CHAPTER 4

GRAPH THEORY

Semester 1 – 2014/2015

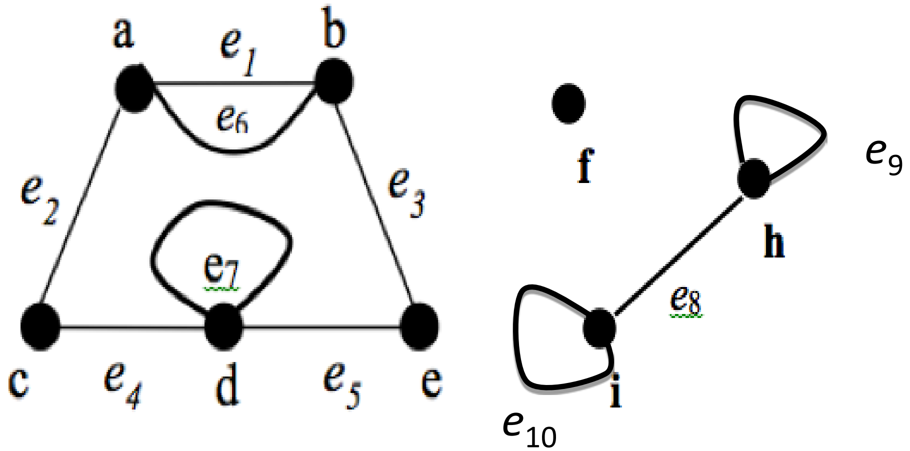
[Part 1]

Definition of Graph

- A graph G consists of two finite sets:
  - a nonempty set $V(G)$ of vertices.
  - a set $E(G)$ of edges, where each edge is associated with a set consisting of either one or two vertices called its endpoints.
  - f is a function, called an incidence function, that assign to each edge, $e \in E$, a one element subset $\{v\}$ or two elements subset $\{v, w\}$, where v and w are vertices.
- We can write G as (V, E, f) or (V, E) or simply as G.

Example 1

Given a graph as shown below,



a) Write a vertex set and the edge set, and give a table showing the edge-endpoint function.

a) Find all edges that are incident on **a**, all vertices that are adjacent to **a**, all edges that are adjacent to e_2, all loops, all parallel edges, all vertices that are adjacent to themselves and all isolated vertices.

Example 1 - Solution

Solution:

a) Vertex set, V = {a, b, c, d, e, f, i, h} and the set of edges, E = {e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}`

Edge	Endpoints
e_1	{a, b}
e_2	{a, c}
e_3	{b, c}
e_4	{c, d}
e_5	{d, e}
e_6	{a, b}
e_7	{d}
e_8	{i, h}
e_9	{h}
e_{10}	{i}

Alternatively, we can write the edge-endpoints function as follows:

$f(e_1) = \{\mathbf{a}, \mathbf{b}\}$, $f(e_2) = \{\mathbf{a}, \mathbf{c}\}$, $f(e_3) = \{\mathbf{b}, \mathbf{e}\}$, $f(e_4) = \{\mathbf{c}, \mathbf{d}\}$, $f(e_5) = \{\mathbf{d}, \mathbf{e}\}$, $f(e_6) = \{\mathbf{a}, \mathbf{b}\}$, $f(e_7) = \{\mathbf{d}\}$, $f(e_8) = \{\mathbf{i}, \mathbf{h}\}$, $f(e_9) = \{\mathbf{h}\}$, $f(e_{10}) = \{\mathbf{i}\}$.

b) e_1, e_2 and e_6 are incident on **a**.

c and **b** are adjacent to **a**.

e_1, e_4 and e_6 are adjacent to e_2.

e_7 is a loop.

e_1 and e_6 are parallel.

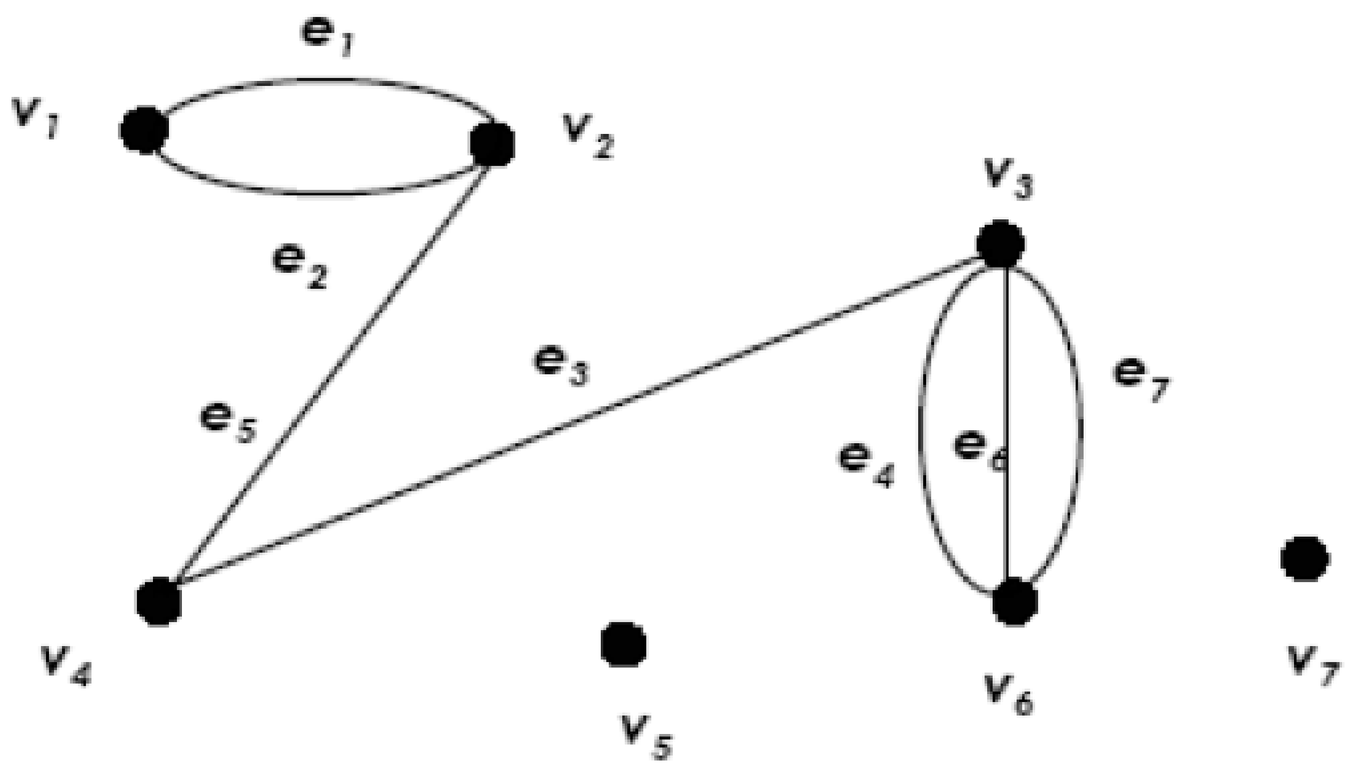
i and **h** are adjacent to themselves.

f is an isolated vertex.

Example 2

- Let,
  - $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$
  - $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$
- And *f* be defined by:
  - $f(e_1) = f(e_2) = \{v_1, v_2\}$
  - $f(e_3) = \{v_4, v_3\}$
  - $f(e_4) = f(e_6) = f(e_6) = \{v_6, v_3\}$
  - $f(e_5) = \{v_2, v_4\}$

- Then, $G = (V, E, f\,)$ is a graph as shown below.

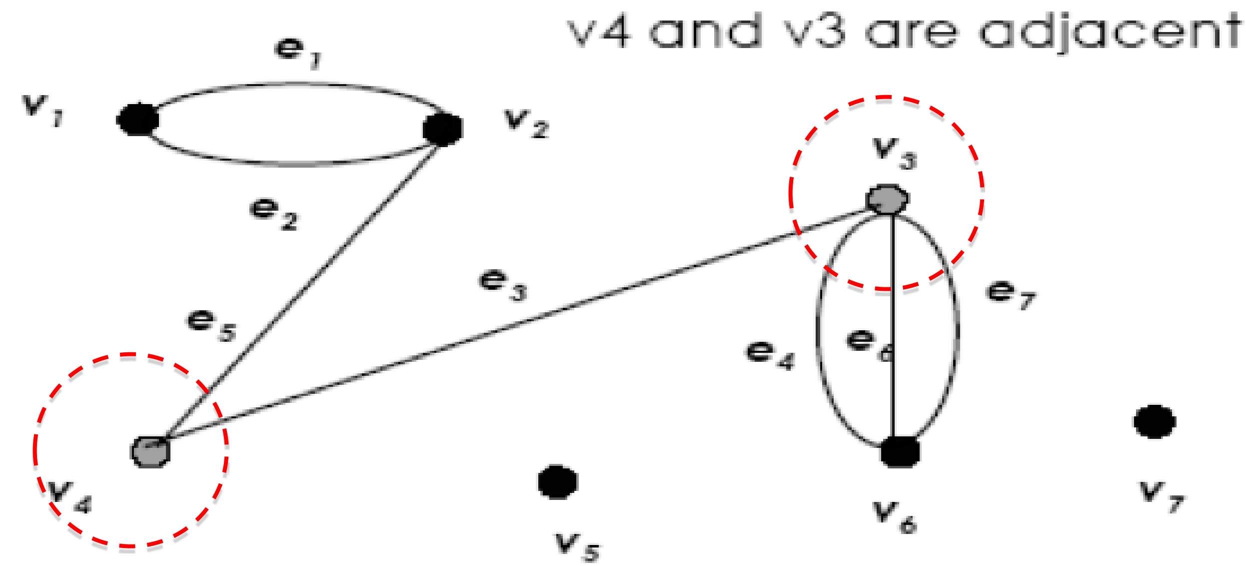


Characteristics of Graph

(a)Adjacent Vertices

Two vertices that are connected by an edge are called adjacent; and a vertex that is an endpoint of a loop is said to be adjacent to itself.

Example

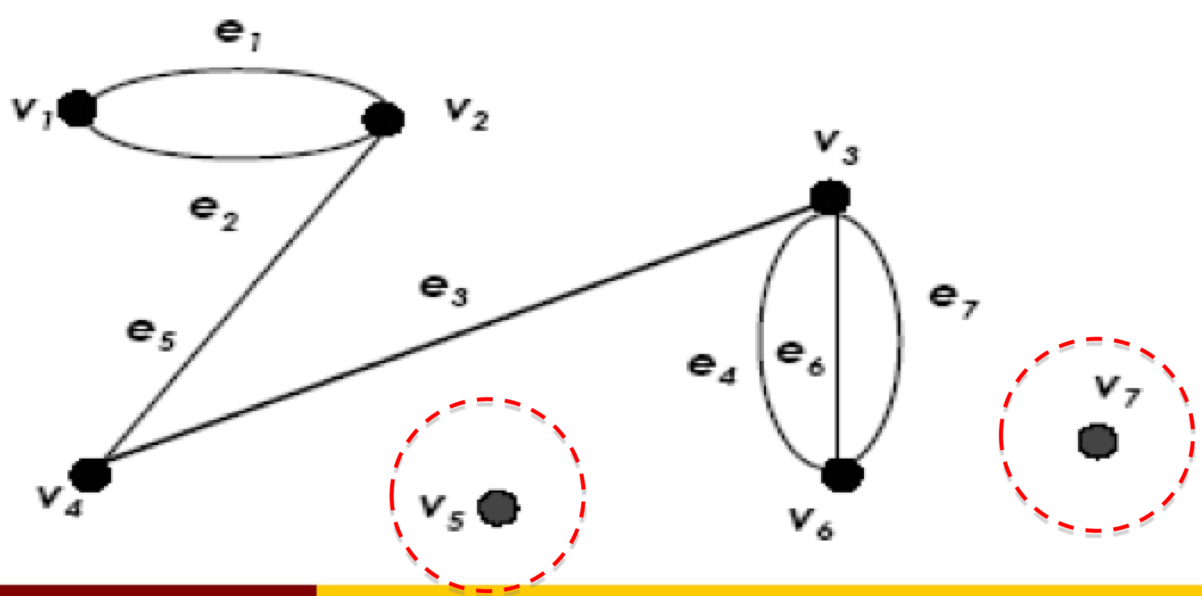


b) Isolated Vertex

Let G be a graph and v be a vertex in G. We say that v is an isolated vertex if it is not incident with any edge.

- v5 and v7 are isolated vertices.

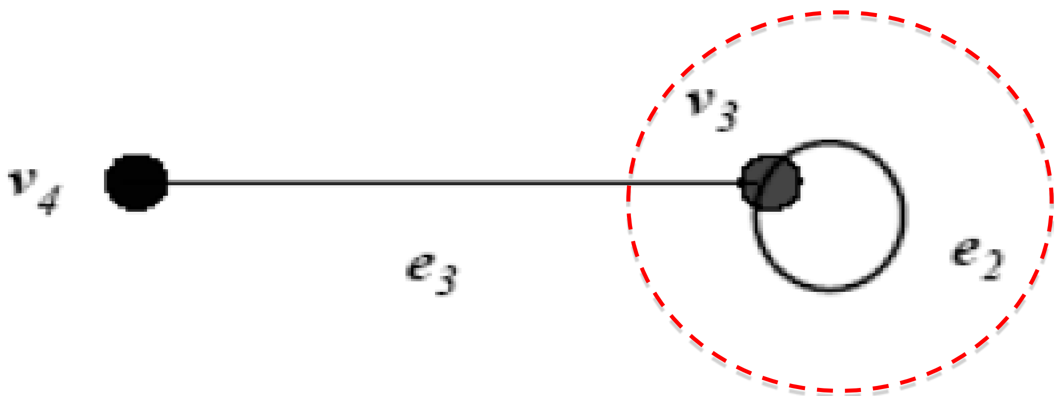
Example



c) Loop

An edge with just one endpoint is called a loop.

Example



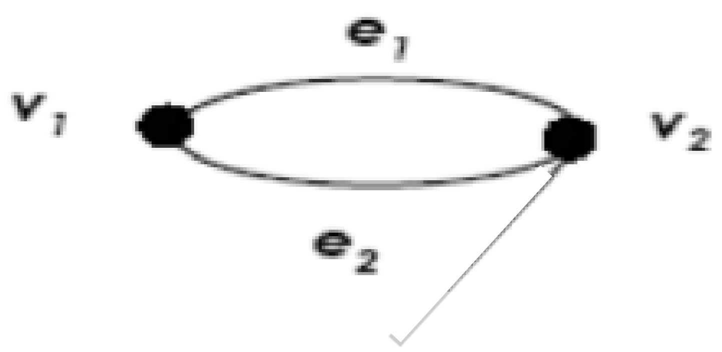
- e2 is a loop

d) Parallel Edges

Two or more distinct edges with the same set of endpoints are said to be parallel.

- e_1 and e_2 are parallel.

Example



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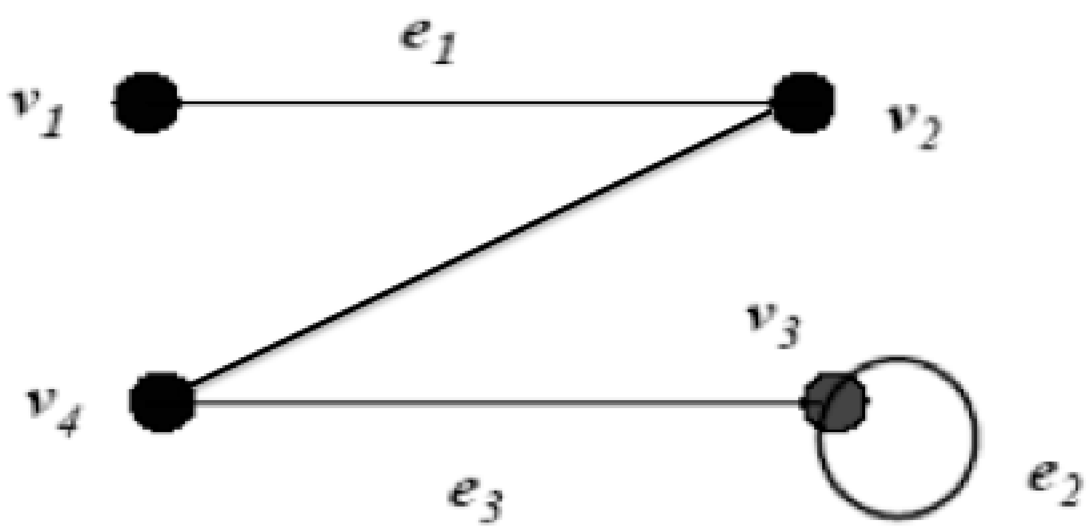
The Concept of Degree

- Let G be a graph and v be a vertex in G.
- The degree of v, written deg(v) or d(v) is the number of edges incident with v.
- Each **loop** on a vertex v contributes 2 to the degree of v.

Example

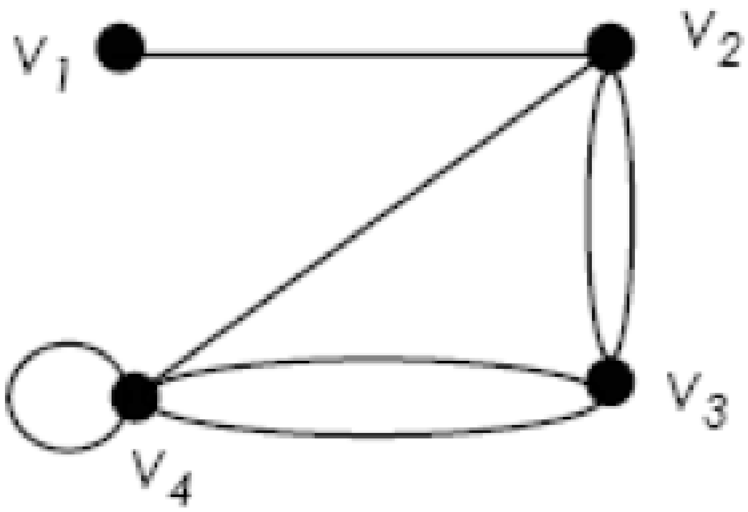
The degree of each vertex for the following graph is:

$\deg(v_1) = 1$; $\deg(v_2) = 2$; $\deg(v_3) = 3$; $\deg(v_4) = 2$



Exercise

- Find the degree of each vertex in the graph.



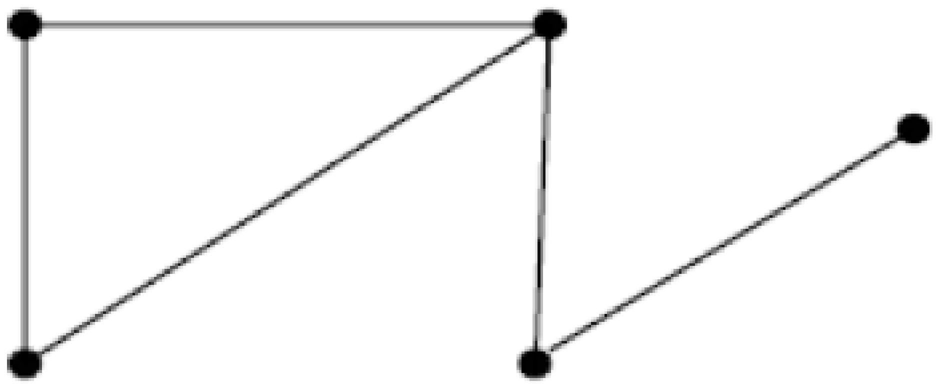
Solution: deg(v_1)= 1; deg(v_2)= 4; deg(v_3)= 4; deg(v_4)= 5

Types of Graphs

a) Simple Graph

A graph G is called a simple graph if G does not contain any parallel edges and any loops.

Example



b) Regular Graph

Let G be a graph and k be a nonnegative integer. G is called a k-regular graph if the degree of each vertex of G is k.

Example

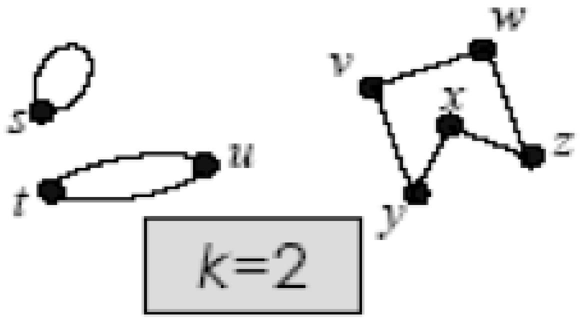


Fig.1: Graph A

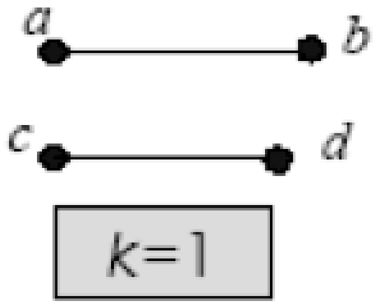


Fig.2: Graph B

c) Connected Graph

A graph G is connected if given any vertices *v* and *w* in G, there is a path from *v* to *w*.

Example

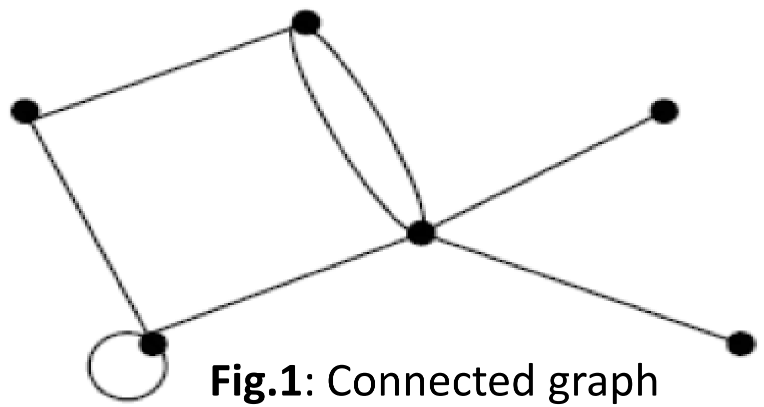


Fig.1: Connected graph

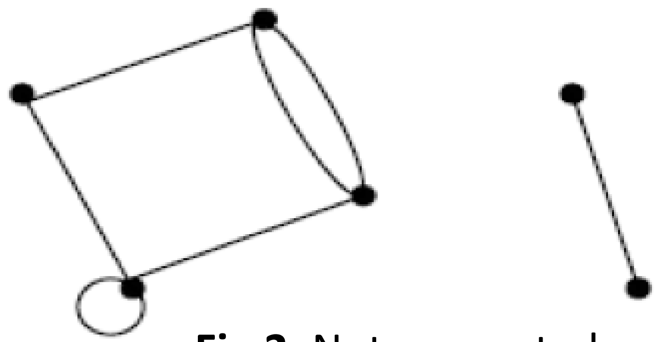


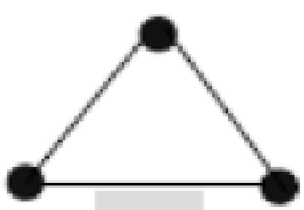
Fig.2: Not connected graph

d) Complete Graph

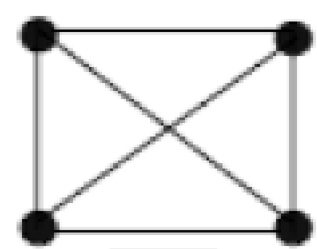
A simple graph with n vertices in which there is an edge between every pair of distinct vertices is called a complete graph on n vertices. This is denoted by k_n .

Example

K_2



K_3

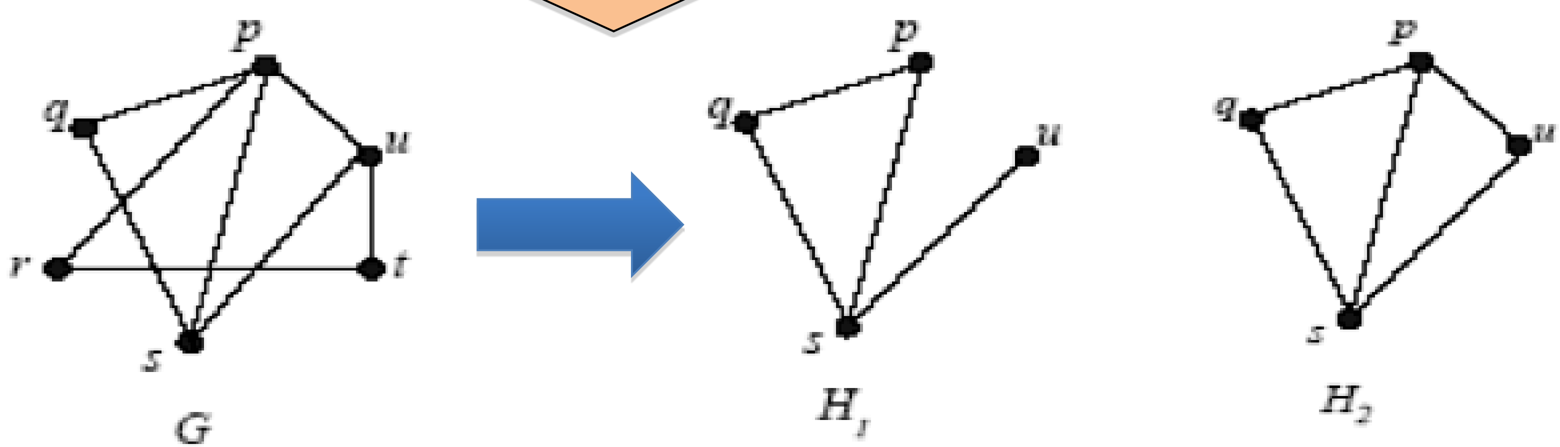


K_4

e) Subgraph

A graph H is said to be a subgraph of a graph G if, every vertex in H is also a vertex in G, every edge in H is also an edge in G, and every edge in H has the same endpoints as it has in G.

