15	∞	∞	∞
2	{a , v1}	{v2, v3, v4, v5, v6, z}	0	3	4	10	9	∞	∞	∞
3	{a , v1,v2}	{v3, v4, v5, v6, z}	0	3	4	10	9	8	∞	13
4	{a , v1,v2,v5}	{v3, v4, v6, z}	0	3	4	10	9	8	∞	13
5	{a , v1,v2,v5, v4}	{v3, v6, z}	0	3	4	10	9	8	16	13

Example –Solution (cont'd)

$S = \{a, v1, v2, v5, v4, v3\}$

$N = \{v6, z\}$

$L(v3) + W[v3, v6] < L(v6)$
$10+4 = 14 < 16$
$L(v6) = 14$

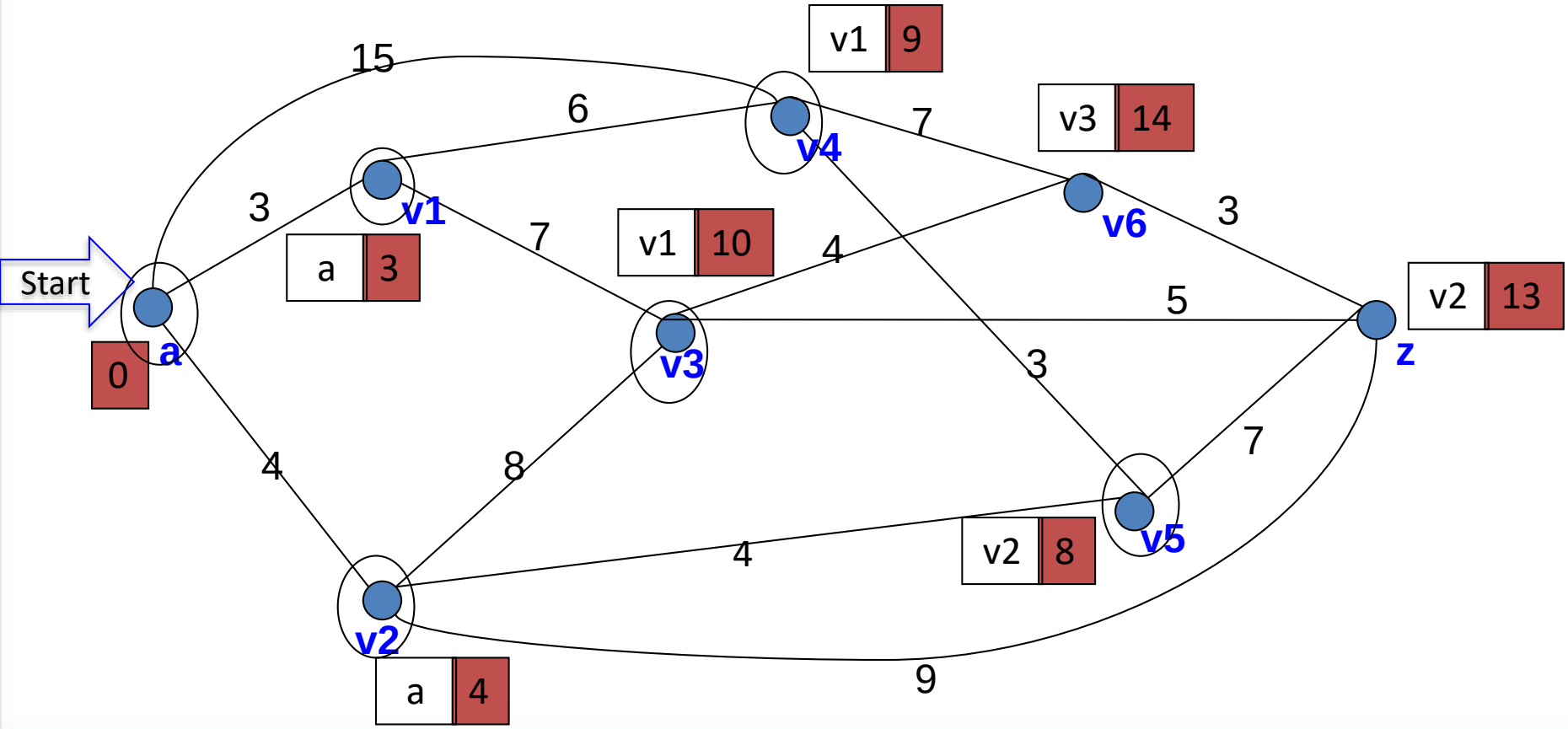


Example –Solution (cont'd)

