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CHAPTER 1

QUANTIFIERS & PROOF TECHNIQUE
[Part 4]

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QUANTIFIERS

- Most of the statements in mathematics and computer science are not described properly by the propositions.

- Since most of the statements in mathematics and computer science use **variables**, the system of logic must be extended to include statements with the variables.

QUANTIFIERS (cont'd)

- Let $P(x)$ is a statement with variable x and **A** is a set.
- *P* a **propositional function** or also known as **predicate** if for each x in **A**, $P(x)$ is a proposition.
- Set **A** is the **domain of discourse** of P.
- Domain of discourse: the particular domain of the variable in a propositional function.

QUANTIFIERS (cont'd)

A predicate is a statement that contains variables.

Example:

$$P\,(x): x > 3$$

$$Q\,(x, y): x = y + 3$$

$$R\,(x, y, z): x + y = z$$

Example:

The following are propositional functions:

1) $x^2 + 4x$ is an odd integer

   (domain of discourse is set of positive numbers).

2) $x^2 - x - 6 = 0$

   (domain of discourse is set of real numbers).

3) The university rated as Research University in Malaysia

   (domain of discourse is set of research university in Malaysia).

QUANTIFIERS (cont'd)

A predicate becomes a proposition if the variable(s) contained is(are)

- Assigned specific value(s)
- Quantified

Example:

i) $P(x)$: $x > 3$

What are the truth values of $P(4)$ and $P(2)$?

ii) $Q(x,y)$: $x = y + 3$

What are the truth values of $Q(1,2)$ and $Q(3,0)$?

QUANTIFIERS (cont'd)

Two types of quantifiers:

a) **Universal**

b) **Existential**

Universal Quantifier

- Let **A** be a propositional function with domain of discourse **B**. The statement:

  for every x, **A**(x)

  is **universally quantified statement**
- Symbol **∀** called a **universal quantifier** is used "**for every**".
- Can be read as *"for all"*, *"for any"*.

Universal Quantifier (cont'd)

- The statement can be written as

$$\forall x \, \mathbf{A}(x)$$

- Above statement is **true** if **A**(x) is true for every x in **B** and it **false** if **A**(x) is false for at least one x in **B**.

- A value x in the domain of discourse that makes the statement **A**(x) false is called as a **counterexample** to the statement.

Example:

Let the universally quantified statement is

$$\forall x\,(x^2 \geq 0)$$

Domain of discourse is the set of real numbers. Determine the truth value of this statement.

Answer:

This statement is true because for every real number x, it is true that the square of x is positive or zero.

Example:

Let the universally quantified statement is

$$\forall x \, (x^2 \leq 9)$$

Domain of discourse is a set B = {1, 2, 3, 4}

Determine the truth value for this statement.

Answer:

This statement is false and the counterexample is 4, because when x = 4, the statement produce false value.

Universal Quantifier (cont'd)

- Easy to prove a universally quantified statement is true or false if the domain of discourse is not too large.
- What happen if the domain of discourse contains a large number of elements?
- For example, a set of integer from 1 to 100, the set of positive integers, the set of real numbers or a set of students in Faculty of Computing. It will be hard to show that every element in the set is true.
- Therefore, we need to use existential quantifier.

Existential Quantifiers (cont'd)

- Let **A** be a propositional function with domain of discourse **B**. The statement

  There exist x**, A(**x**)**

  is **existentially quantified statement**

- Symbol **∃** called an **existential quantifier** is used "**there exist**".

- Can be read as "*for some*", "*for at least one*".

Existantial Quantifiers (cont'd)

- The statement can be written as

$$\exists x\ \mathbf{A}(x)$$

- Above statement is true if it possible to find at least one x in **B** that makes **A**(x) true, and it will be false if every x in **B** that makes the statement **A**(x) false.

- Just find one x that makes **A**(x) true!

Example:

Let the existentially quantified statement is

$$\exists x\left(\frac{x}{x^2+1}=\frac{2}{5}\right)$$

Domain of discourse is the set of real numbers.

Determine the truth value for this statement.

Answer:

Statement is true because it is possible to find at least one real number x to make the proposition true.

For example, if x = 2, we obtain the true proposition as below.

$$\left(\frac{x}{x^2+1}=\frac{2}{5}\right)=\left(\frac{2}{2^2+1}=\frac{2}{5}\right)$$

Negation of Quantifiers

Distributing a negation operator across a quantifier changes a universal to an existential and vice versa.

$$\neg\,(\forall x\, P(x)) \Rightarrow\; \exists x\, \neg P(x)$$

$$\neg\,(\exists x\, P(x)) \Rightarrow\; \forall x\, \neg P(x)$$

Example:

Let $P(x)$ = x is taking Discrete Structure course with the domain of discourse is the set of all students.

- $\forall x\, P(x)$: All students are taking Discrete Structure course.
- $\exists x\, P(x)$: There is some students who are taking Discrete Structure course.
- $\forall x\, \neg P(x)$: All students are not taking Discrete Structure course.
- $\exists x\, \neg P(x)$: There are some students who are not taking Discrete Structure course.