Example



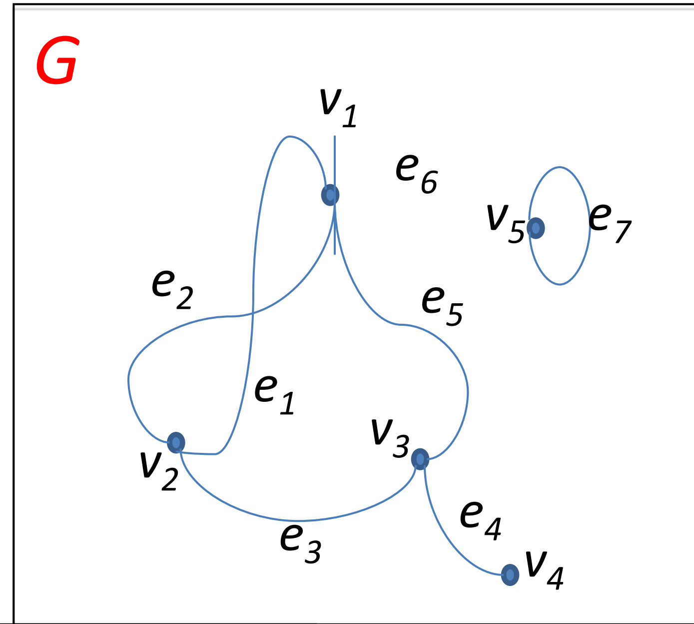
Exercise

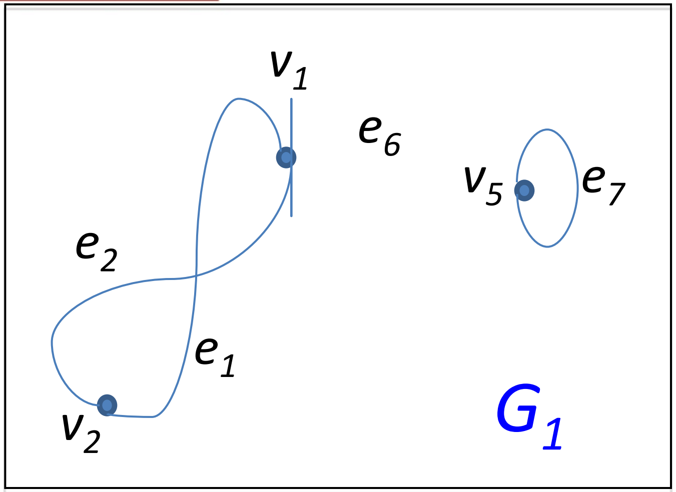
- Draw

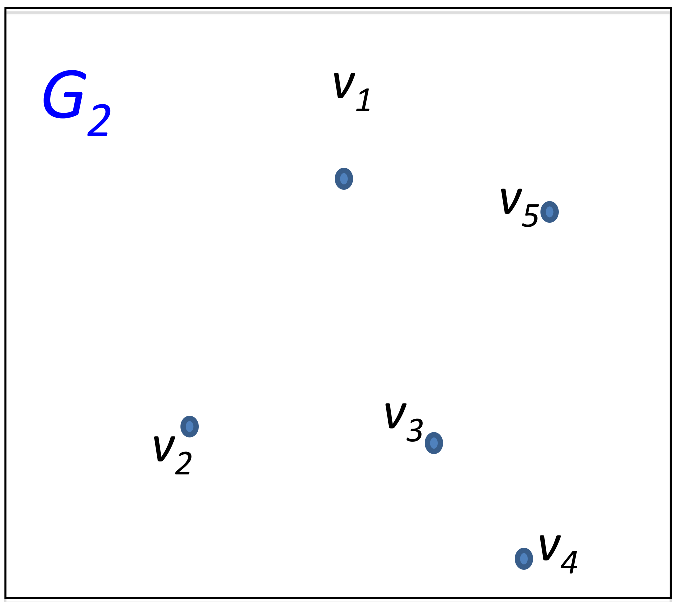
  i) K_5

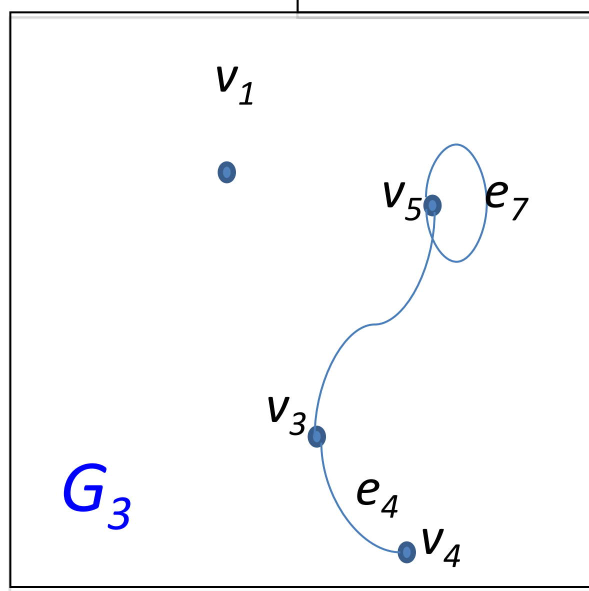
  ii) K_6

■Given G is a graph. Are G_1, G_2, and G_3 subgraphs of G?









Graph Representation

- To write programs that process and manipulate graphs, the graphs must be stored, that is, represented in computer memory.
- A graph can be represented (in computer memory) in several ways.
- 2-dimensional array: adjacency matrix and incidence matrix.

Adjacency Matrix

- Let G be a graph with n vertices.

- The adjacency matrix, A_G is an nxn matrix [a_{ij}] such that,

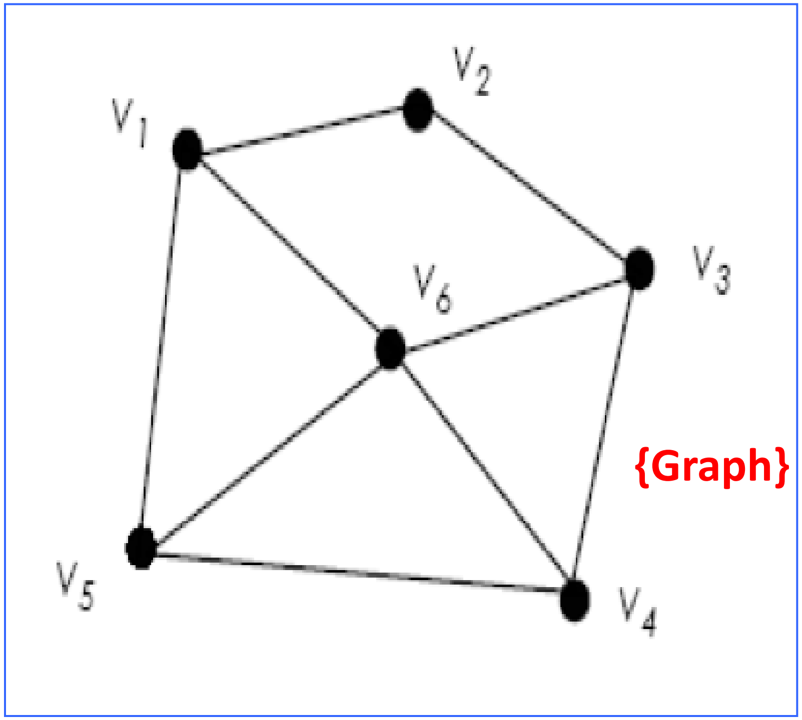
  a_{ij}= the number of edges from v_i to v_j, {undirected G}

  or,

  a_{ij}= the number of arrows from v_i to v_j, {directed G}

  for all i, j = 1,2,….., n.

Example 1

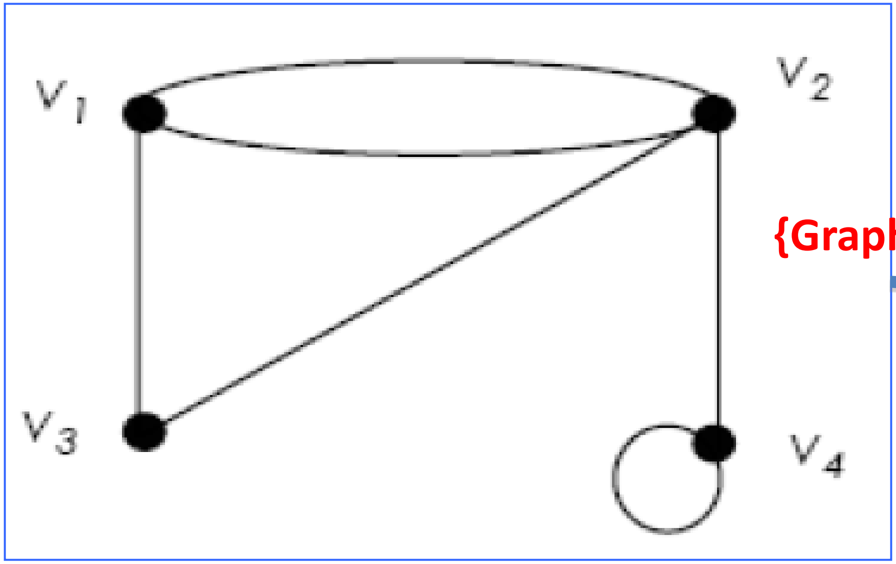


{Graph}

[Matrix]

$$A_G = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Example 2



[Matrix]

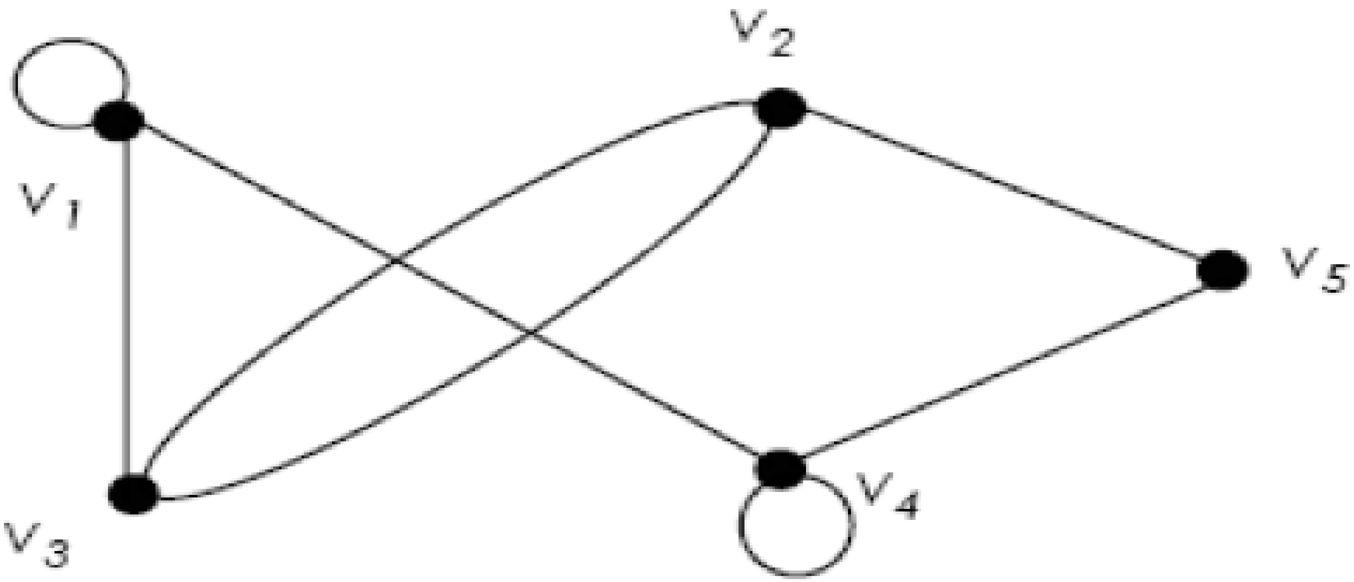
$$A_G = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Example 3

Draw the graph based on the following matrix:

$$A_G = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 1 \\ 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Example 3 - Solution



- Adjacency matrix is a **symmetric matrix** if it is representing an undirected graph, where

$$a_{ij} = a_{ji}$$

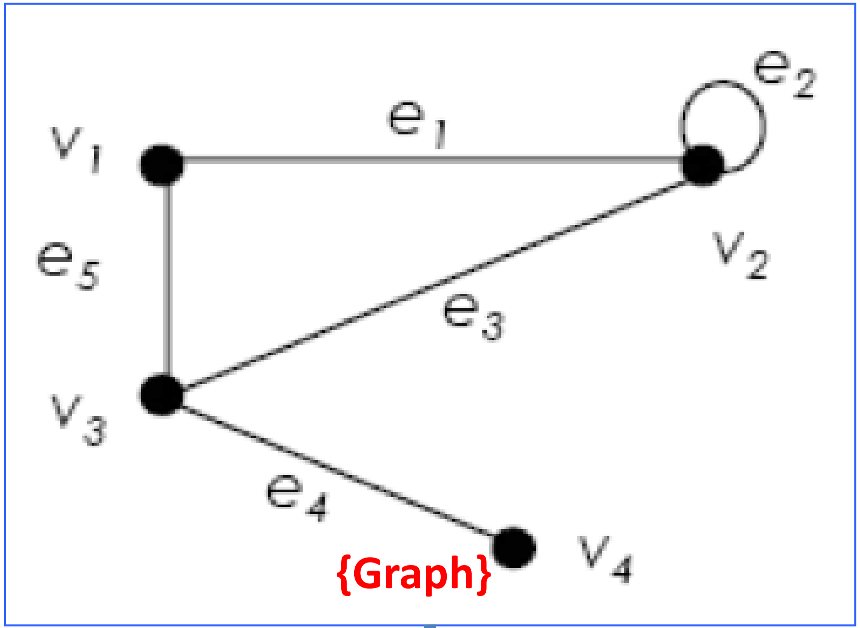
- If the graph is directed graph, the presented matrix is not symmetrical.

Incidence Matrix

- Let G be a graph with n vertices and m edges.
- The incidence matrix, I_G is an $n \times m$ matrix $[a_{ij}]$ such that,

$$a_{ij} = \begin{cases} 0 & \text{if } v_i \text{ is not an end vertex of } e_j, \\ 1 & \text{if } v_i \text{ is not an end vertex of } e_j, \text{but } e_j \text{ is not a loop} \\ 2 & \text{if } e_j \text{ is a loop at } v_i \end{cases}$$

Example



[Matrix]

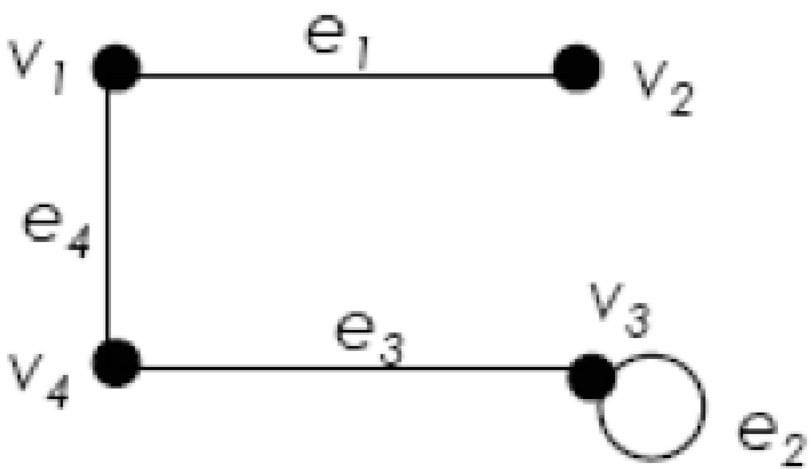
$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

deg(v_1)= 2;
deg(v_2)= 4;
deg(v_3)= 3;
deg(v_4)= 1

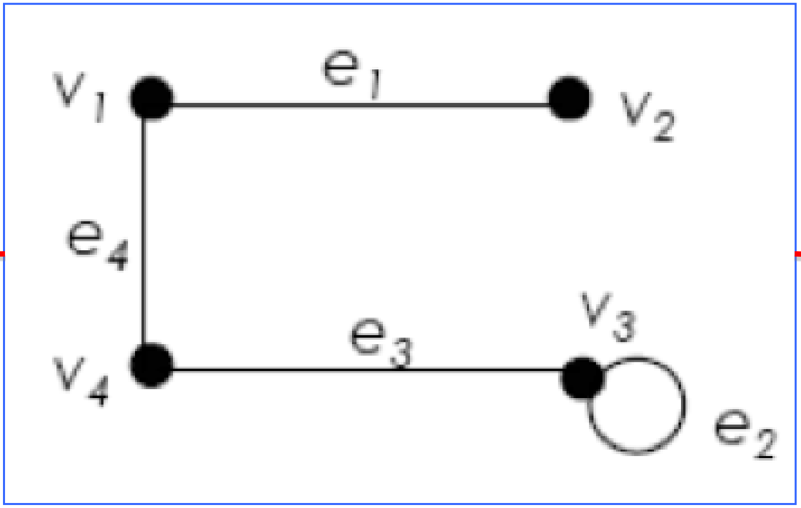
Notice that the sum of the *i*-th row is the degree of v_i.

Exercise

- Find the adjacency matrix and the incidence matrix of the graph.



Exercise - Solution



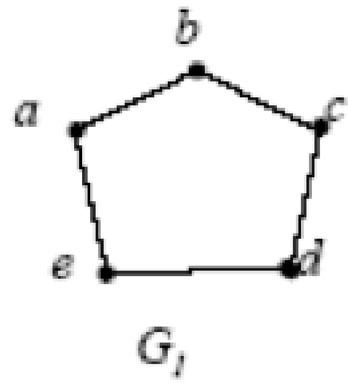
Adjacency matrix

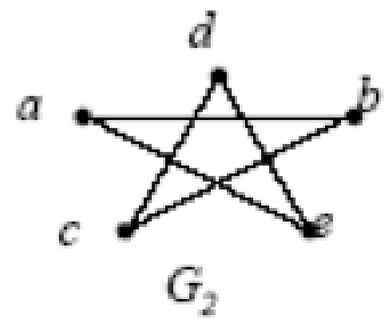
Incidence matrix

$$A_G = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$I_G = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Isomorphisms





- Are these two graphs (G_1 and G_2) are same?
- When we say that 2 graphs are the same mean they are isomorphic to each other.

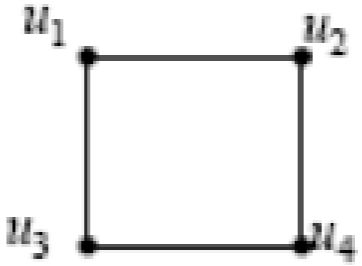
- Graphs G_1 and G_2 are isomorphic if there is:
  - a one-to-one, onto function *f* from the vertices of G_1 to the vertices of G_2, AND
  - a one-to-one, onto function *f* from the edges of G_1 to the edges of G_2.
  - For some ordering of their vertices, their adjacency matrices are equal.

- An edge e is incident on v and w in G_1 if and only if the edge $g(e)$ is incident on $f(v)$ and $f(w)$ in G_2.

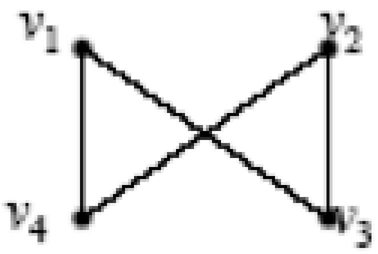
- The pair of functions *f* and *g* is called an isomorphism of G_1 onto G_2.

Example 1

- Determine whether G is isomorphic to *H*.



G



H

Example - Solution

- Both graphs are simple and have the same number of vertices and the same number of edges.

- All the vertices of both graphs have degree 2.

- Define $f : U \rightarrow V$, where $U = \{ u_1, u_2, u_3, u_4 \}$ and $V = \{ v_1, v_2, v_3, v_4 \}$;
  $f(u_1) = v_1$; $f(u_2) = v_4$; $f(u_3) = v_3$; $f(u_4) = v_2$.

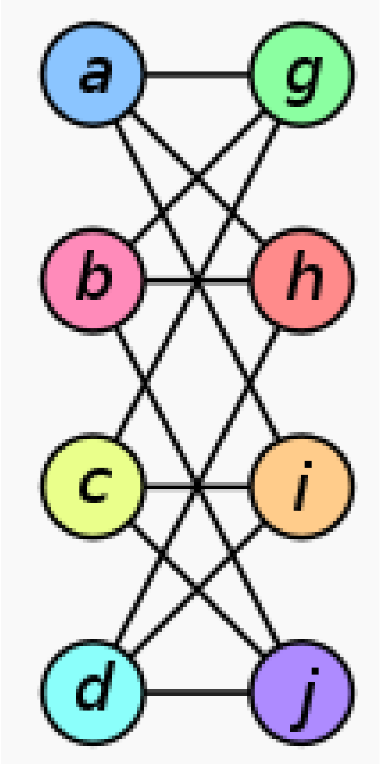
- To verify whether G and H are isomorphic, we examine the adjacency matrix A_G with rows and columns labeled in the order u_1, u_2 , u_3, u_4 , and the adjacency matrix A_H with rows and columns labeled in the order v_1, v_2 , v_3, v_4 .

- A_G and A_H are the same, G and *H* are isomorphic.

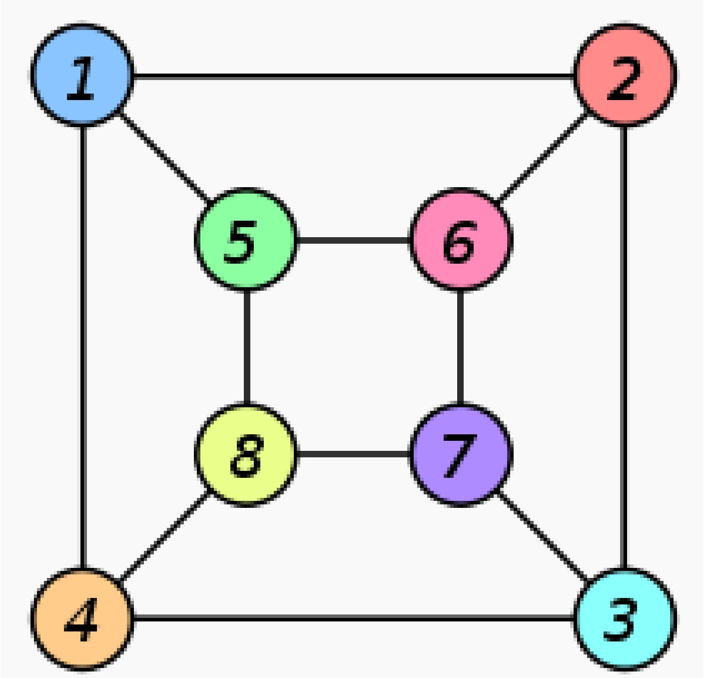
$$A_G = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array} \qquad A_H = \begin{array}{c} \\ v_1 \\ v_4 \\ v_3 \\ v_2 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_4 & v_3 & v_2 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

Example 2

- Is these two graphs are isomorphic?