$S = \{a, v1, v2, v5, v4, v3\}$

$N = \{v6, z\}$

$L(v3) + W[v3, z] < L(z)$
$10+5 = 15 > 13$
$L(z)$ remains the same



Example –Solution (cont'd)

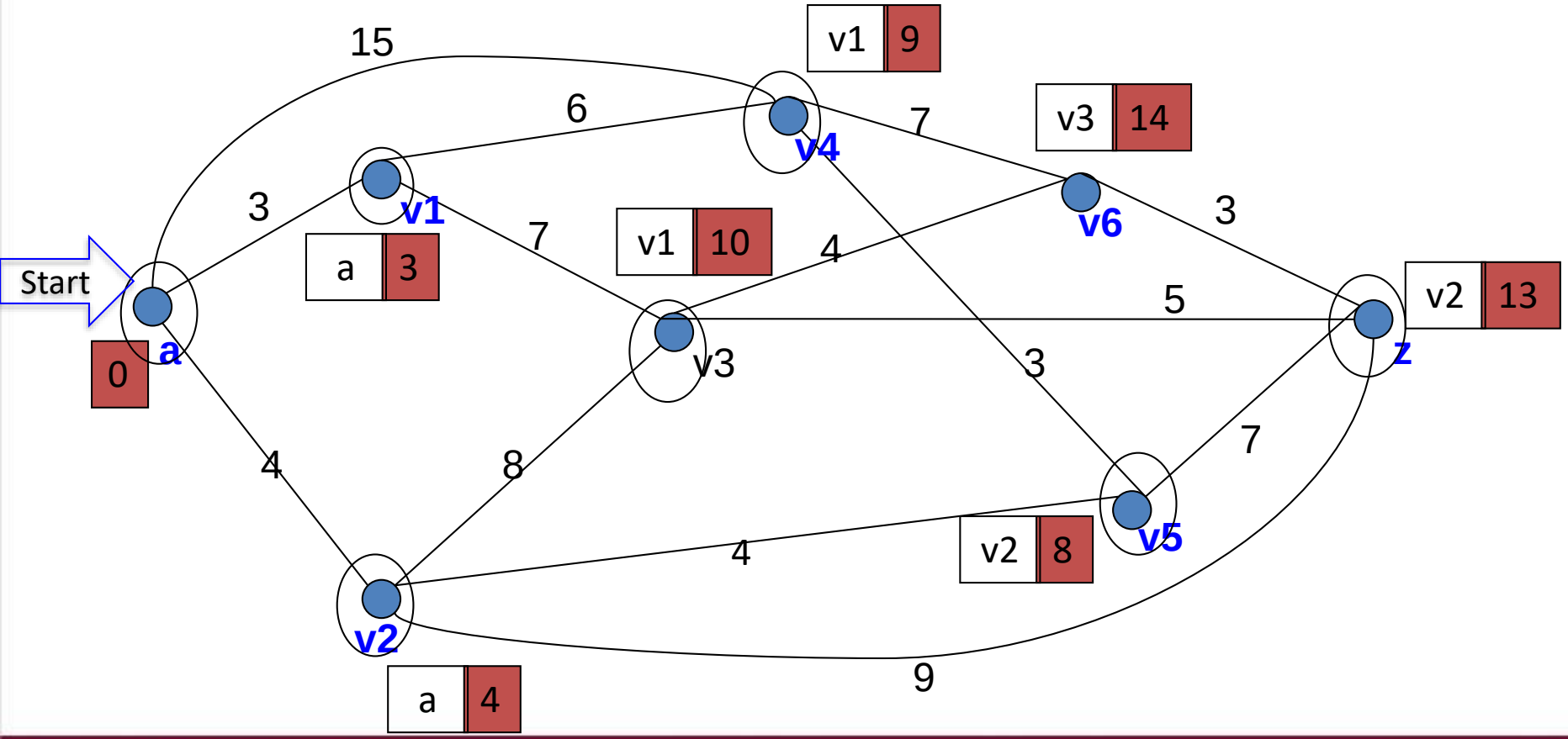
Iteration	S	N	L(a)	L(v1)	L(v2)	L(v3)	L(v4)	L(v5)	L(v6)	L(z)
0	{ }	{a, v1, v2, v3, v4, v5, v6, z}	0	∞	∞	∞	∞	∞	∞	∞
1	{a }	{v1, v2, v3, v4, v5, v6, z}	0	3	4	∞	15	∞	∞	∞
2	{a , v1}	{v2, v3, v4, v5, v6, z}	0	3	4	10	9	∞	∞	∞
3	{a , v1,v2}	{v3, v4, v5, v6, z}	0	3	4	10	9	8	∞	13
4	{a , v1,v2,v5}	{v3, v4, v6, z}	0	3	4	10	9	8	∞	13
5	{a , v1,v2,v5, v4}	{v3, v6, z}	0	3	4	10	9	8	16	13
6	{a , v1,v2,v5, v4, v3}	{v6, z}	0	3	4	10	9	8	14	13