PROOF TECHNIQUES

Mathematical systems consists:

- ✓ Axioms - assumed to be true.
- ✓ Definitions - used to create new concepts.
- ✓ Undefined terms - some terms that are not explicitly defined.
- ✓ Theorem – proposition that has been proved to be true.

Theorem:

- It can be stated as follows:
  - ➢ As facts
  - ➢ As implications
  - ➢ As bi-implications
- An argument that establishes the truth of a theorem is called a **proof.**

PROOF TECHNIQUES (cont'd)

There are several techniques to proof theorem:

i. Direct proof
ii. Indirect proof
iii. Proof by contradiction

Direct Proof

- Direct proof or proof by direct method assumes that $p(x_1, x_2, \ldots, x_n)$ is true and then using $p(x_1, x_2, \ldots, x_n)$ as well as other axioms, definitions and theorems, show directly that $q(x_1, x_2, \ldots, x_n)$ is true.

Symbol: $\forall x\ (P(x) \rightarrow Q(x))$

- Procedure for direct proof:
  - ✓ Let **D** is the domain of discourse, select a particular, but arbitrarily chosen, member of the domain **D**.
  - ✓ Then show that the statement $\forall x\ (P(x) \rightarrow Q(x))$ is true by assuming that $P(x)$ is true and then show that $Q(x)$ is also true.

Example:

Use direct proof to prove the following theorem.

For all integer x, if x is odd, then x^2 is odd.

Solution:

We can verify the theorem by substituting x with certain values. For example,

$x = 3 \rightarrow x^2 = 9$ (both odd);

$x = 511 \rightarrow x^2 = 261121$ (both odd)

However, verifying a given theorem for a particular value is not a proof. Therefore, we must prove that the theorem is true for an arbitrary value. So, let

$P(x) =$ x is an odd integer

$Q(x) = x^2$ is an odd integer

Solution (cont'd)

Symbolically, $\forall x\big(P(x) \rightarrow Q(x)\big)$ with domain of discourse is the set of all integers. Let a is an odd integer.

$$\Rightarrow a = 2n + 1$$ for some integer n

$$\Rightarrow a^2 = (2n + 1)^2$$

$$\Rightarrow a^2 = 4n^2 + 4n + 1$$

$$\Rightarrow a^2 = 2(2n^2 + 2n) + 1$$

$$\Rightarrow a^2 = 2m + 1$$ where $\boldsymbol{m} = 2n^2 + 2n$ is an integer

$$\Rightarrow a^2$$ is an odd integer

Indirect Proof

Consider the implication $p \rightarrow q$ which is equivalent to the implication $\neg q \rightarrow \neg p$.

Therefore, in order to show that $p \rightarrow q$ is true, we can also show that the implication $\neg q \rightarrow \neg p$ is true.

To show that $\neg q \rightarrow \neg p$ is true, assume that the negation of q is true and prove that the negation of p is true.

Example:

Use indirect proof to prove the following:

Let n be an integer, if n^2+3 is odd, then n is even.

Solution:

Let,

$P(n) = n^2+3$ is an odd integer

$Q(n) = n$ is an even integer.

Symbolically, $\forall x\big(P(x) \rightarrow Q(x)\big)$ with domain of discourse is the set of all integers.

Assume n is a particular integer but arbitrary chosen element from the domain of discourse.

Solution (cont'd)

For this n, suppose $\neg Q(n)$ is true, we need to show that $\neg P(n)$ is true.

Because $\neg Q(n)$ is true, n is not even. Then, n is odd.

So, $n = 2k + 1$ for some integer k.

$$n = 2k + 1$$
$$n^2 = (2k + 1)^2$$
$$n^2 = 4k^2 + 4k + 1$$
$$n^2 + 3 = (4k^2 + 4k + 1)+3$$
$$n^2 + 3 = 4k^2 + 4k + 4$$
$$n^2 + 3 = 2(2k^2 + 2k + 2)$$

Because k is integer, thus $t = 2k^2 + 2k + 2$ is an integer. So, $n^2 + 3 = 2t$, which is multiple of 2. Therefore, $n^2 + 3$ is even integer, $\neg P(n)$ is true.

Proof by Contradiction

In a proof by contradiction, we assume that the conclusion is not true then arrive at a contradiction.

Example:

Prove the following theorem:

There are infinitely many prime numbers.

Solution:

Assume to the contrary that there are only finite many prime numbers and all of them are listed as: $p_1, p_2, \ldots\ldots, p_n$

Consider the number $q = p_1, p_2, \ldots\ldots, p_{n+1}$. The number q is either prime or not divisible. If we divide any of the listed primes, p_i into q, there would result a remainder of 1 for each $i = 1, 2, \ldots, n$. Thus q is not divisible, but not listed above. Therefore, q is a prime.

Contradiction! Therefore, there are infinitely many primes numbers.

Exercise

Let $P(x,y)=(x*y)^2 \geq 1$. Given the domain of discourse for x and y is set of integer, Z. Determine the truth value of the following statements. Give the value of x and y that make the statement TRUE or FALSE.

$a)\ \exists x \exists y\ P(x,y)$

$b)\ \forall x \forall y\ P(x,y)$