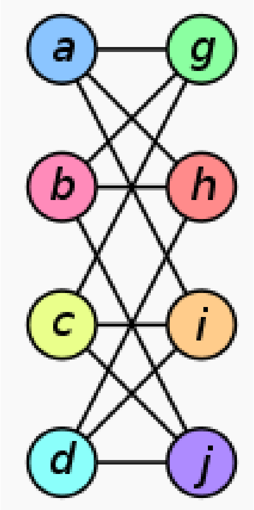
Graph G



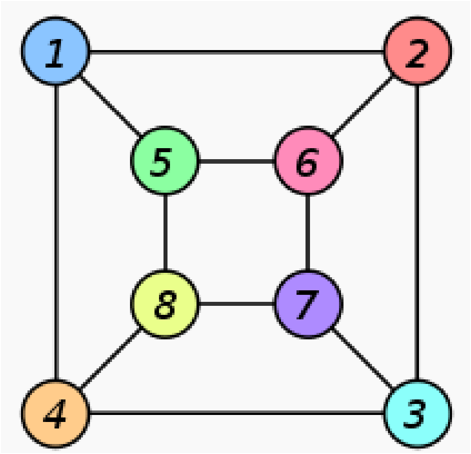
Graph H

Example 2 – Solution

- Count the vertices:
  - G_G = 8 vertices
  - G_H = 8 vertices
- Count the edges:
  - G_G = 12 edges
  - G_H = 12 edges

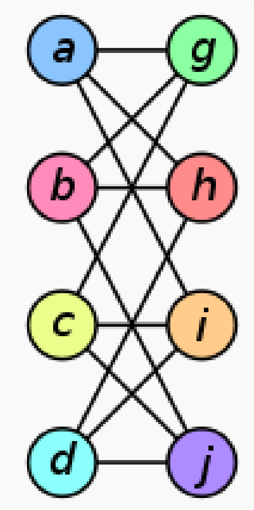


Graph G

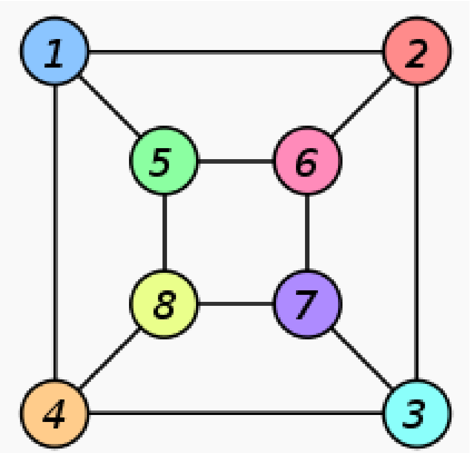


Graph H

- Find the degree of each vertex:
  - G_G = all vertices has degree of 3
  - G_H = all vertices has degree of 3



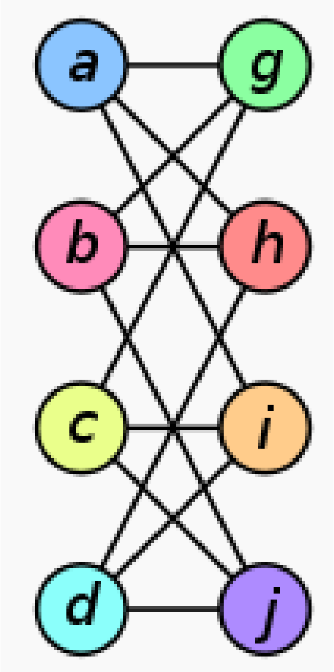
Graph G



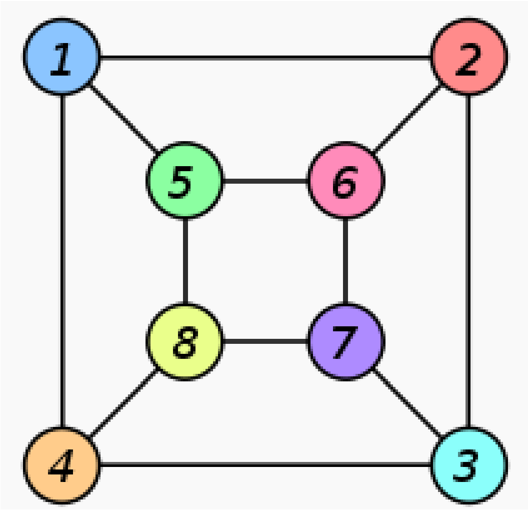
Graph H

- Define the incident function:

An isomorphism between G and H
$f(a) = 1$
$f(b) = 6$
$f(c) = 8$
$f(d) = 3$
$f(g) = 5$
$f(h) = 2$
$f(i) = 4$
$f(j) = 7$



Graph G



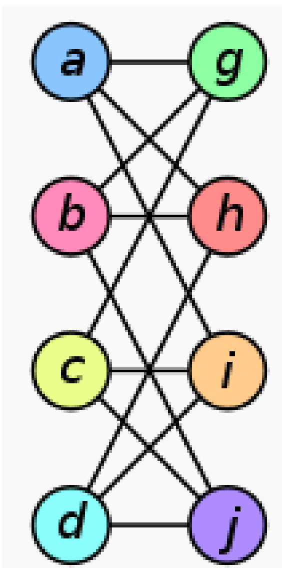
Graph H

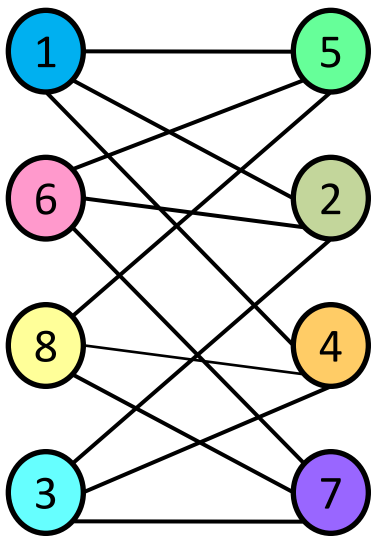
- Check the adjacency matrix for both graphs:

$$A_G = \begin{array}{c} \\ a \\ b \\ c \\ d \\ g \\ h \\ i \\ j \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & g & h & i & j \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

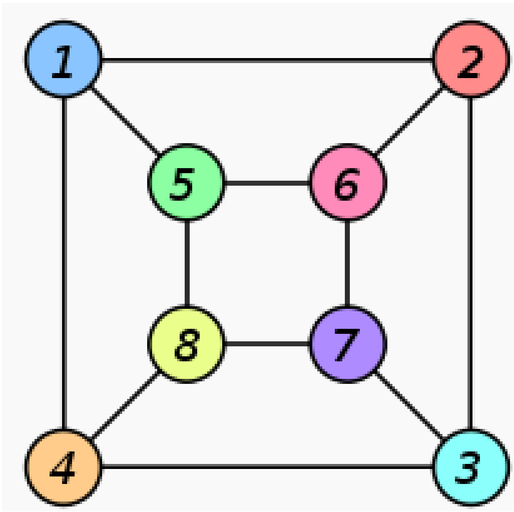
$$A_H = \begin{array}{c} \\ 1 \\ 6 \\ 8 \\ 3 \\ 5 \\ 2 \\ 4 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 6 & 8 & 3 & 5 & 2 & 4 & 7 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

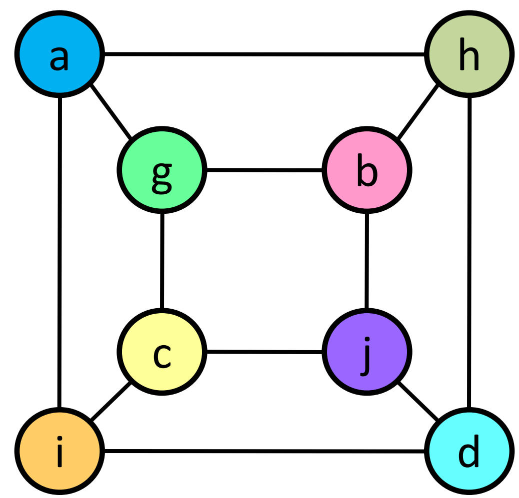
- Try to redraw the graphs:





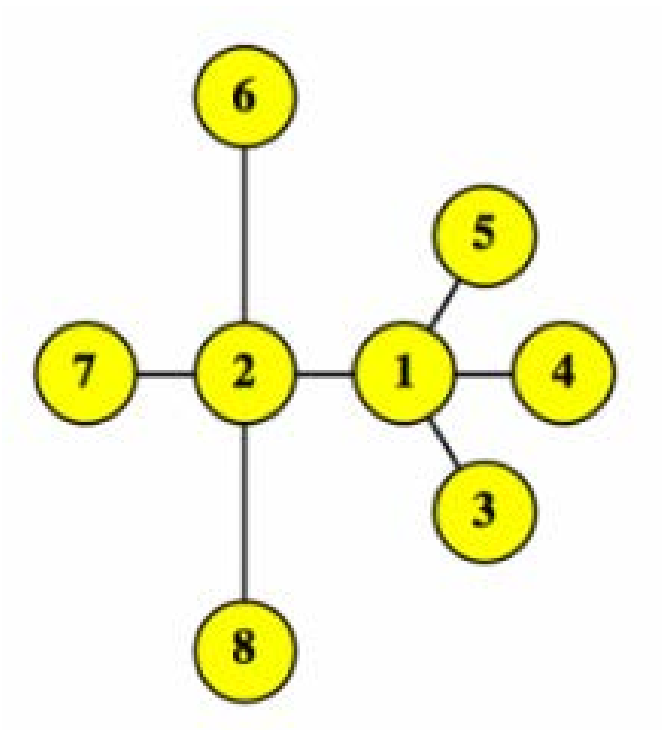
OR



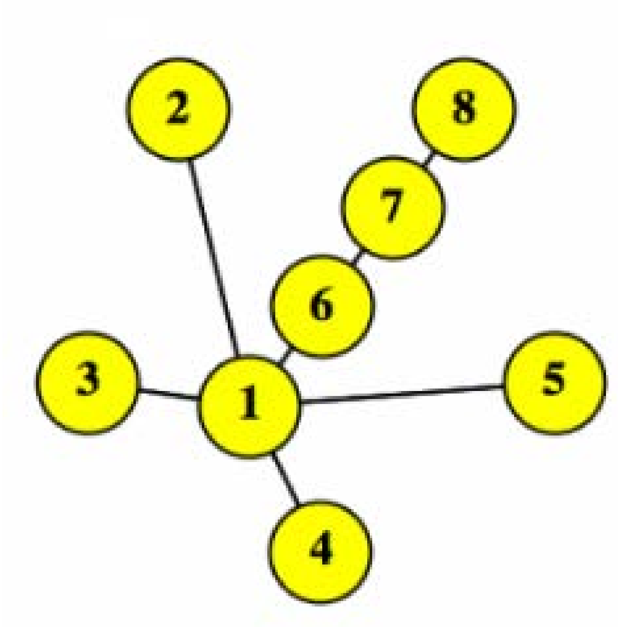


Example 3

- Determine if these graphs are isomorphic.



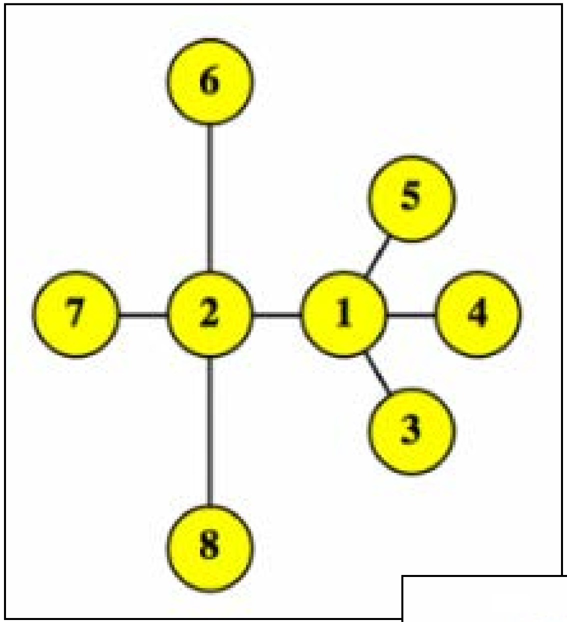
Graph A



Graph B

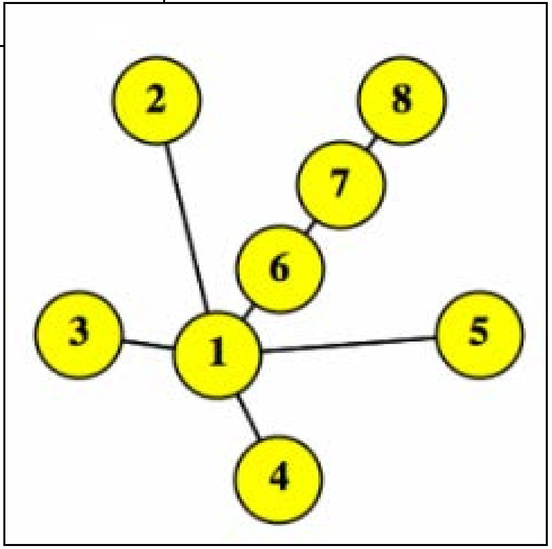
Example 3 – Solution

- Count the vertices:
  - G_A = 8 vertices
  - G_B = 8 vertices
- Count the edges:
  - G_A = 7 edges
  - G_B = 7 edges



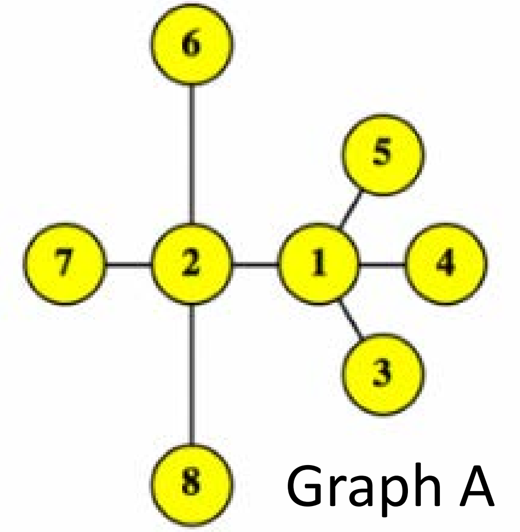
Graph A

Graph B

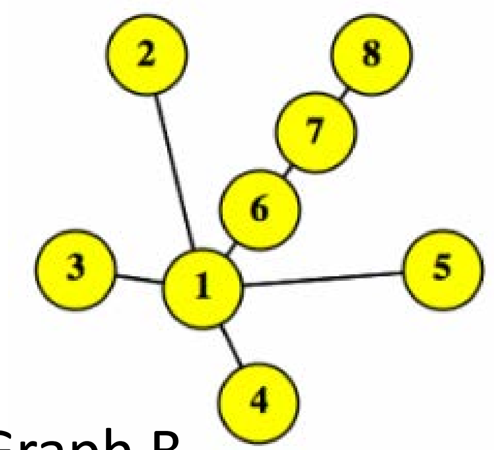


Example 10 – Solution (cont.)

- Find the degree of each vertex:
  - G_A
    - 2 vertices having degree of 4
    - 6 vertices having degree of 1
  - G_B
    - 1vertex having degree of 5
    - 2 vertices having degree of 2
    - 5 vertices having degree of 1

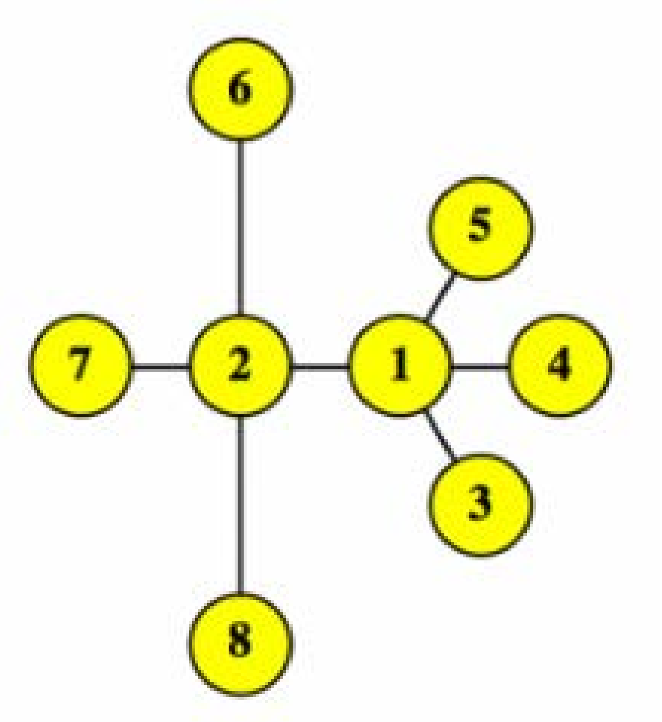


Graph A

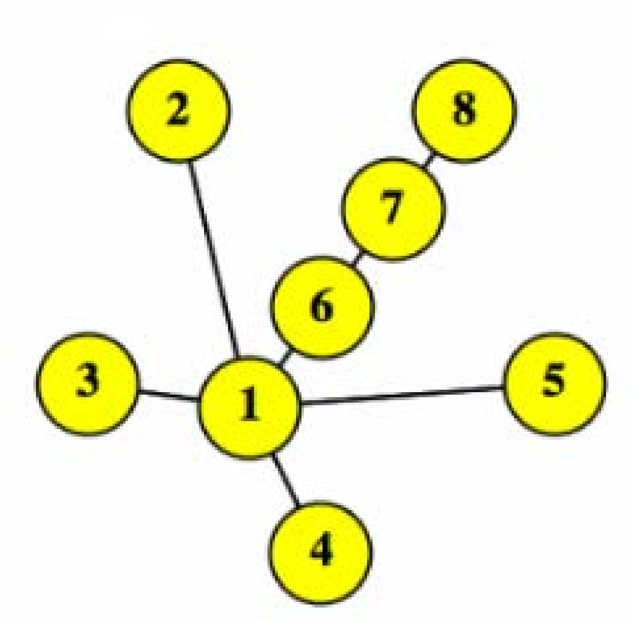


Graph B

- Hence, these graphs are not isomorphic due to the degree of vertices are different.



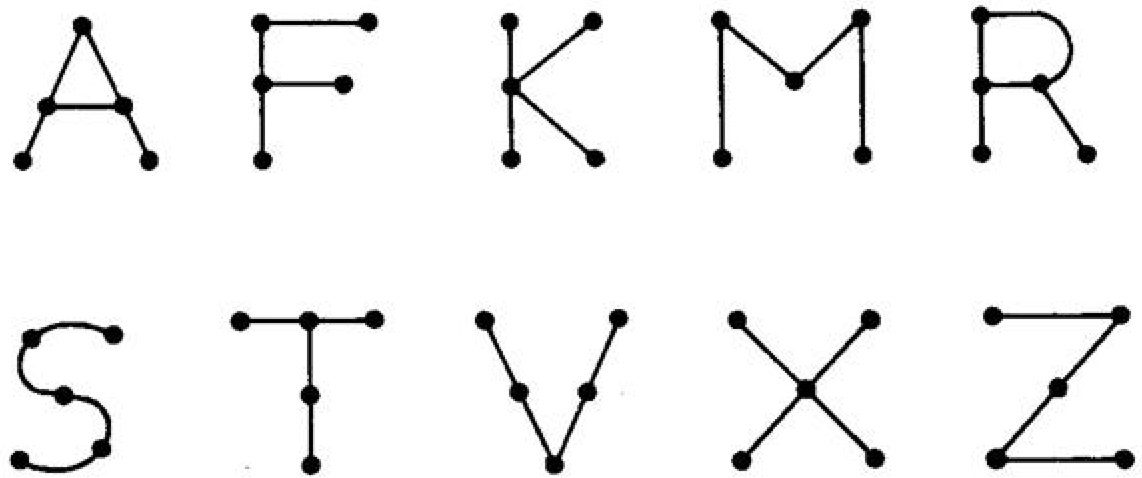
Graph A



Graph B

Exercise

- There are 10 graphs pictured as letters. Find isomorphic graphs between them.



[Part 2]

Trails, Paths & Circuits

Term and Description

- A walk from v to w is a finite alternating sequence of adjacent vertices and edges of G. Thus a walk has the form

$$(v_0, e_1, v_1, e_2, v_2, \ldots\ldots, v_{n-1}, e_n, v_n)$$

where the v's represent vertices, the e's represent edges, $v = v_0$, $w = v_n$, and for $i = 1,2,\ldots,n$. v_{i-1} and v_i are the endpoints of e_i.

- A trivial walk from v to w consist of the single vertex v.

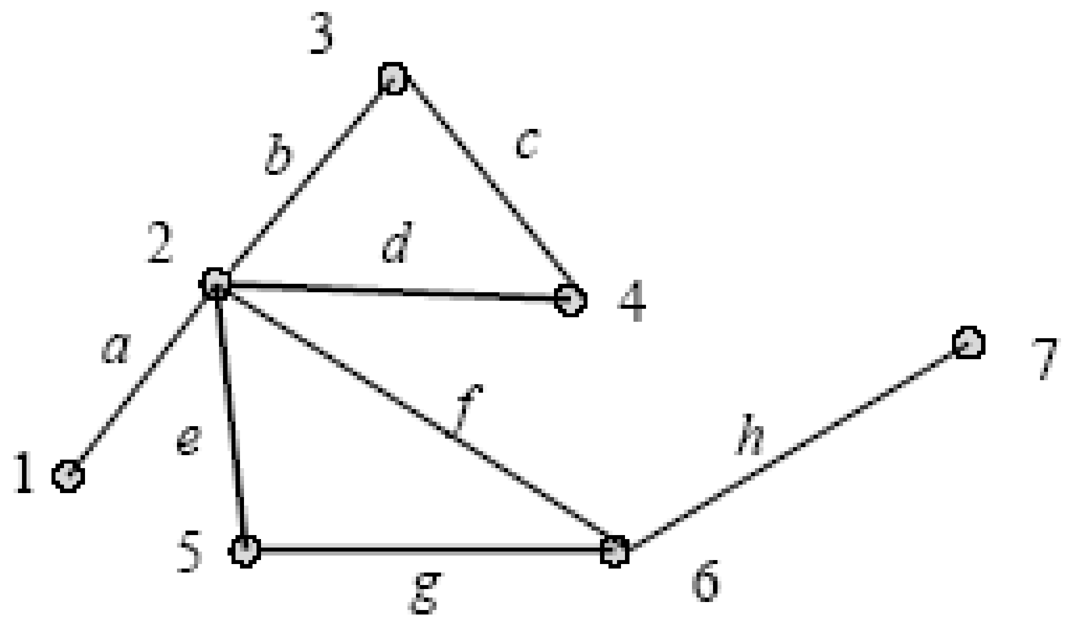
- The length of a walk is the number of edges it has.

Term and Description (cont.)

- A trail from v to w is a walk from v to w that does not contain a repeated edge.
- A path from v to w is a trail from v to w that does not contain a repeated vertex.
- A closed walk is a walk that start and ends at the same vertex.
- A circuit/cycle is a closed walk that contains at least one edge and does not contain a repeated edge.
- A simple circuit is a circuit that does not have any other repeated vertex except the first and the last.

Example 1 – Trail & Path

- (1, a, 2, b, 3, c, 4, d, 2, e, 5) is a trail.
- (6, g, 5, e, 2, d, 4) is a path.



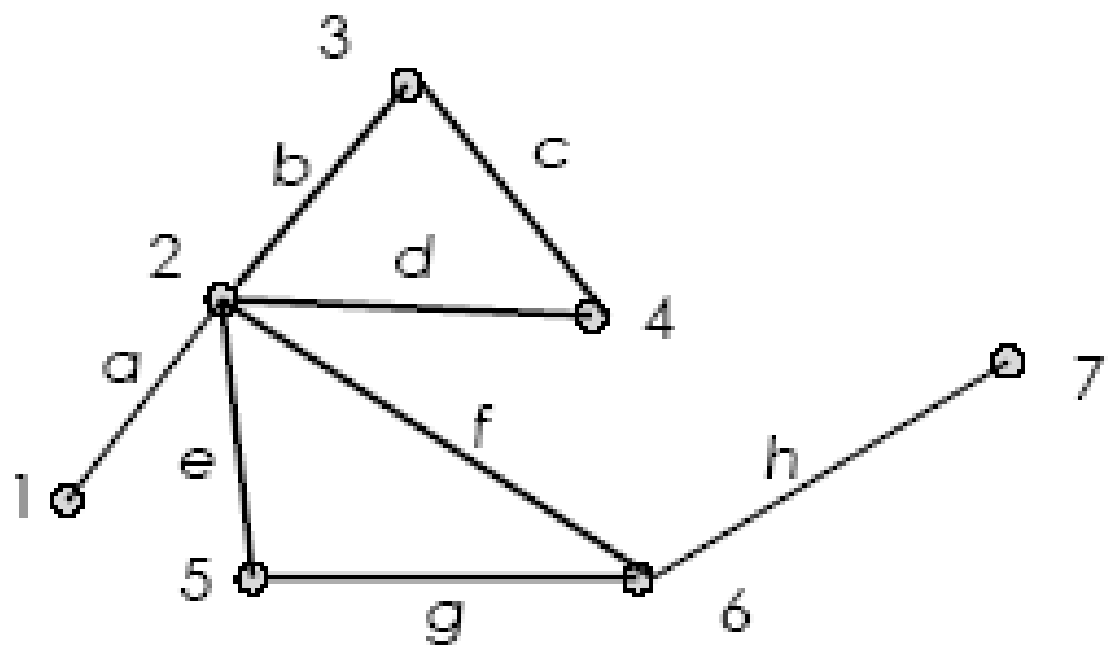
Note:

Trail: No repeated edge (can repeat vertex).

Path: No repeated vertex and edge.