Example –Solution (cont'd)

$S = \{a, v1, v2, v5, v4, v3\}$

$N = \{v6, z\}$

choose z because
$L(z) = 13 = \min\{L(u) | u \in N\}$

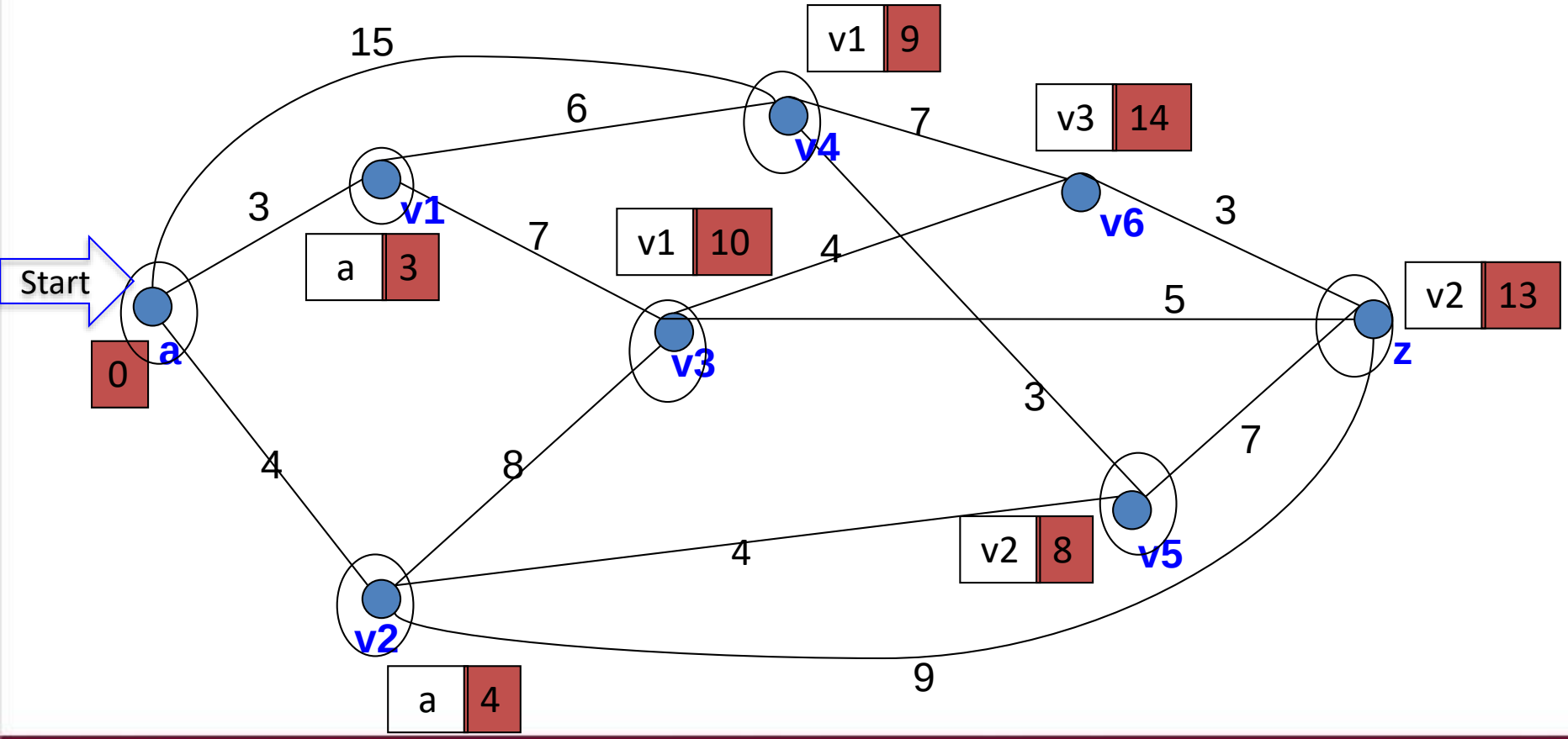


Example –Solution (cont'd)