Example 2 – Cycle/circuit

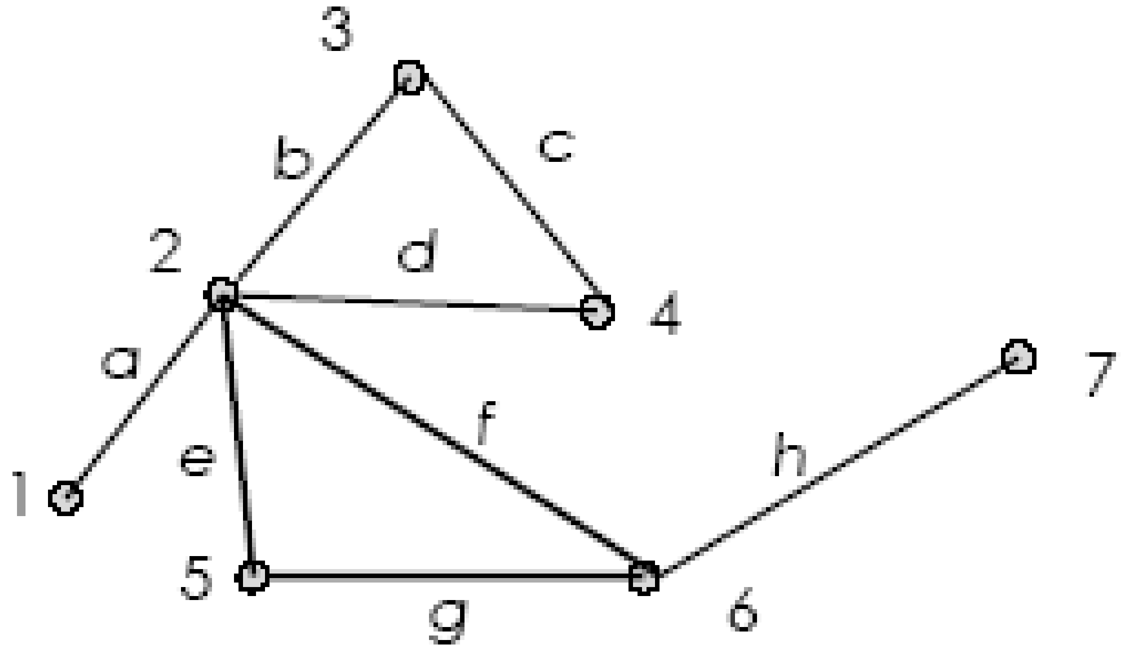
- (2, f, 6, g, 5, e, 2, d, 4, c, 3, b, 2) is a cycle.



Note: cycle –> start and end at same vertex, no repeated edge.

Example 3 – Simple Cycle

- (5, g, 6, f, 2, e, 5) is a simple cycle.

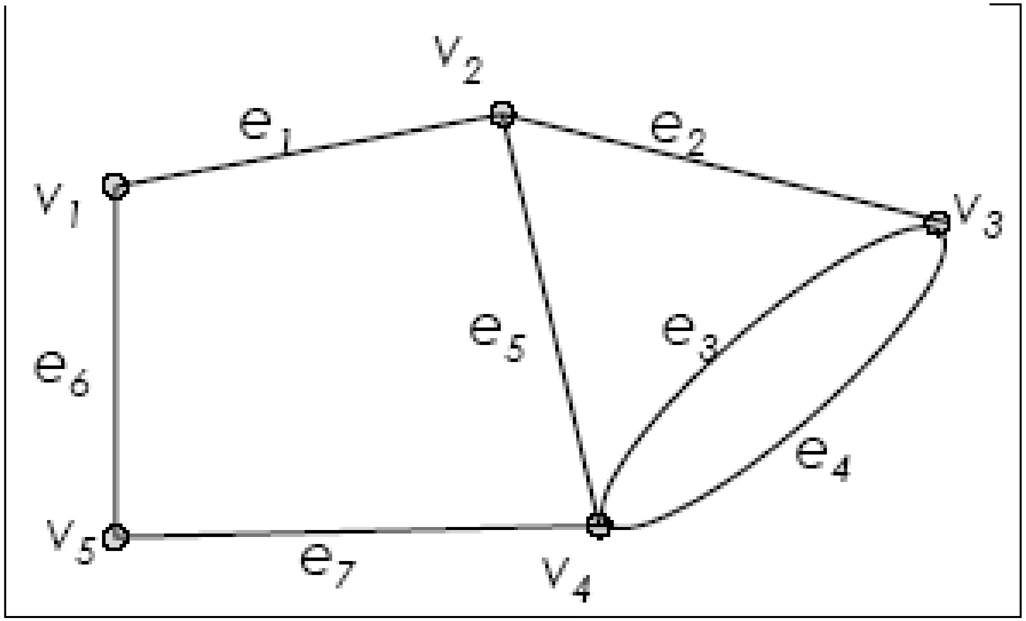


Note: Simple cycle –> start and end at same vertex, no repeated edge or vertex except for the start and end vertex.

Exercise

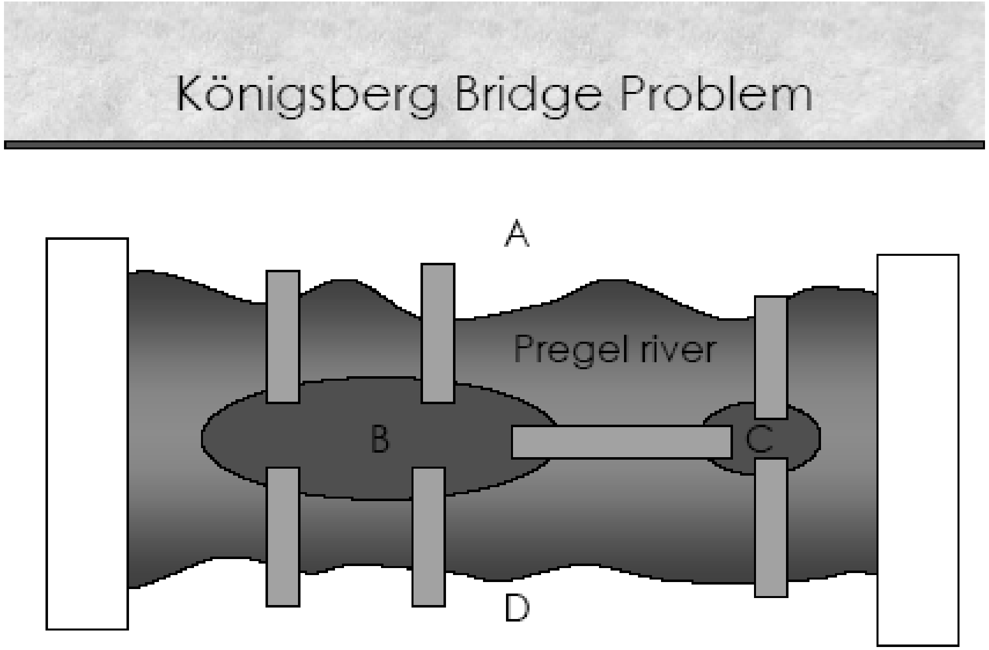
Tell whether the following is either a walk, trail, path, cycle, simple cycle, closed walk or none of these.

- (v_1, e_1, v_2) Trail
- $(v_2, e_2, v_3, e_3, v_4, e_4, v_3)$ Walk; Trail
- $(v_4, e_7, v_5, e_6, v_1, e_1, v_2, e_2, v_3, e_3, v_4)$ Simple cycle
- $(v_4, e_4, v_3, e_3, v_4, e_5, v_2, e_1, v_1, e_6, v_5, e_7, v_4)$ Cycle



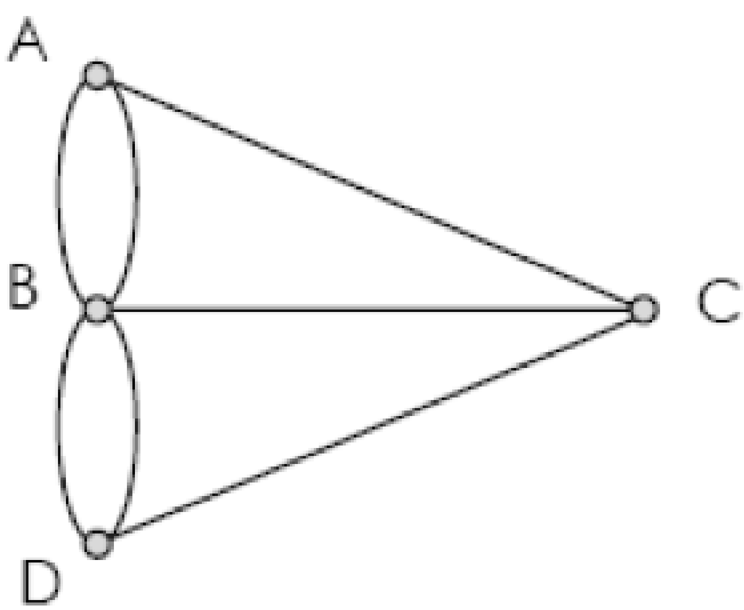
Euler Path & Circuit

The town of Konigsberg in Prussia (now Kaliningrad in Russia) was built at a point where two branches of the Pregel River came together. It consisted of an island and some land along the river banks. These were connected by seven bridges as shown in figure below:

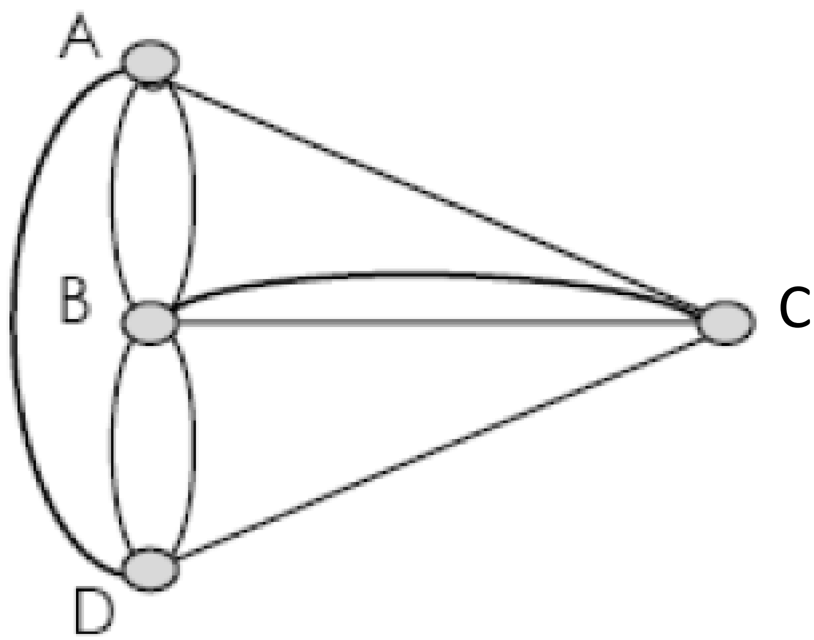


Problem: Starting at one land area, is it possible to walk across all of the bridges exactly once and return to the starting land area?

- Graph of the Königsberg Bridge Problem



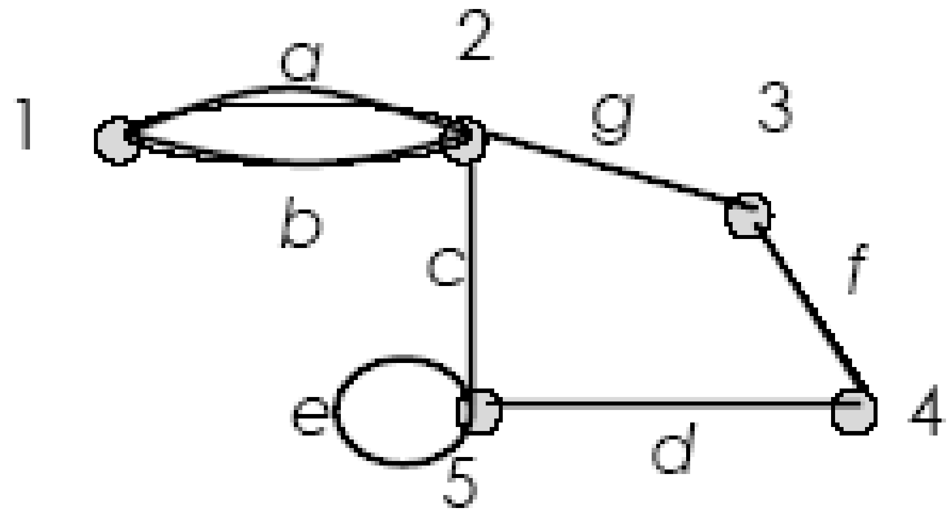
Solution: It is not possible to walk across all of the bridges exactly once and return to the starting land area. Since 1736, two additional bridges have been constructed on the Pregel river.



Euler Circuit

Let G be a graph. An Euler circuit for G is a circuit that contains every vertex and every edges of G. That is, an Euler circuit for G is a sequence of adjacent vertices and edges in G that has at least one edges, starts and ends at the same vertex, uses every vertex of G at least once, and uses every edge of G exactly once.

Example 1

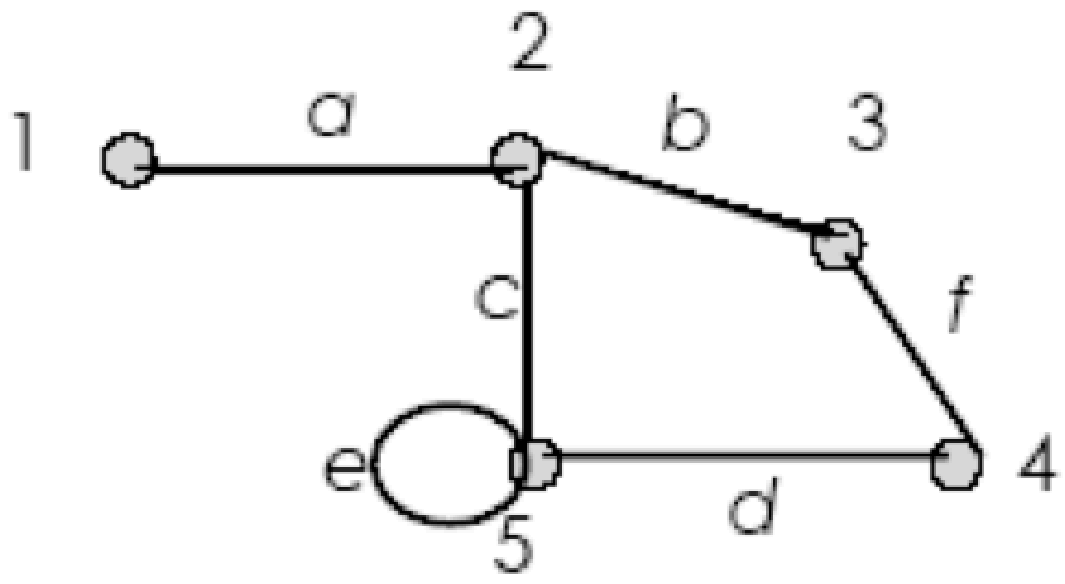


(1, *a*, 2, *c*, 5, *e*, 5, *d*, 4, *f*, 3, *g*, 2, *b*, 1)
is an Euler cycle

Euler Trail

Let G be a graph, and let v and w be two distinct vertices of G. An Euler trail from v to w is a sequence of adjacent vertices and edges that starts at v and ends at w, passes through every **vertex** of G at least once, and traverses every **edge** of G exactly once.

Example 2

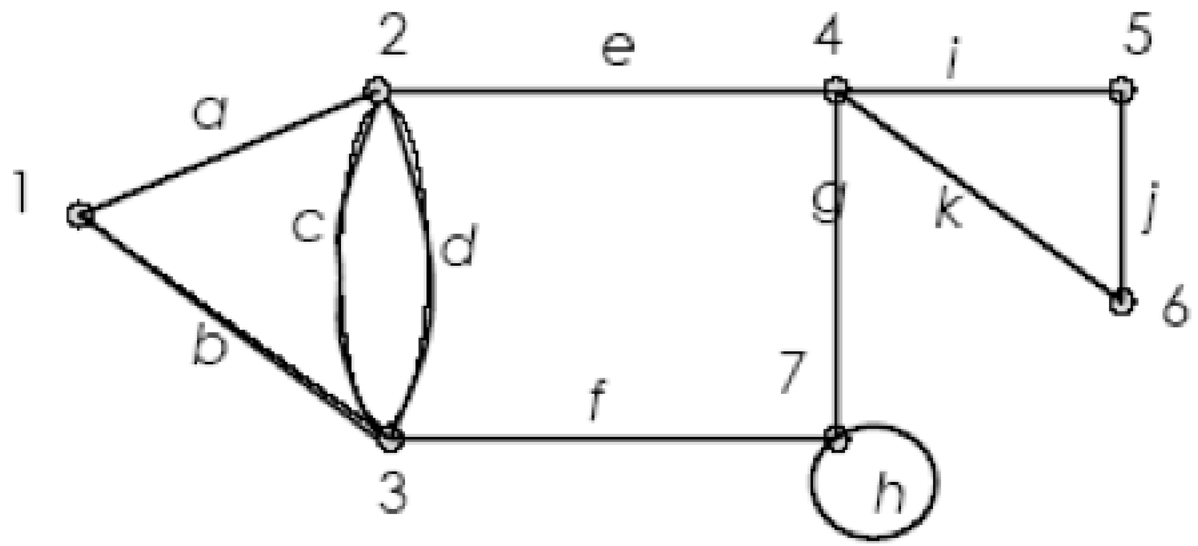


(1, a, 2, c, 5, e, 5, d, 4, f, 3, b, 2)
is an Euler trail

Theorem - Euler

- If G is a connected graph and every vertex has even degree, then G has an Euler circuit.
- A graph has an Euler trail from v to w ($v \neq w$) if and only if it is connected and v and w are the only vertices having odd degree.

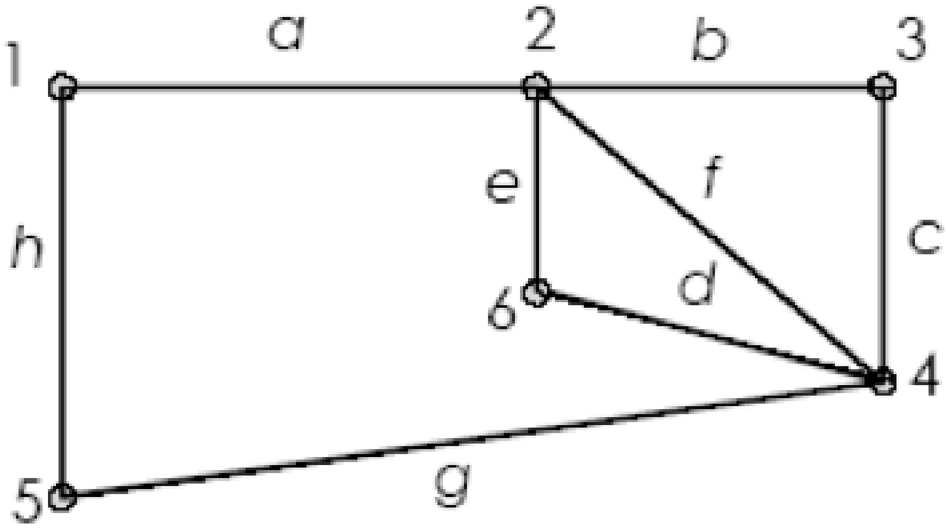
Example 3



This graph has an Euler cycle

Vertex	1	2	3	4	5	6	7
Degree	2	4	4	4	2	2	4

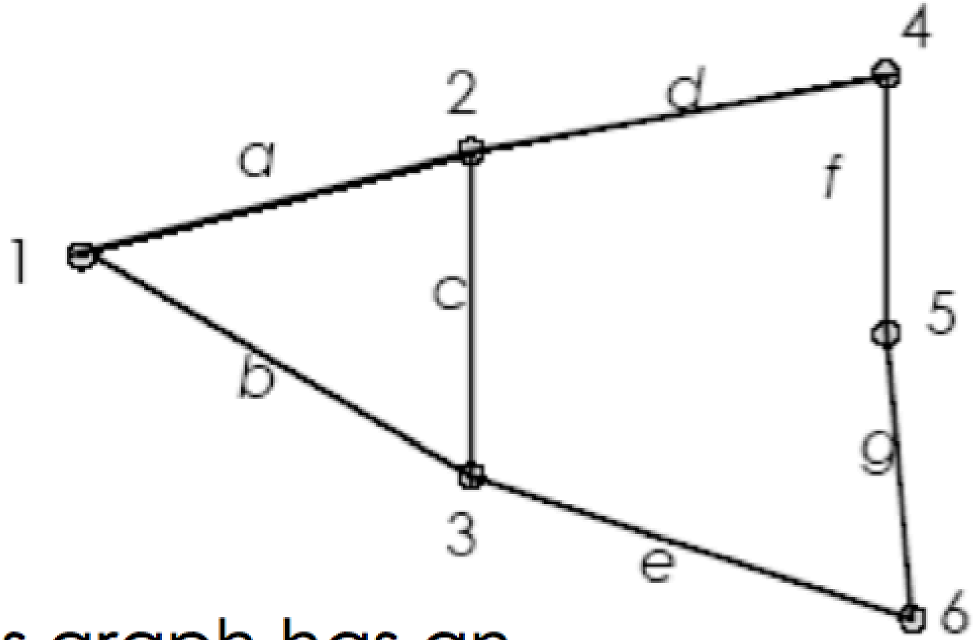
Example 4



This graph has an Euler cycle

Vertex	1	2	3	4	5	6
Degree	2	4	2	4	2	2

Example 5



This graph has an Euler trail.

Vertex	1	2	3	4	5	6
Degree	2	3	3	2	2	2

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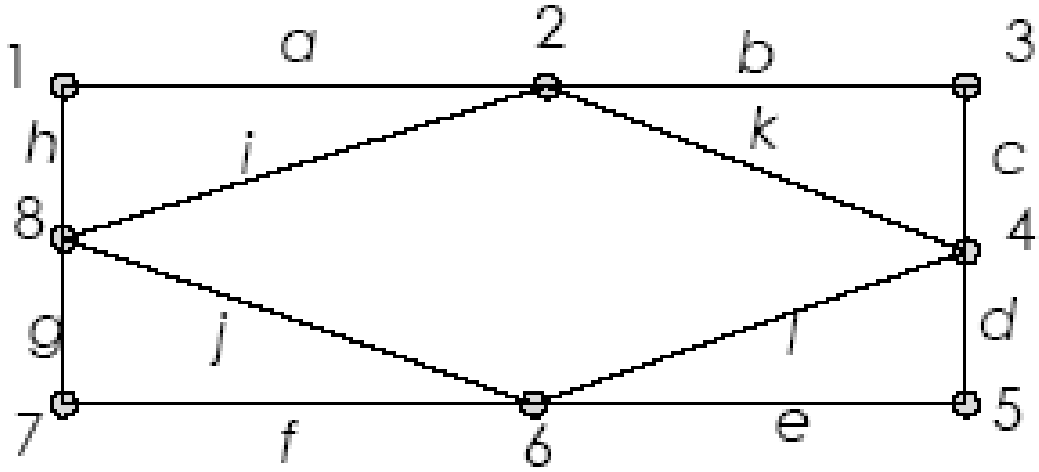
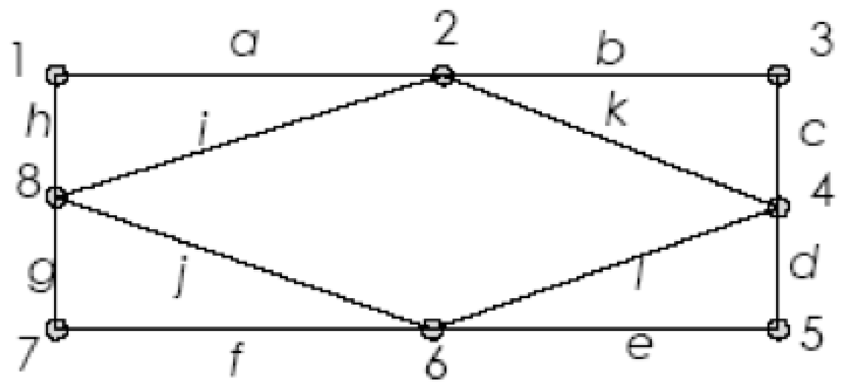
Exercise

- Decide whether the graph has an Euler cycle. If the graph has an Euler cycle, exhibit one.



Exercise - Solution



Vertex	1	2	3	4	5	6	7	8
Degree	2	4	2	4	2	2	2	4

- This graph has an Euler cycle because all the vertices has an even degree.
- The cycle is: [1,a,2,i,8,j,6,l,4,k,2,b,3,c,4,d,5,e,6,f,7,g,8,h,1]

Hamilton Circuits

Hamiltonian Circuits

Given a graph G, a Hamiltonian circuit for G is a simple circuit that includes every vertex of G (but doesn't need to include all edges). That is, a Hamiltonian circuit for G is a sequence of adjacent vertices and distinct edges in which every vertex of G appears exactly once, except for the first and the last, which are the same.