$S = \{a, v1, v2, v5, v4, v3, z\}$

$N = \{v6\}$

The loop terminates because $z \in S$

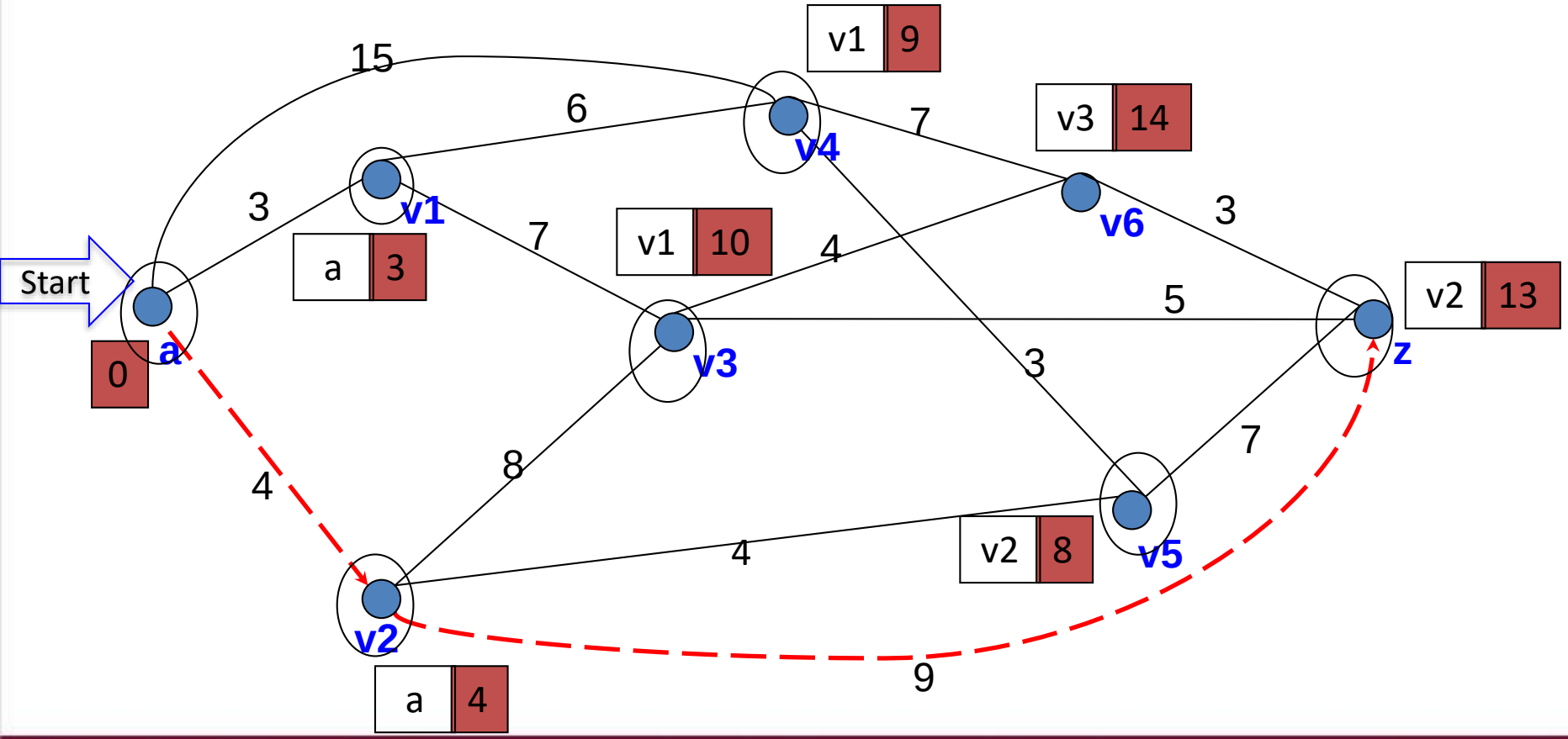


Example –Solution (cont'd)

Iteration	*S*	*N*	*L*(a)	*L*(v1)	*L*(v2)	*L*(v3)	*L*(v4)	*L*(v5)	*L*(v6)	*L*(z)
0	{ }	{a, v1, v2, v3, v4, v5, v6, z}	0	∞	∞	∞	∞	∞	∞	∞
1	{a }	{v1, v2, v3, v4, v5, v6, z}	0	3	4	∞	15	∞	∞	∞
2	{a , v1}	{v2, v3, v4, v5, v6, z}	0	**3**	4	10	9	∞	∞	∞
3	{a , v1,v2}	{v3, v4, v5, v6, z}	0	**3**	**4**	10	9	8	∞	13
4	{a , v1,v2,v5}	{v3, v4, v6, z}	0	**3**	**4**	10	9	**8**	∞	13
5	{a , v1,v2,v5, v4}	{v3, v6, z}	0	**3**	**4**	10	**9**	**8**	16	13
6	{a , v1,v2,v5, v4, v3}	{v6, z}	0	**3**	**4**	**10**	**9**	**8**	14	13
7	{a , v1,v2,v5, v4, v3, z}	{v6}	0	**3**	**4**	**10**	**9**	**8**	14	**13**