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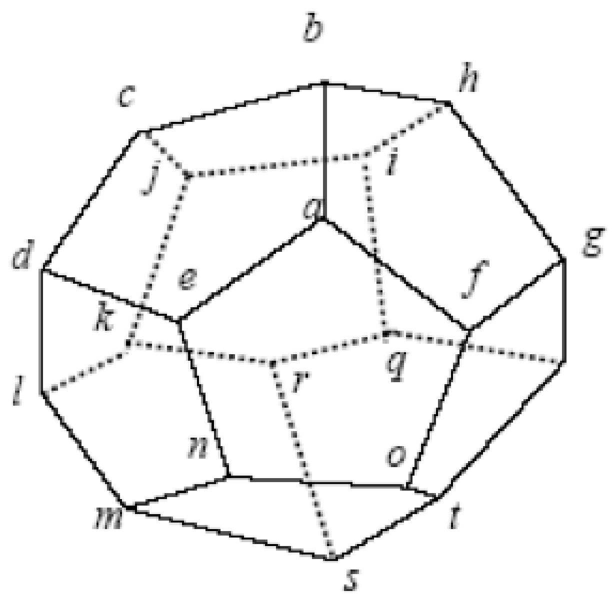
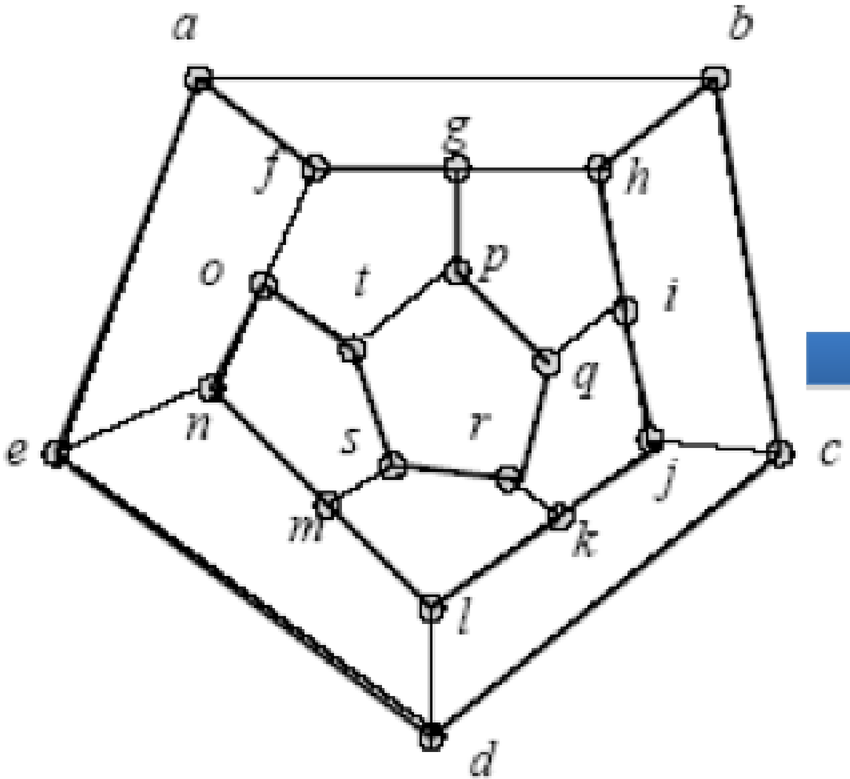
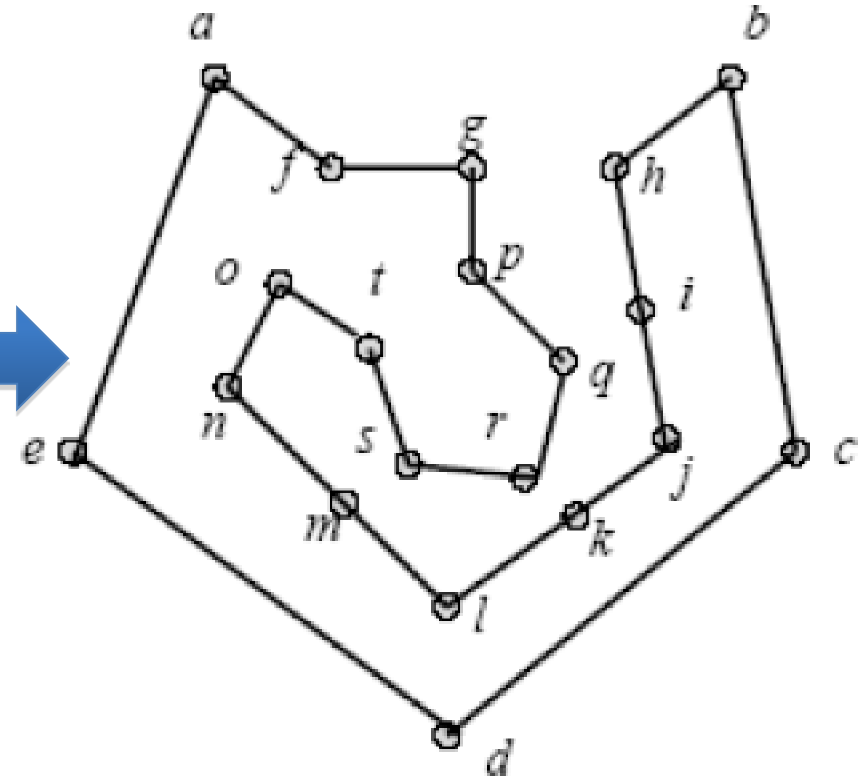
Example 1

- Sir William Rowan Hamilton marketed a puzzle in the mid-1800s in the form of dedocahedron.
- Each corner bore the name of a city.
- The problem was to start at any city, travel along the edges, visit each city exactly one time and return to the initial city.

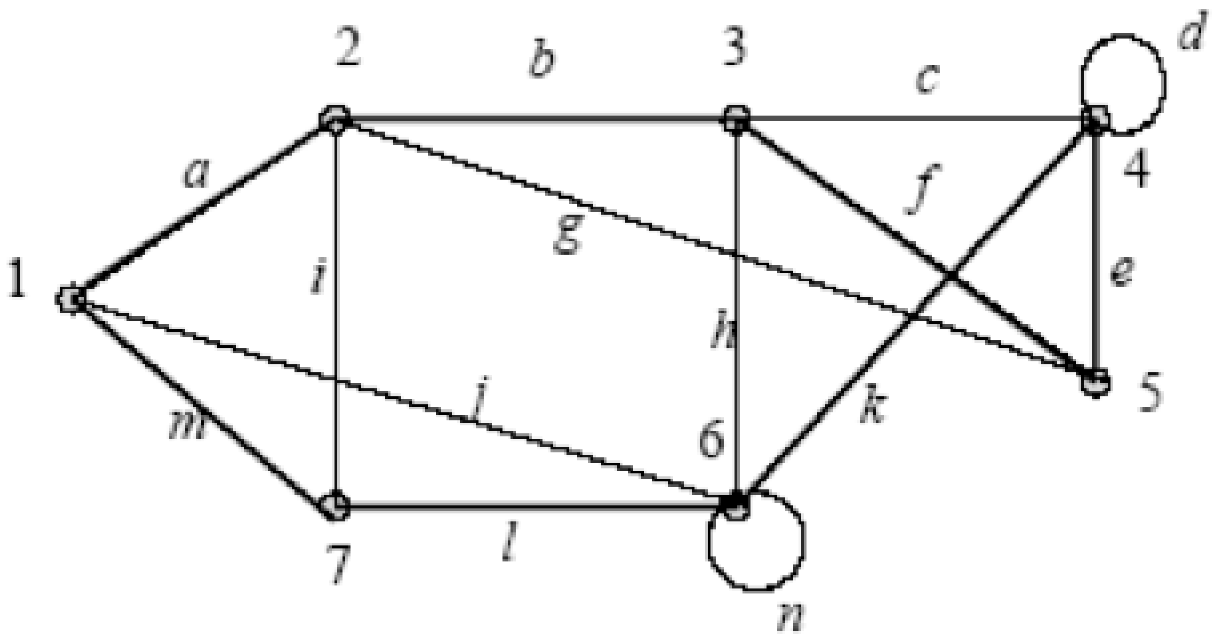






(a): The graph (b): Hamilton circuit

Example 2



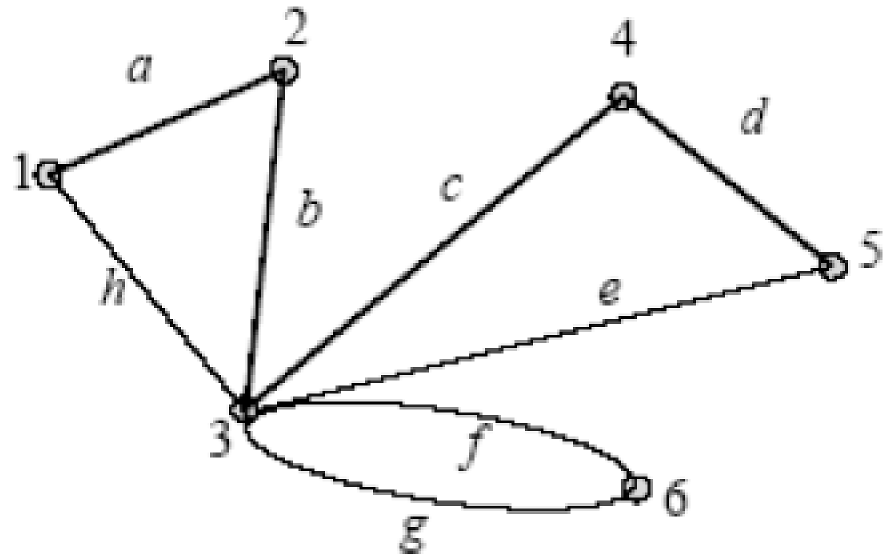
This graph has a Hamilton circuit.

(1, a, 2, b, 3, f, 5, e, 4, k, 6, l, 7, m, 1)

- Visit each vertex just once.

Example 3

This graph does not contain Hamilton circuit.



(1, a, 2, b, 3, g, 6, f, 3, e, 5, d, 4, c, 3, h, 1)

- Vertex (3) has to be visited more than once.

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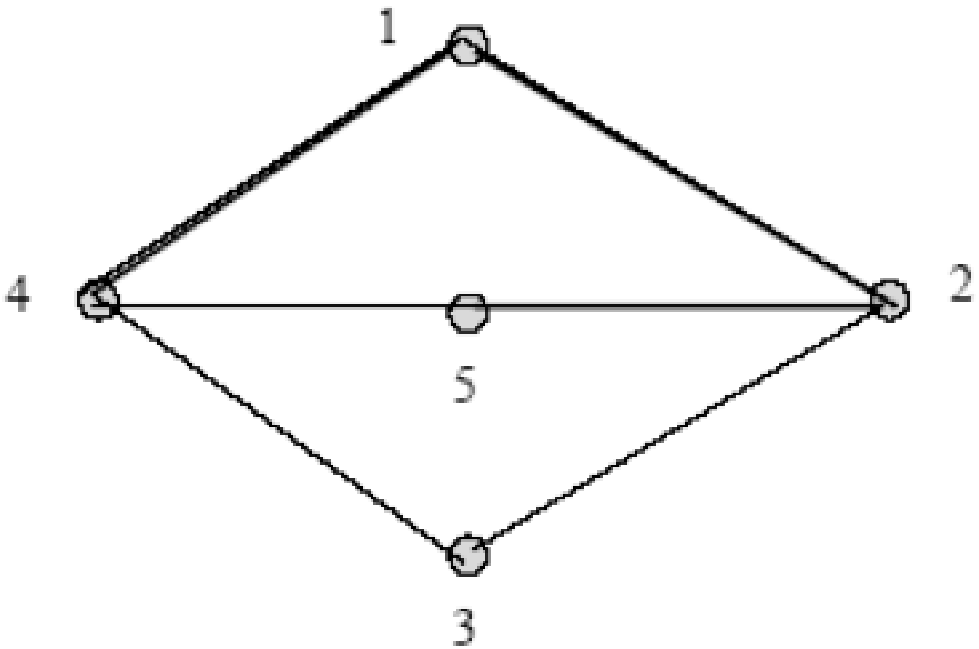
Example 4

Question: Is this graph has Hamiltonian cycle?



Solution: (1, 4, 3, 2, 5, 4, 1)

- Vertex (4) has to be visited more than once.

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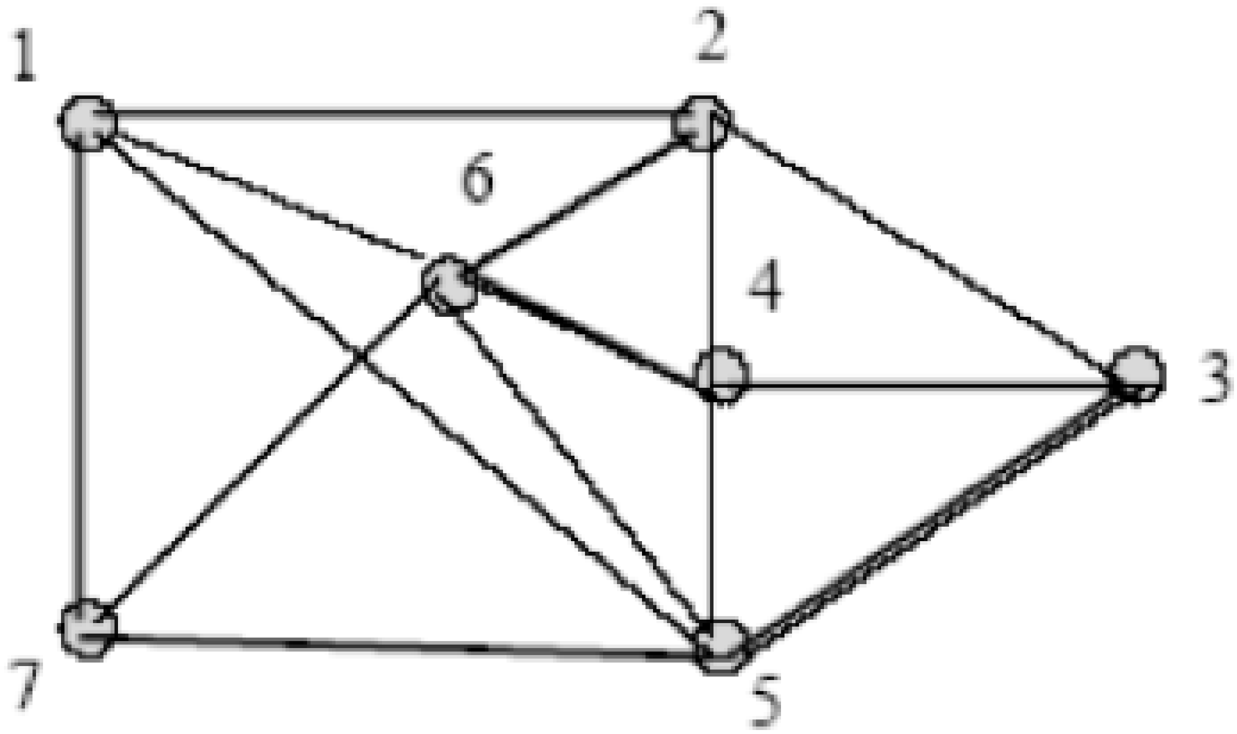
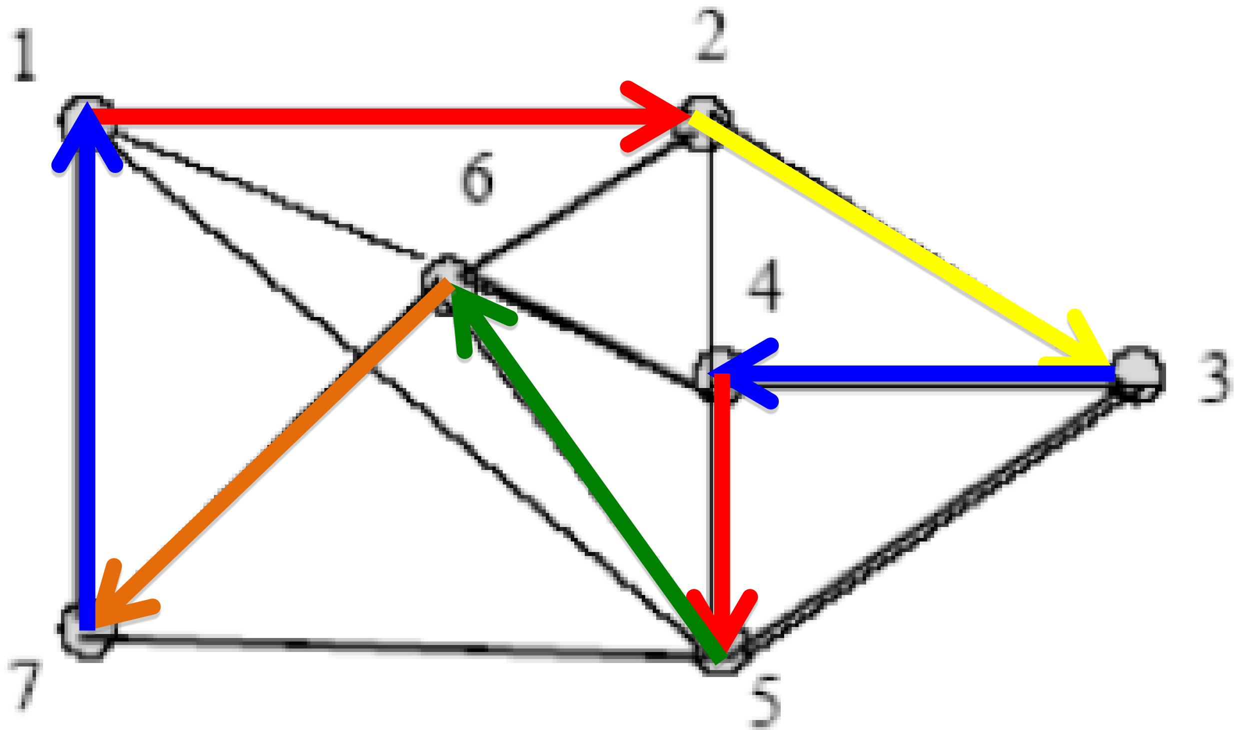
Exercise

Question: Is this graph has Hamiltonian cycle?



Exercise - Solution

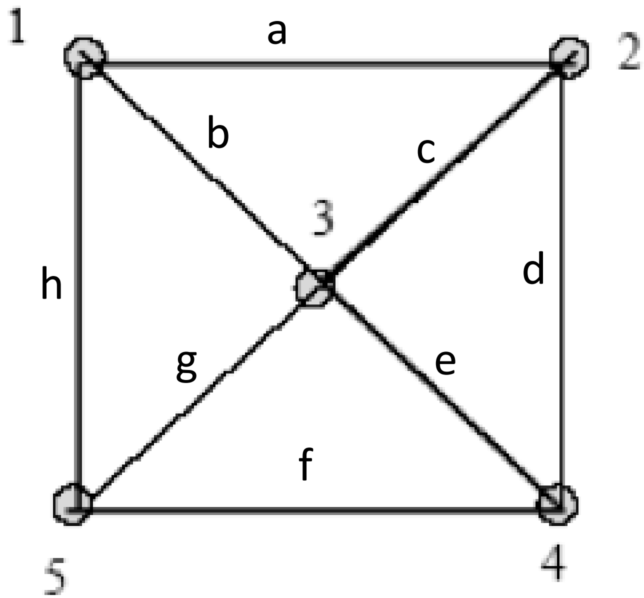
Manually check: This graph has Hamiltonian cycle.



Each vertex has to be visited only once.
For example: (1, 2, 3, 4, 5, 6, 7, 1)

Exercise

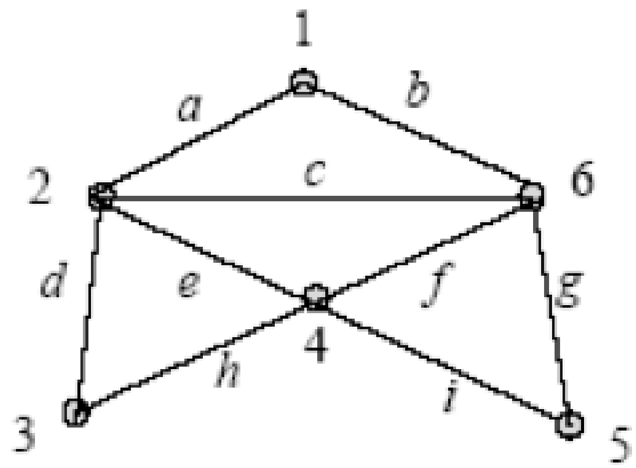
Question: Prove that this graph has Hamiltonian cycle.



Solution: (1, b, 3, c, 2, d, 4, f, 5, h, 1)

Exercise

- Find a Hamiltonian cycle in this graph.



Solution: (1, b, 6, g, 5, i, 4, h, 3, d, 2, a, 1)

Shortest Path Problem

Shortest Path

- Let ***G*** be a weighted graph.
- Let ***u*** and ***v*** be two vertices in ***G***, and let ***P*** be a path in ***G*** from ***u*** to ***v***.
- The length of path ***P***, written ***L*(*P*)**, is the sum of the weights of all the edges on path ***P***.
- A **shortest path** from a vertex to another vertex is a path with the shortest length between the vertices.

Dijkstra’s Shortest Path Algorithm

http://en.wikipedia.org/wiki/Dijkstra%27s_algorithm

- 1. $S := \emptyset$
- 2. $N := V$
- 3. For all vertices, $u \in V$, $u \neq a$, $L(u) := \infty$
- 4. $L(a) := 0$

Dijkstra's Shortest Path Algorithm (cont.)