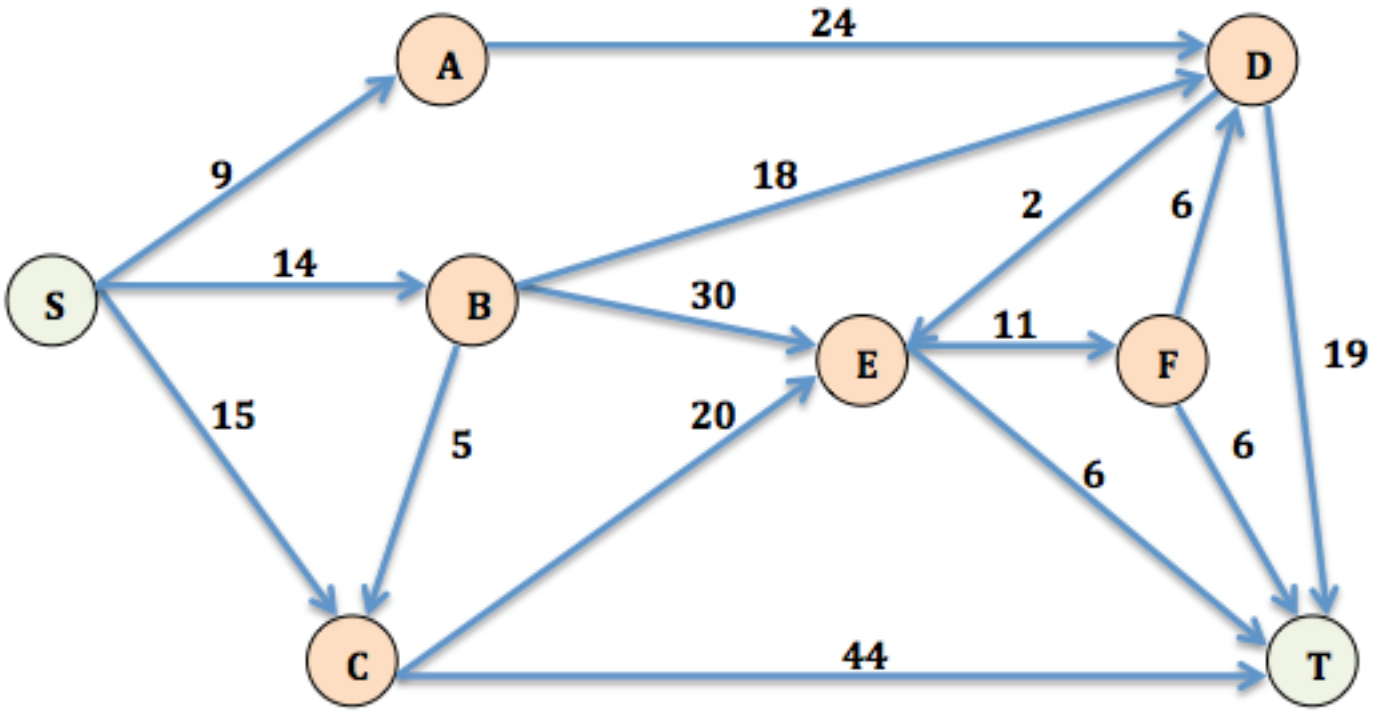
Example –Solution (cont'd)

Shortest path from ***a*** to **z** is **a** --> **v2** --> **z**, with the shortest length is 13.



Exercise #1

Given a weighted digraph, find the shortest path from **S** to **T**, using Djikstra's Algorithm.



Note: Weights are arbitrary numbers (i.e., not necessarily distances).

Exercise #2

Ahmad cycles to lecture every day. He wakes up late today and have only 15 minutes left before the lecture start. Help him to find the shortest route to the lecture hall if Ahmad college is located at **A** and the lecture hall is located at **F.** All possible routes to reach the lecture hall from the college and the time in minute for each path are illustrated in Figure 1.

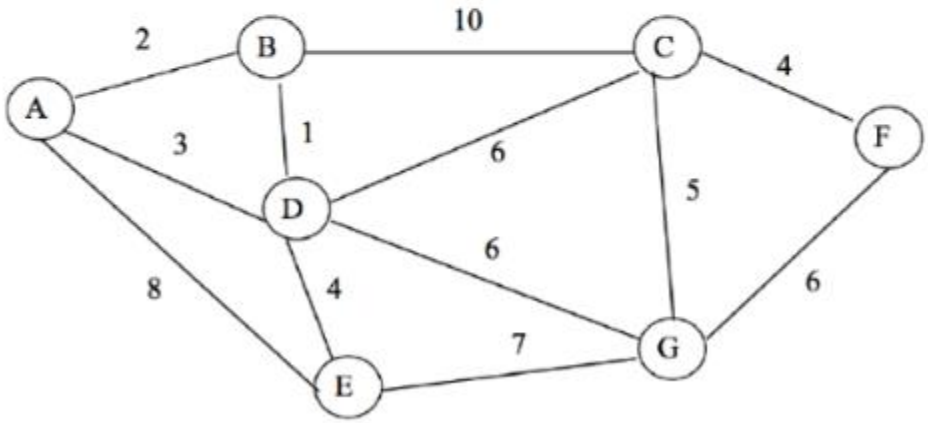


Figure 1