- 5. While $z \notin S$ do,

  5.a :Let $v \in N$ be such that

  $L(v)=\min\{L(u) \mid u \in N\}$

  5.b : $S := S \in \{v\}$

  5.c : $N := N - \{v\}$

  5.d : For all $w \in N$ such that there is an edge

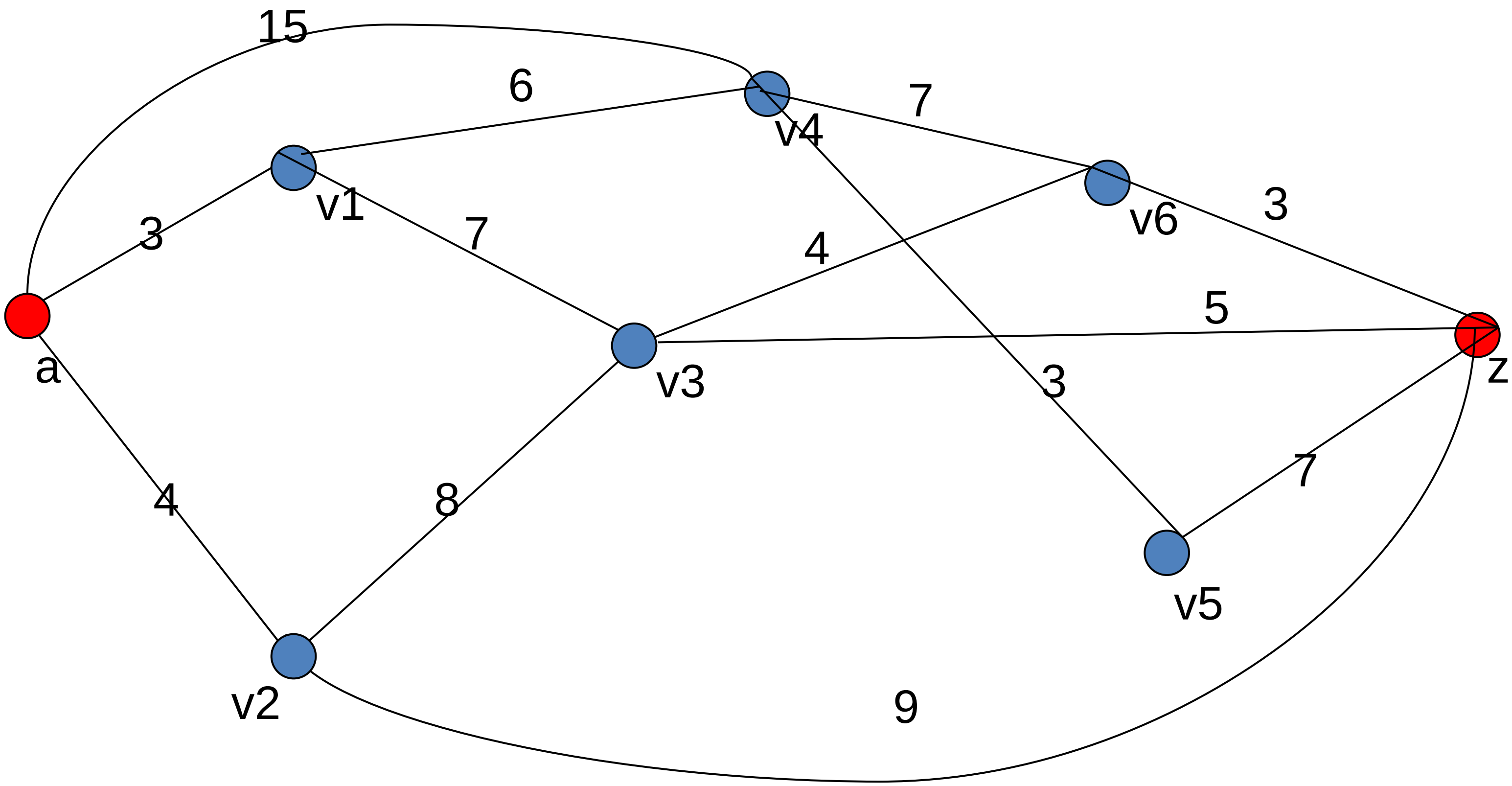
  from v to w

  5.d.1: If $L(v)+W[v,w] < L(w)$ then

  $L(w)=L(v)+W[v,w]$

Example

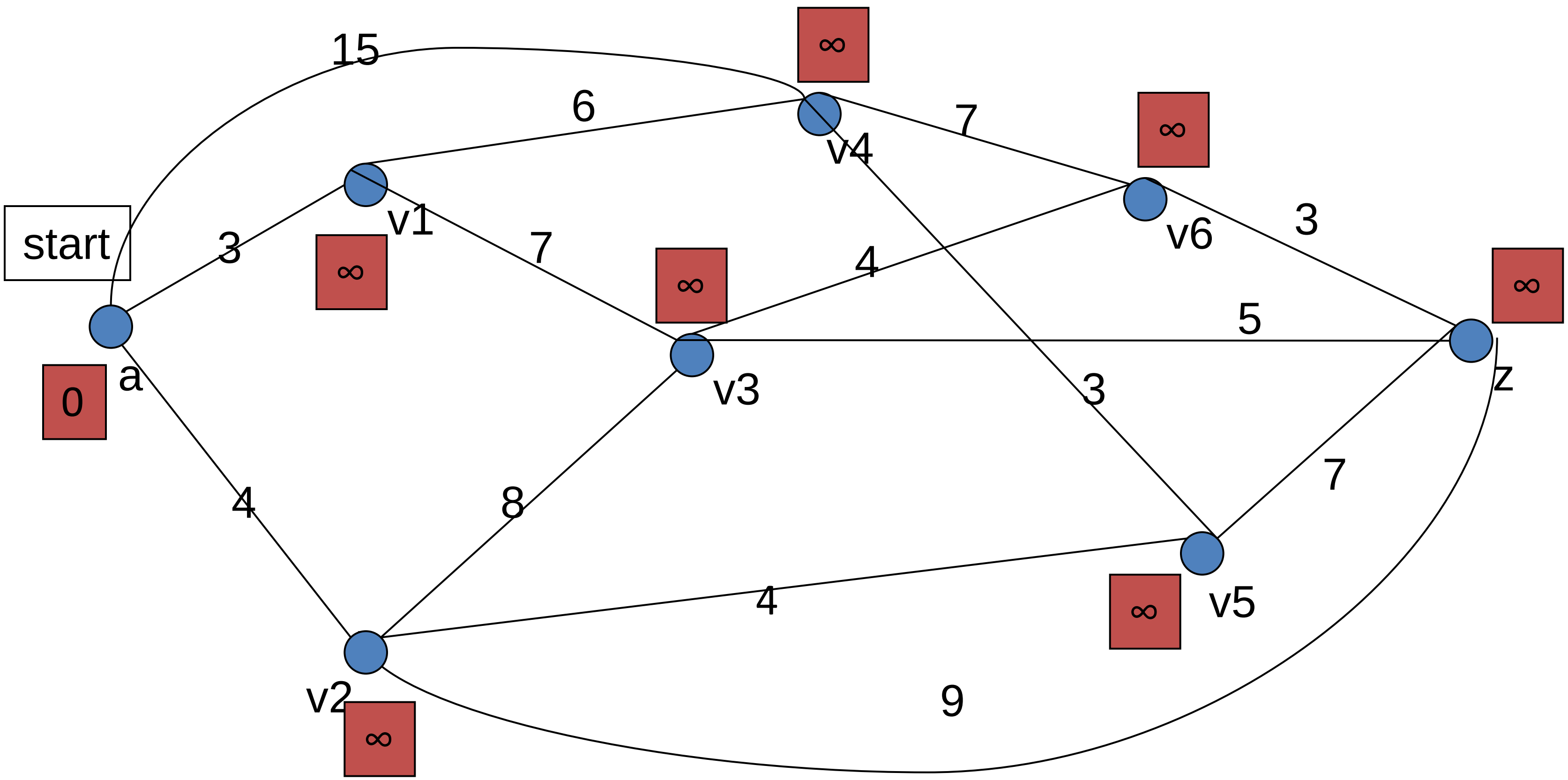
What is the shortest path from **a** to **z**?



Vertex	a	v1	v2	v3	v4	v5	v6	z

$S=\emptyset$

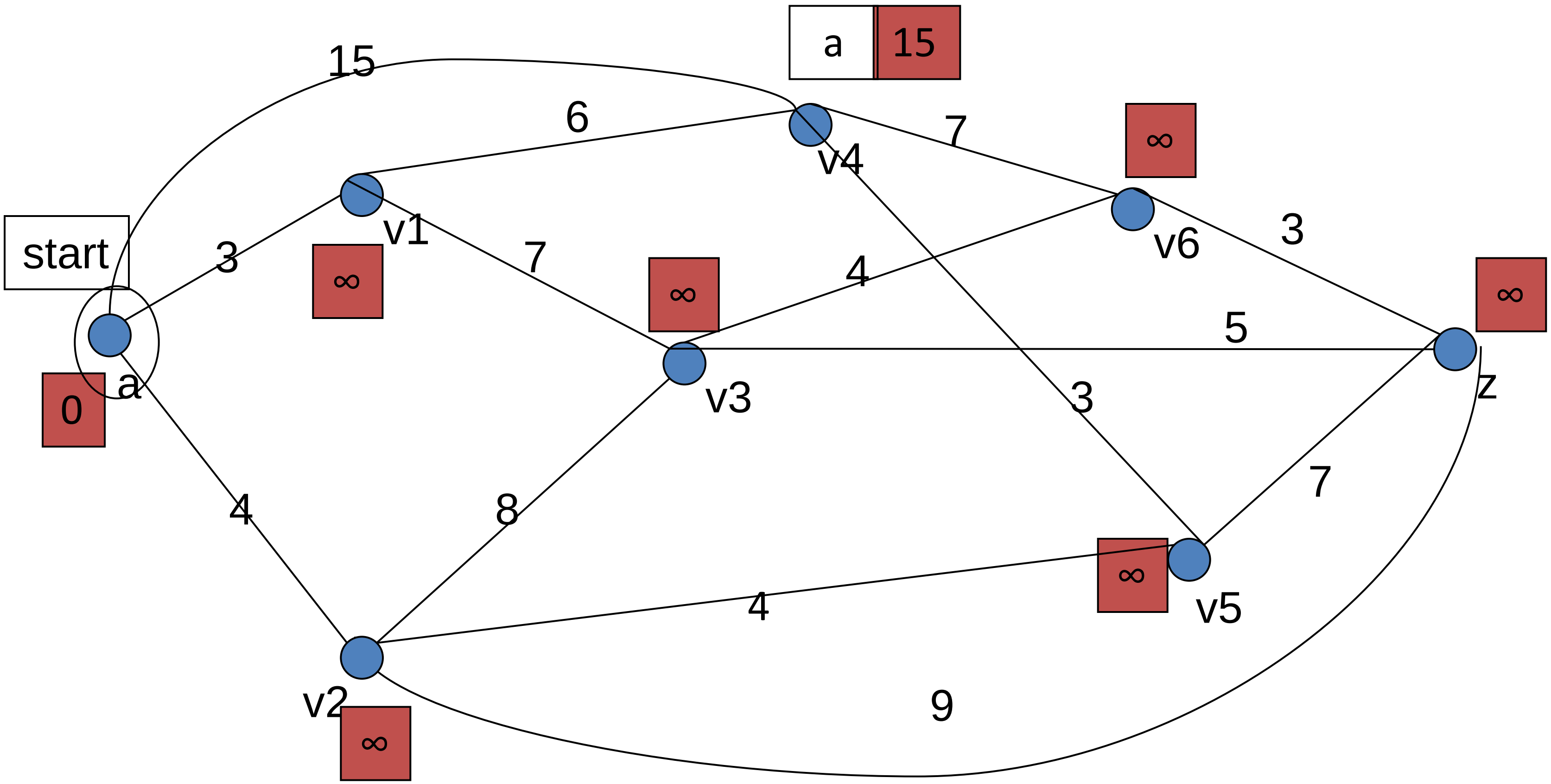
$N=\{a,v1,v2,v3,v4,v5,v6,z\}$



$S=\{a\}$

$N=\{v1,v2,v3,v4,v5,v6,z\}$

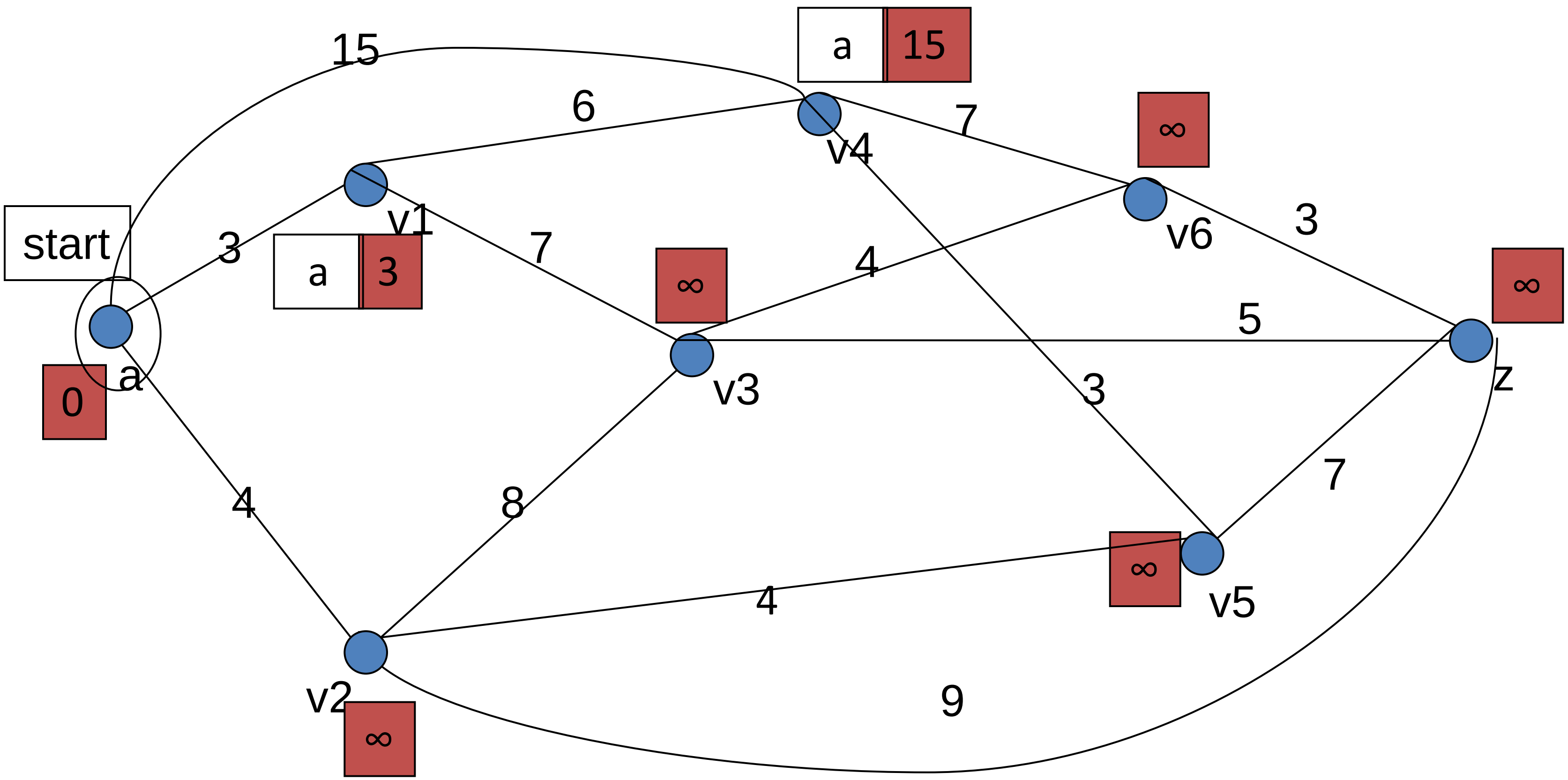
$L(a)+W[a,v4] < L(v4)$
$0+15=15 < \infty$
$L(v4)=15$



$S=\{a\}$

$N= \{v1,v2,v3,v4,v5,v6,z\}$

$L(a)+W[a,v1] < L(v1)$
$0+3= 3 < \infty$
$L(v1)=3$



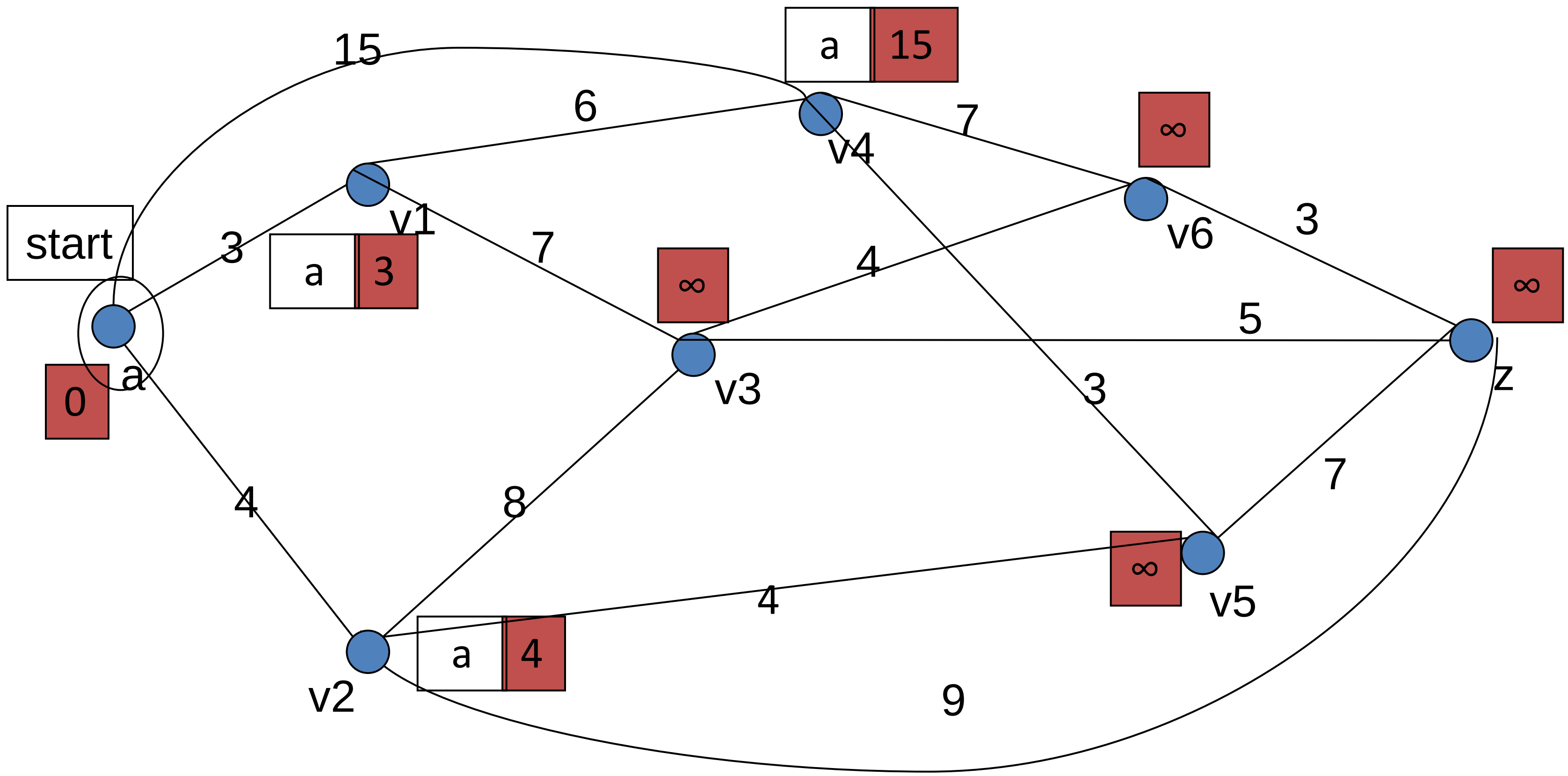
$S=\{a\}$

$N=\{v1,v2,v3,v4,v5,v6,z\}$

$L(a)+W[a,v2] < L(v2)$

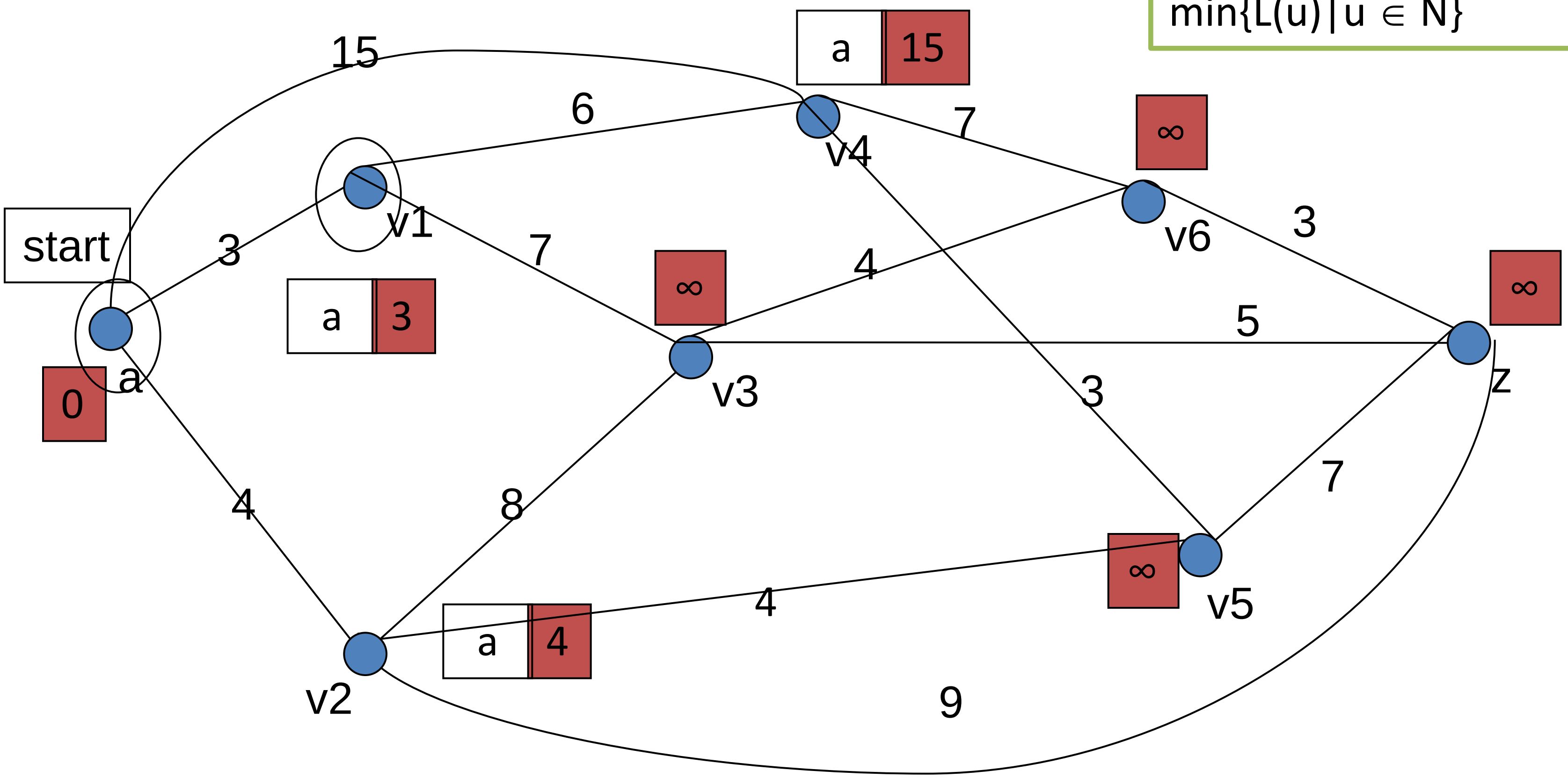
$0+4= 4 < \infty$

$L(v2)=4$



$S=\{a\}$

$N=\{v1,v2,v3,v4,v5,v6,z\}$

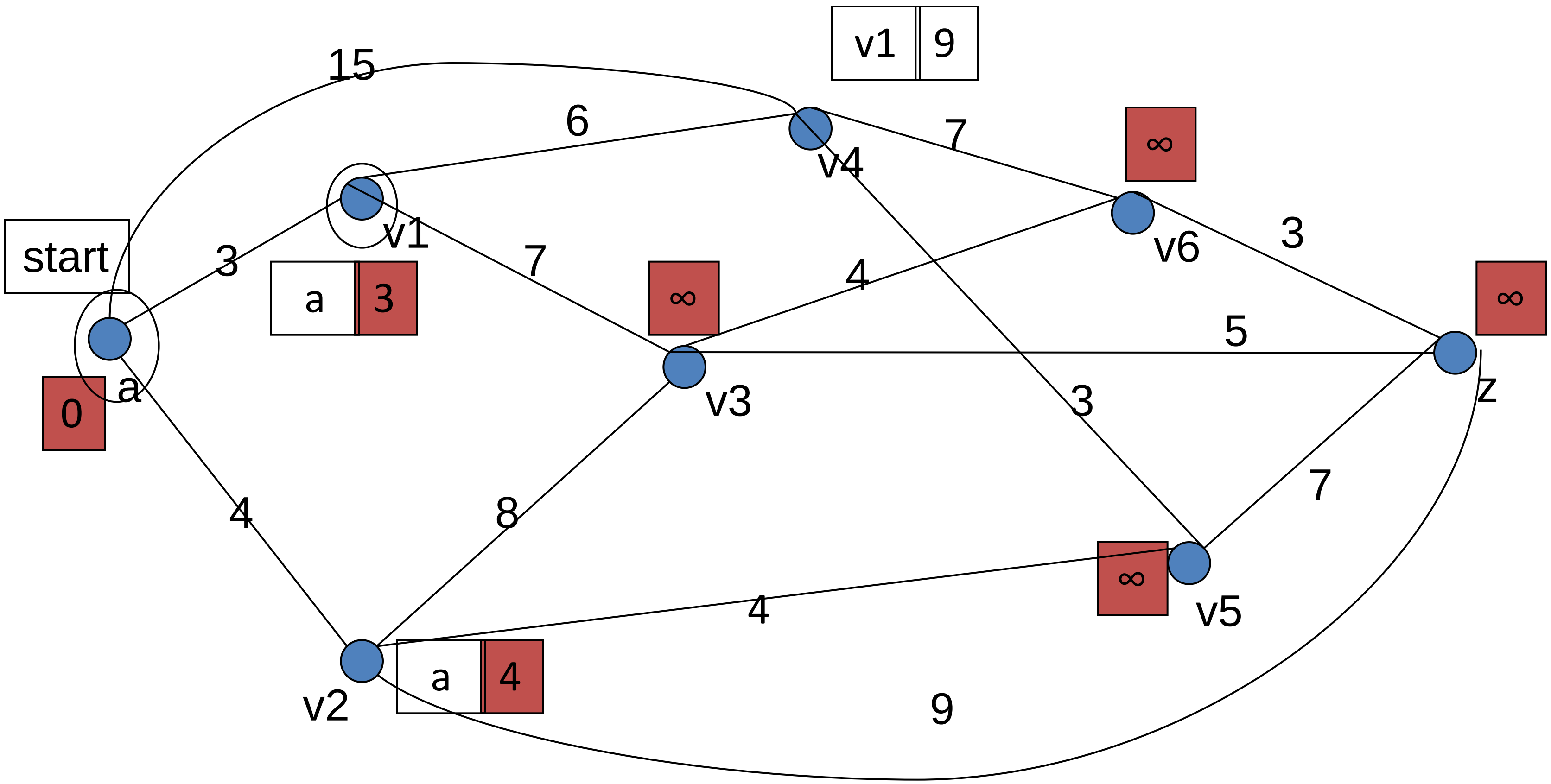


choose $v1$
because
$L(v1)=3 =$
$\min\{L(u) | u \in N\}$

S={*a*, *v1*}

N= {*v2*,*v3*,*v4*,*v5*,*v6*,*z*}

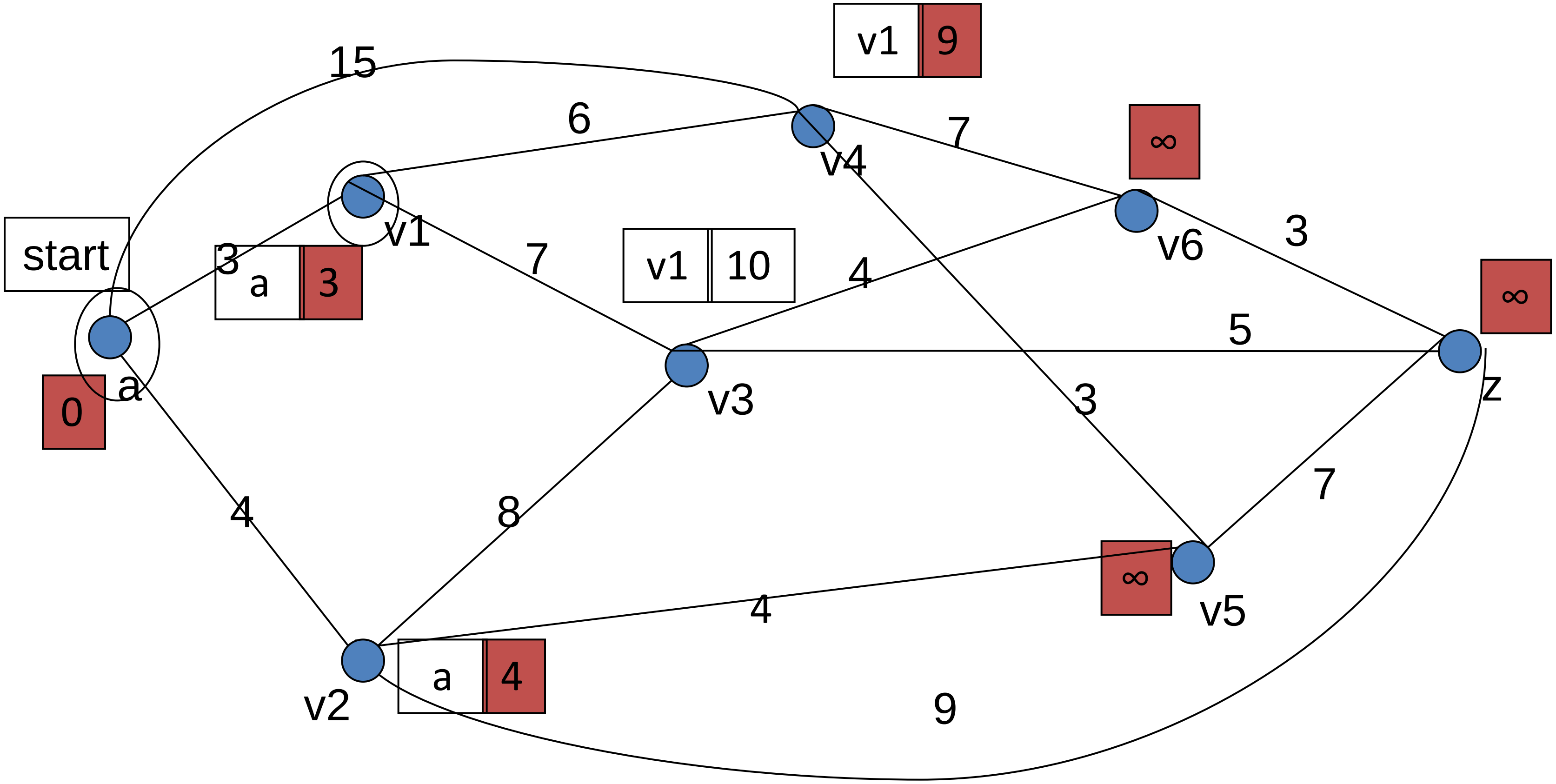
L(*v*1)+*W*[*v*1,*v*4]<*L*(*v*4)
3+6= 9 < 15
L(*v*4)=9



S={a, v1}

N= {v2,v3,v4,v5,v6,z}

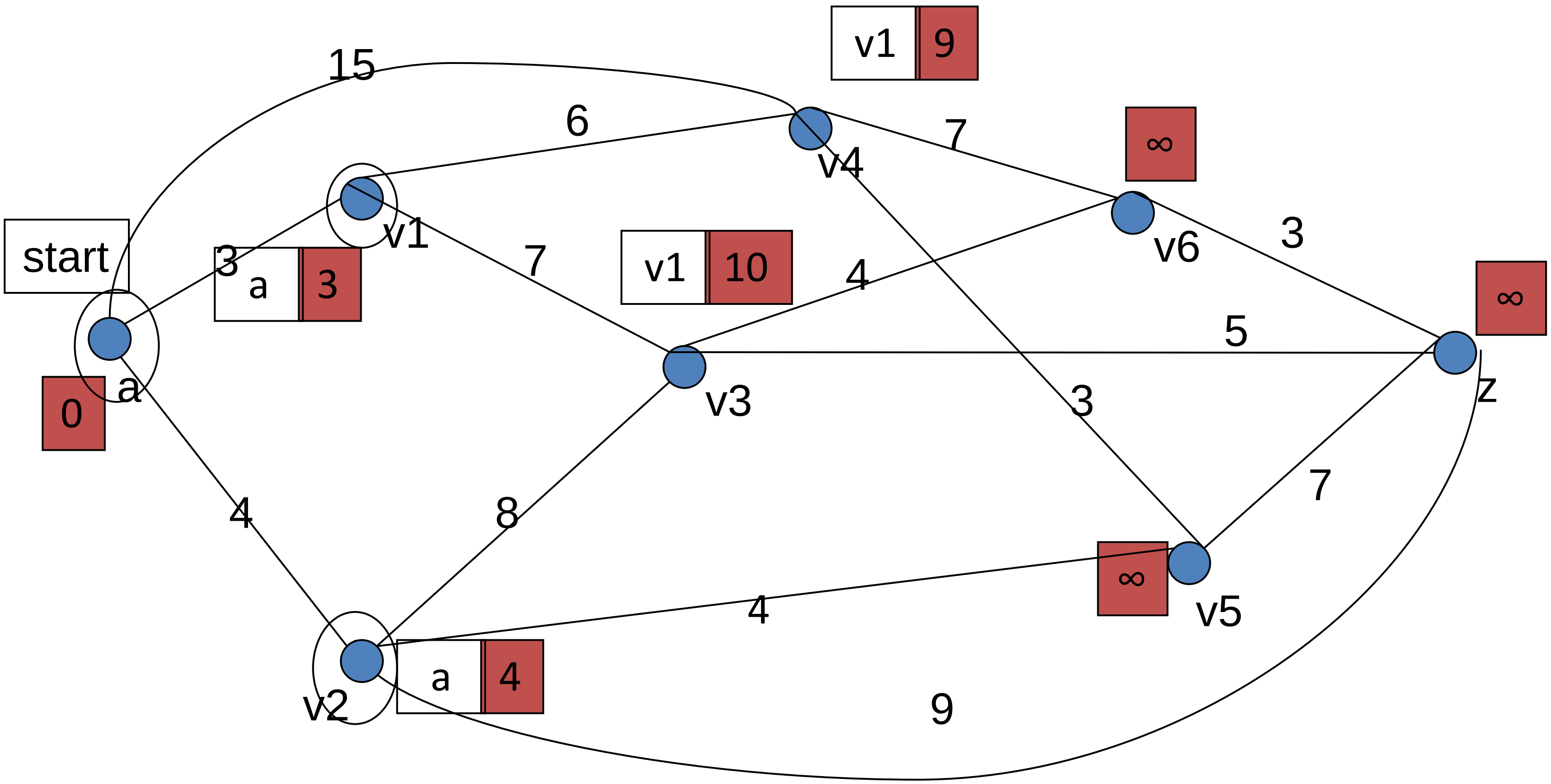
L(*v*1)+*W*[*v*1,*v*3]<*L*(*v*3)
3+7= 10 < ∞
L(*v*4)=10



$S=\{a, v1\}$

$N= \{v2,v3,v4,v5,v6,z\}$

choose *v2*
because
$L(v2)=4 = \min\{L(u) | u \in N\}$



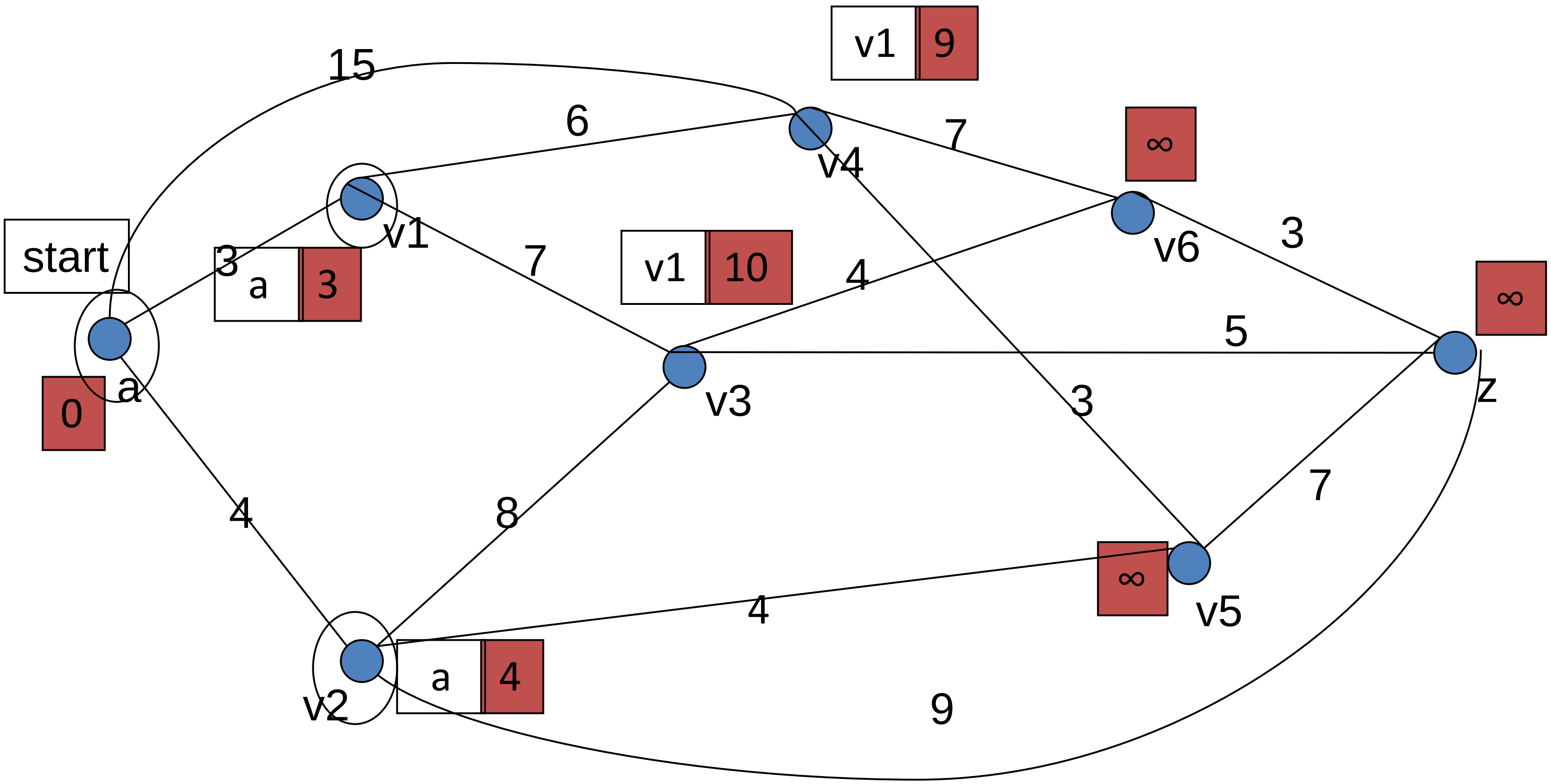
$S=\{a, v1, v2\}$

$N= \{v3,v4,v5,v6,z\}$

$L(v2)+W[v2,v3]<L(v3)$

4+8= 12 > 10

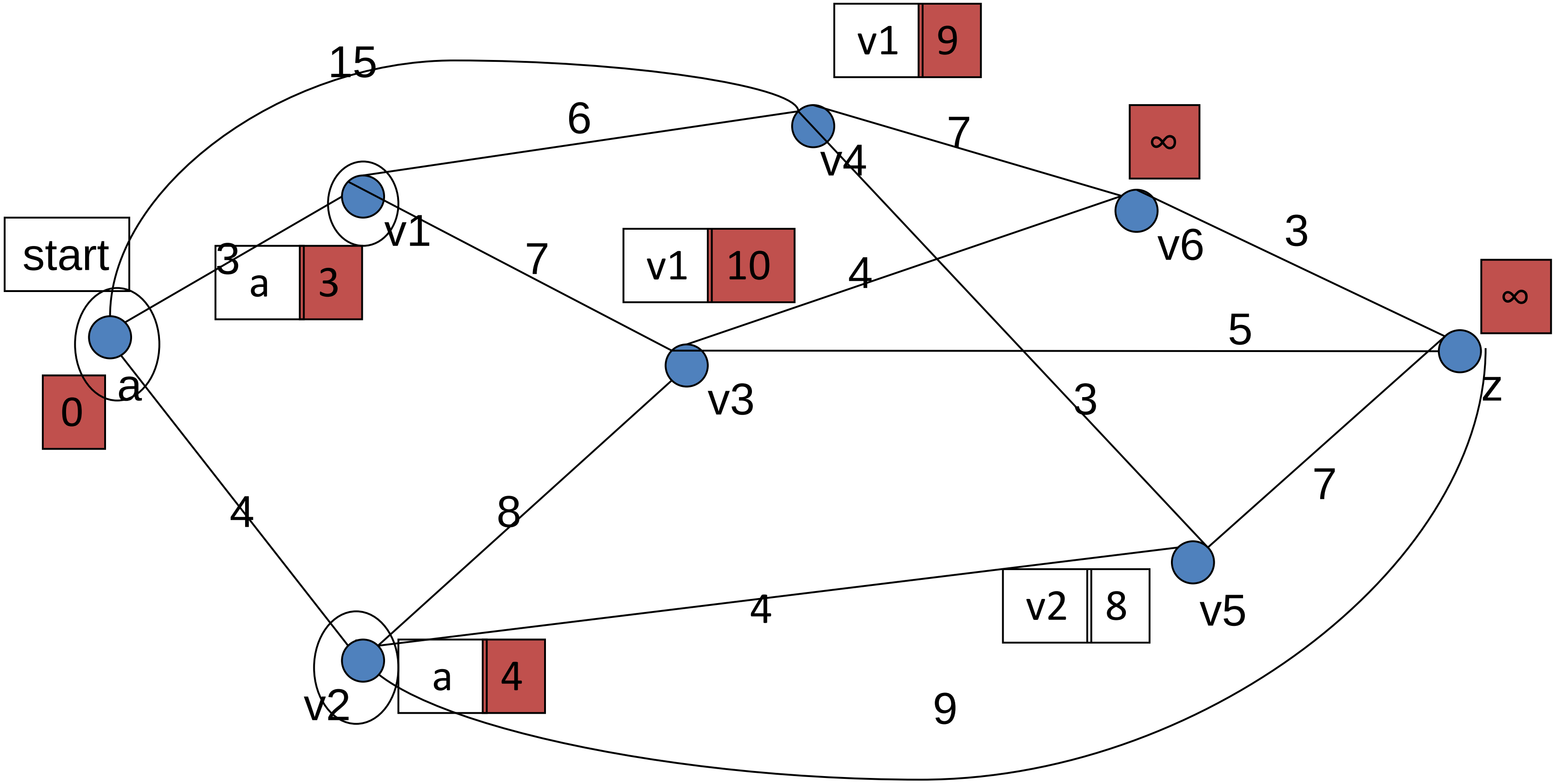
$L(v3)$ remains the same.



$S=\{a, v1, v2\}$

$N=\{v3,v4,v5,v6,z\}$

$L(v2)+W[v2,v5]<L(v5)$
$4+4= 8 < \infty$
$L(v5)=8$



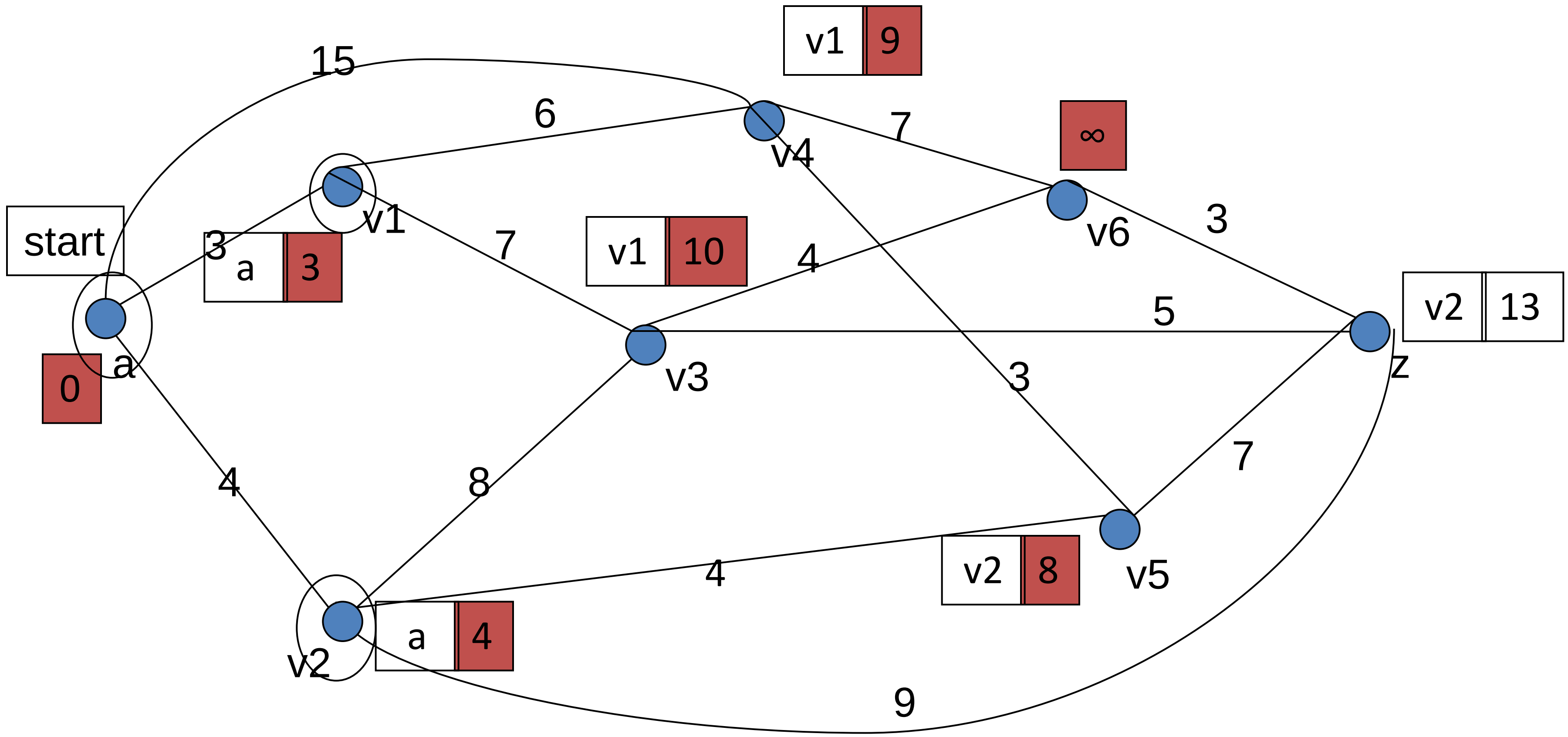
$S=\{a, v1, v2\}$

$N= \{v3,v4,v5,v6,z\}$

$L(v2)+W[v2, z]<L(z)$

$4+9= 13 < \infty$

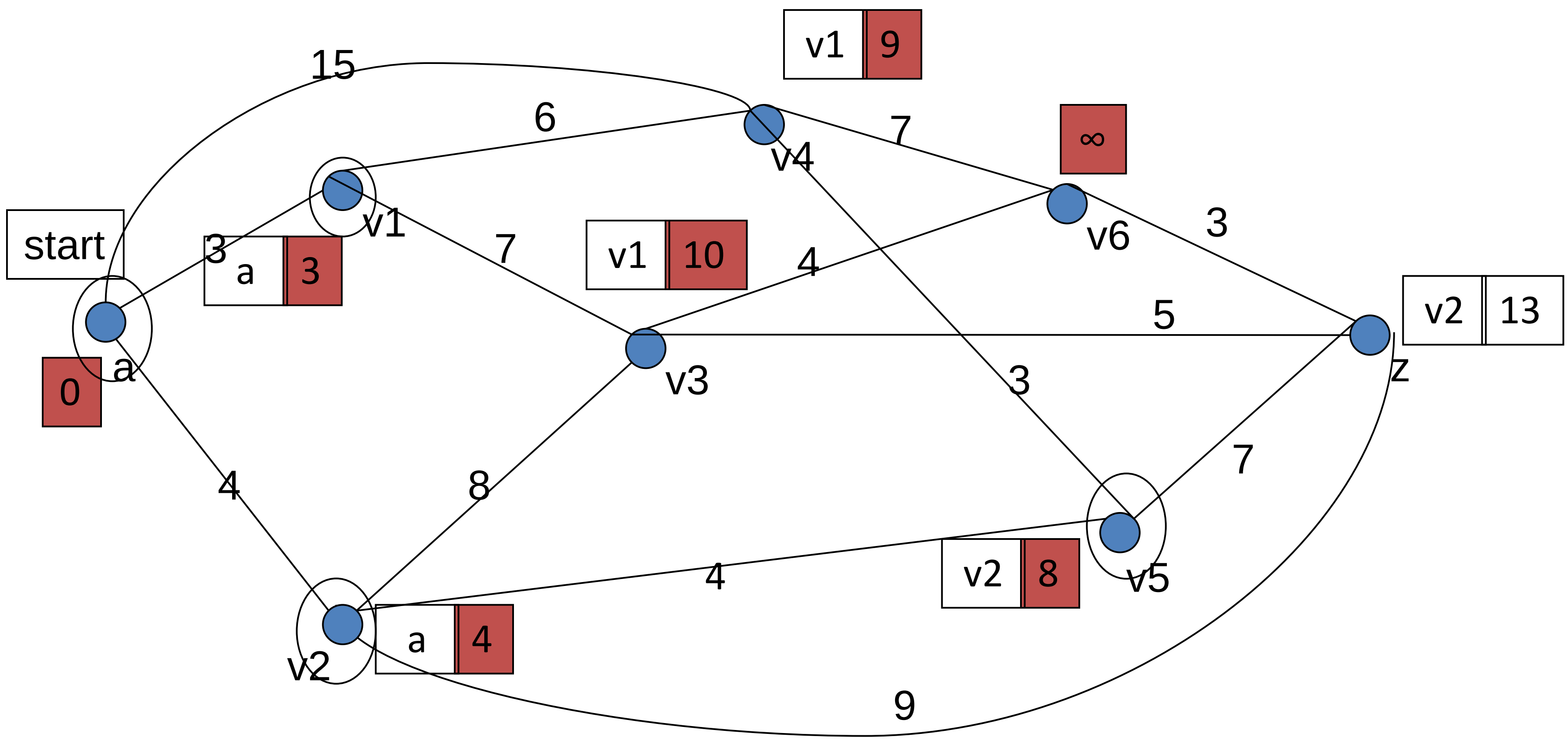
$L(z)=13$



S={*a*, *v1*, *v2*}

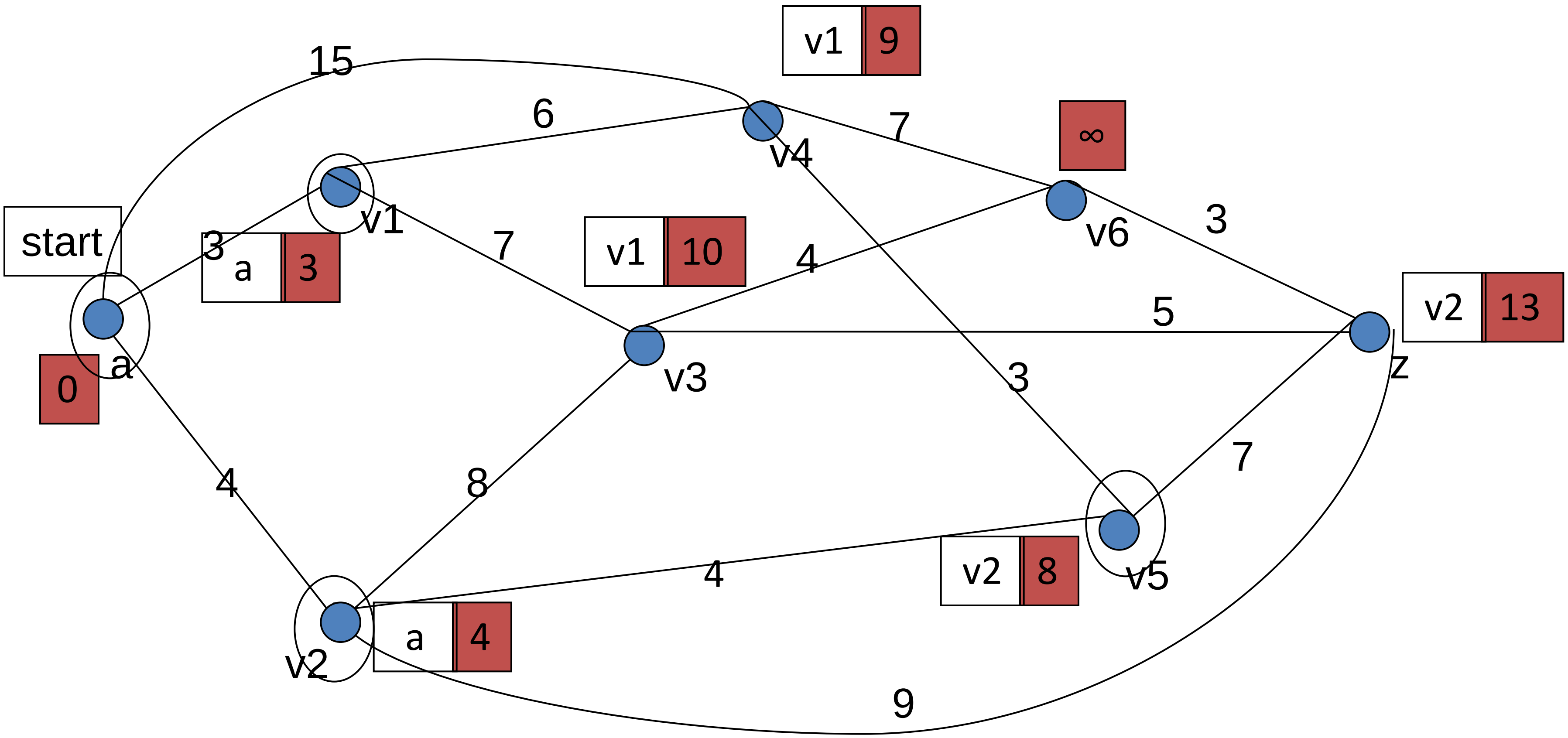
N= {*v3*,*v4*,*v5*,*v6*,*z*}

choose *v*5
because
L(*v*5)=8 = min{L(u)|u ∈ N}



S={a, v1, v2, v5}

N= {v3,v4, v6,z}



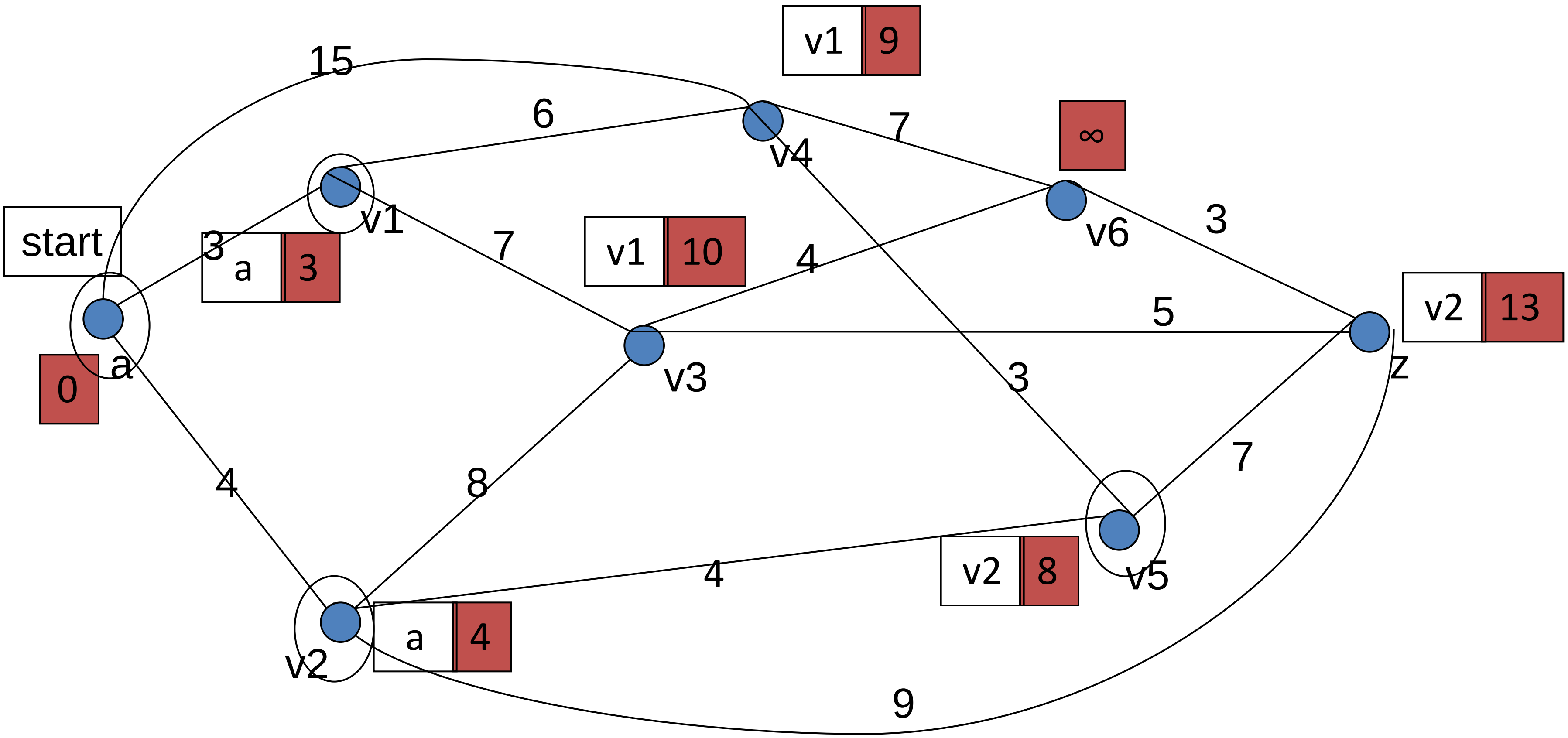
$S=\{a, v1, v2, v5\}$

$N= \{v3, v4, v6, z\}$

$L(v5)+W[v5, v4]<L(v4)$

8+3= 11 > 9

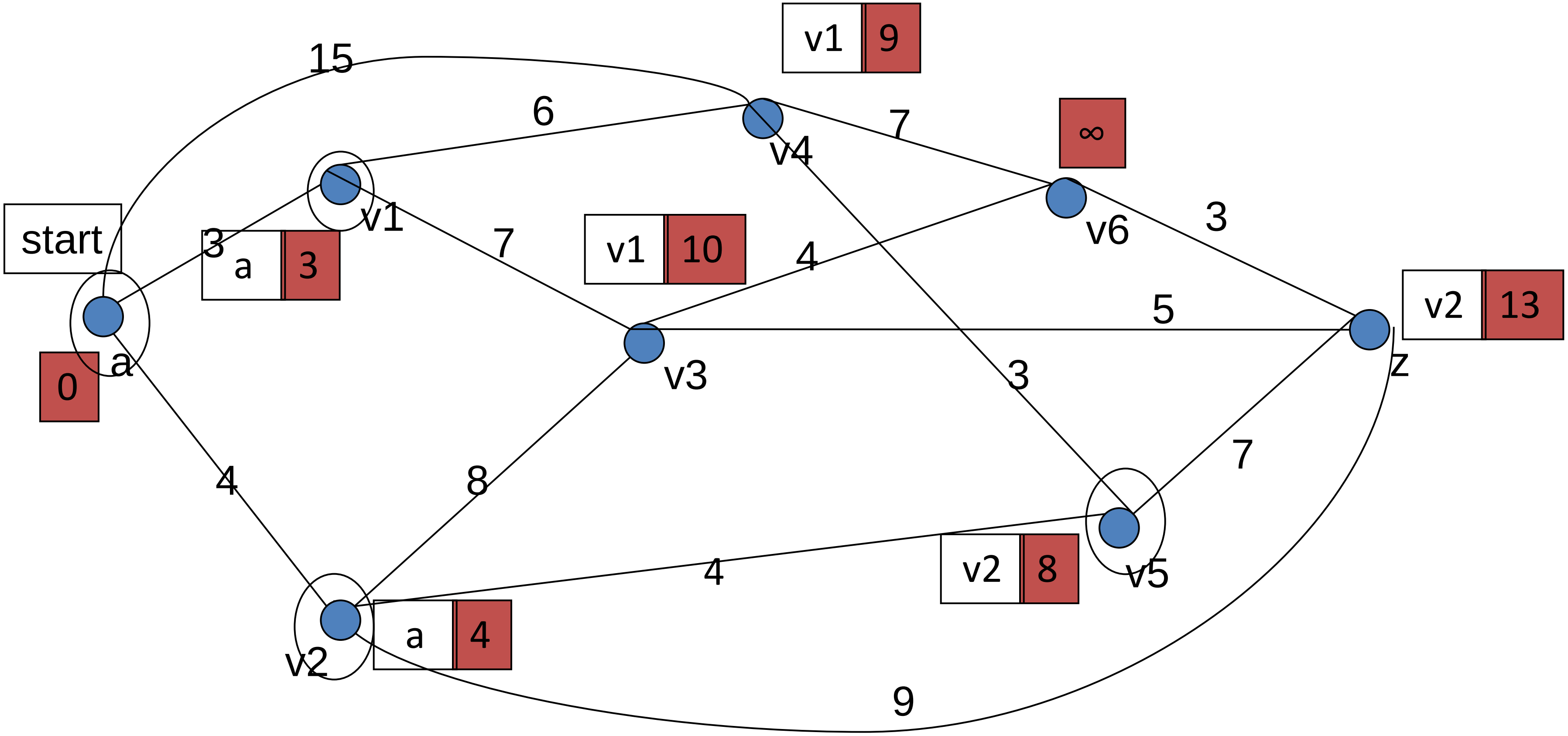
$L(v4)$ remains the same



$S=\{a, v1, v2, v5\}$

$N= \{v3, v4, v6, z\}$

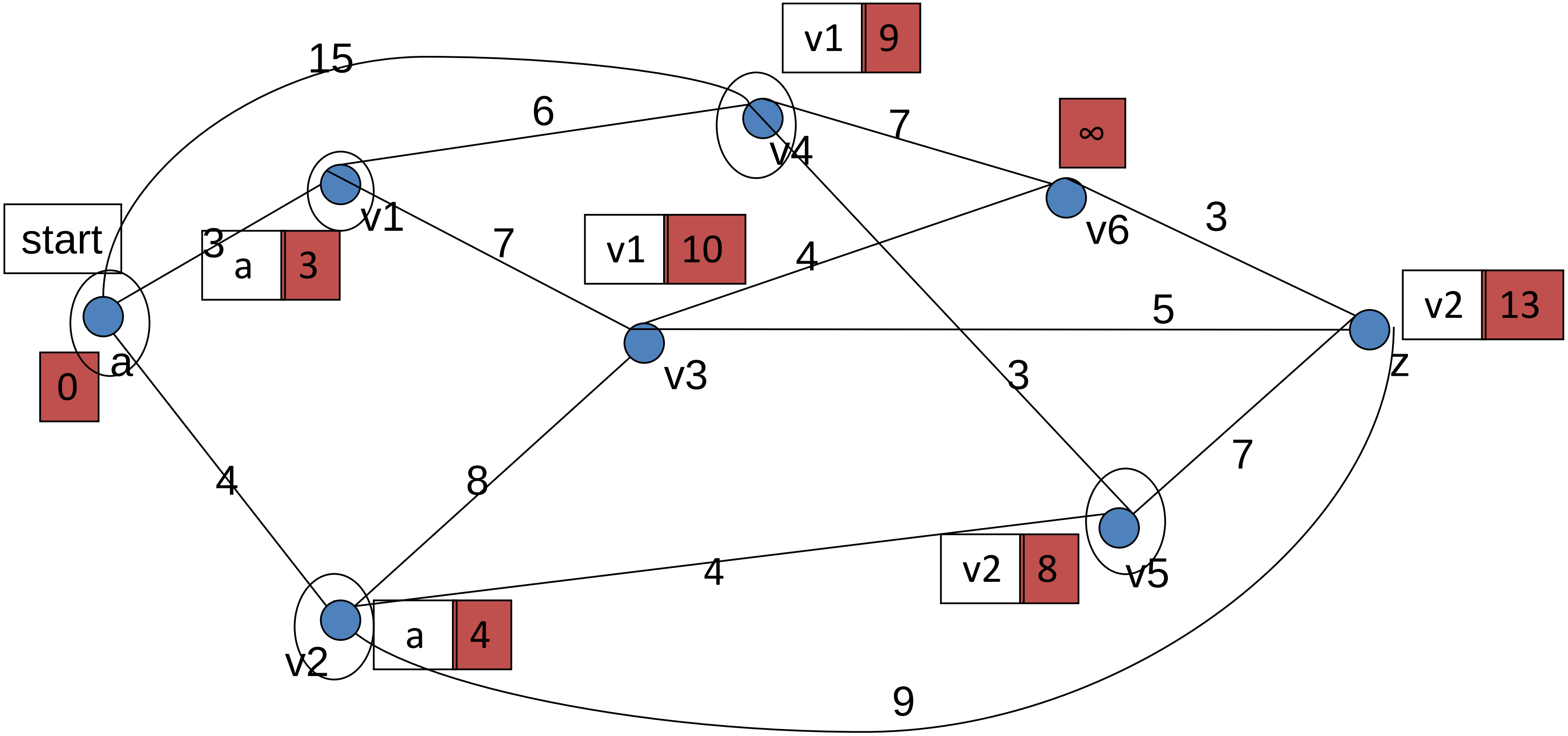
$L(v5)+W[v5, z] < L(z)$
8+7= 15 > 13
$L(z)$ remains the same



S={a, v1, v2, v5}

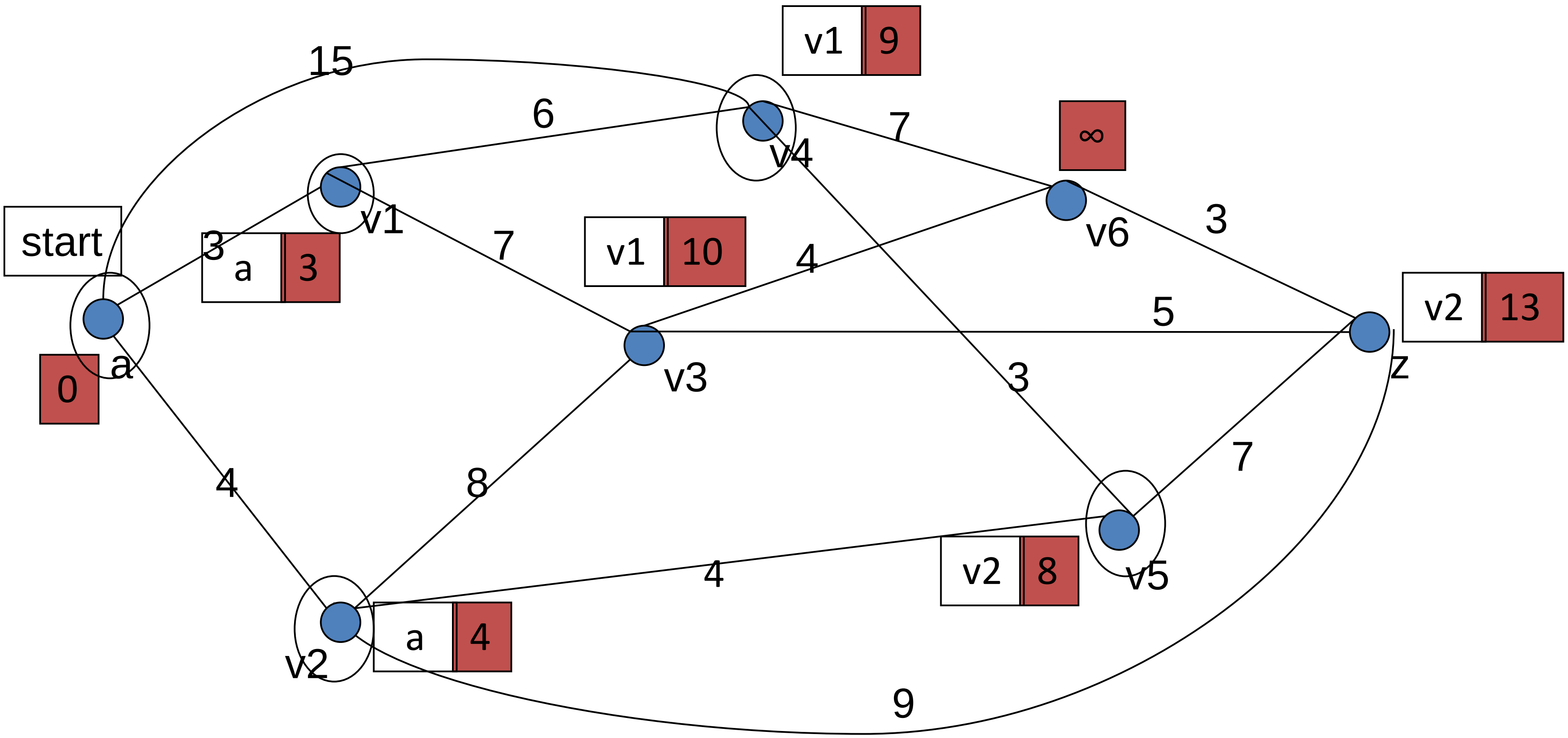
N= {v3,v4, v6,z}

choose *v4*
because
$L(v4)=9 = \min\{L(u) \mid u \in N\}$



S={a, v1, v2, v5}

N= {v3,v4, v6,z}



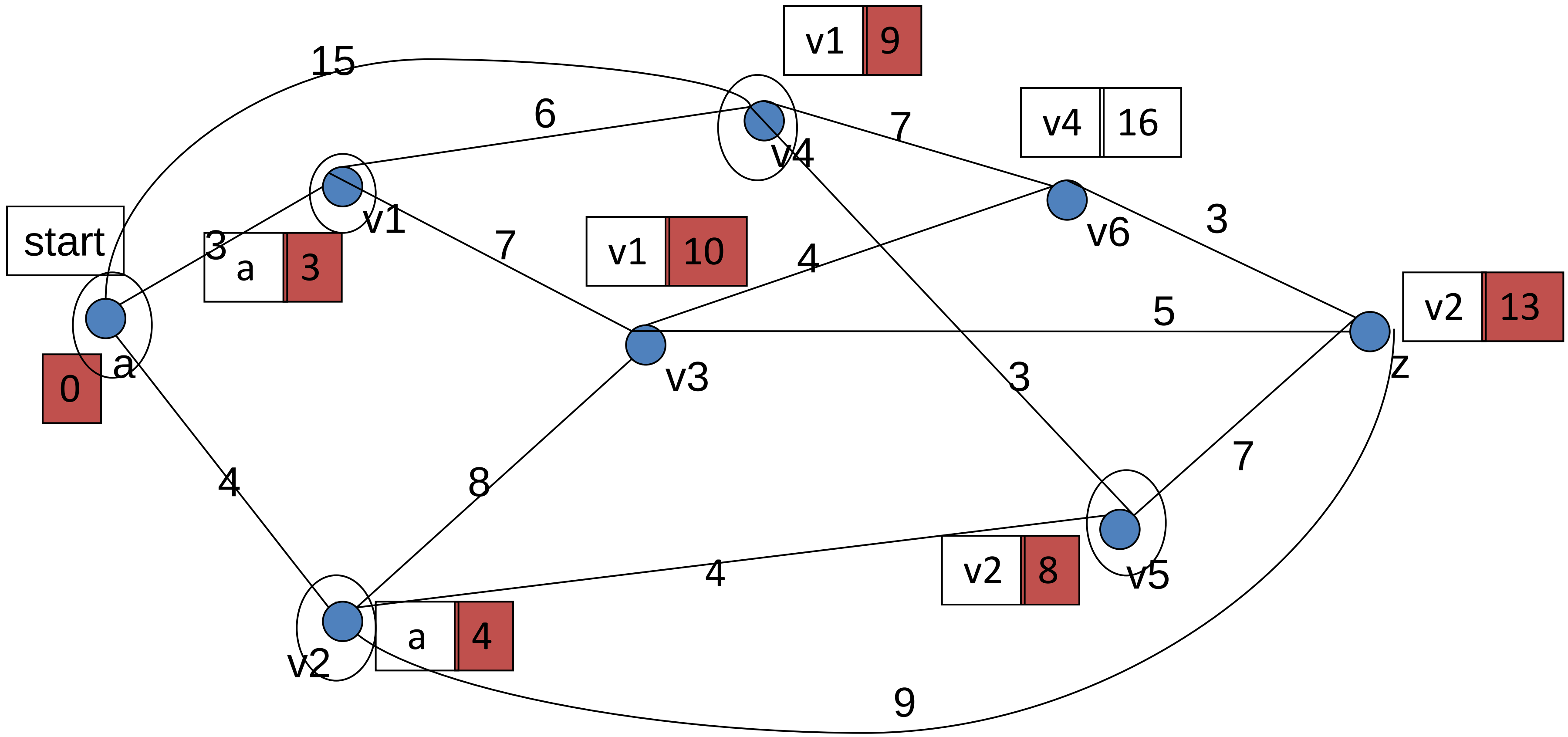
$S=\{a, v1, v2, v5, v4\}$

$N=\{v3, v6, z\}$

$L(v4)+W[v4, v6]<L(v6)$

$9+7= 16 < \infty$

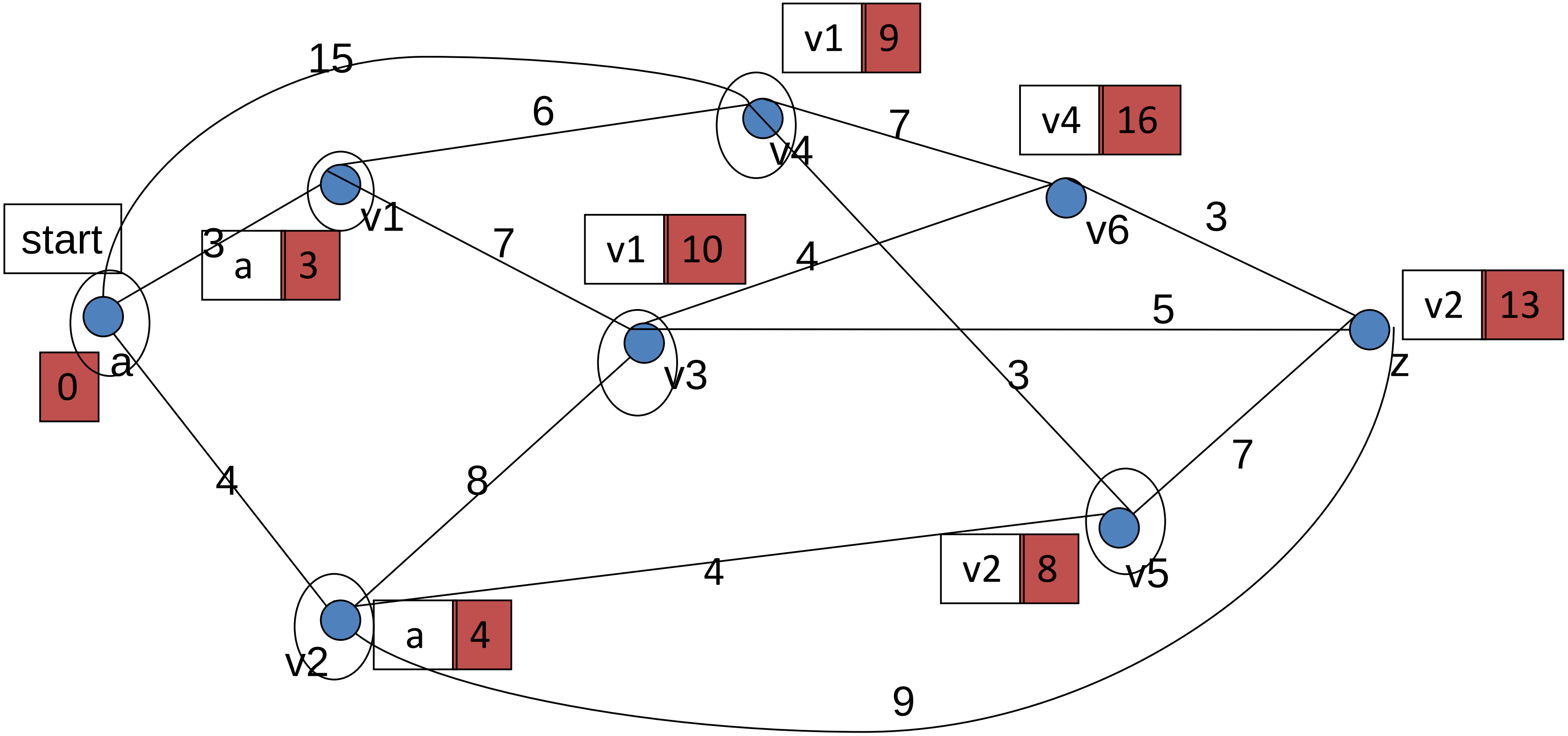
$L(v6) = 16$



S={*a*, *v1*, *v2*, *v5*,*v4*,*v3*}

N= {v6,*z*}

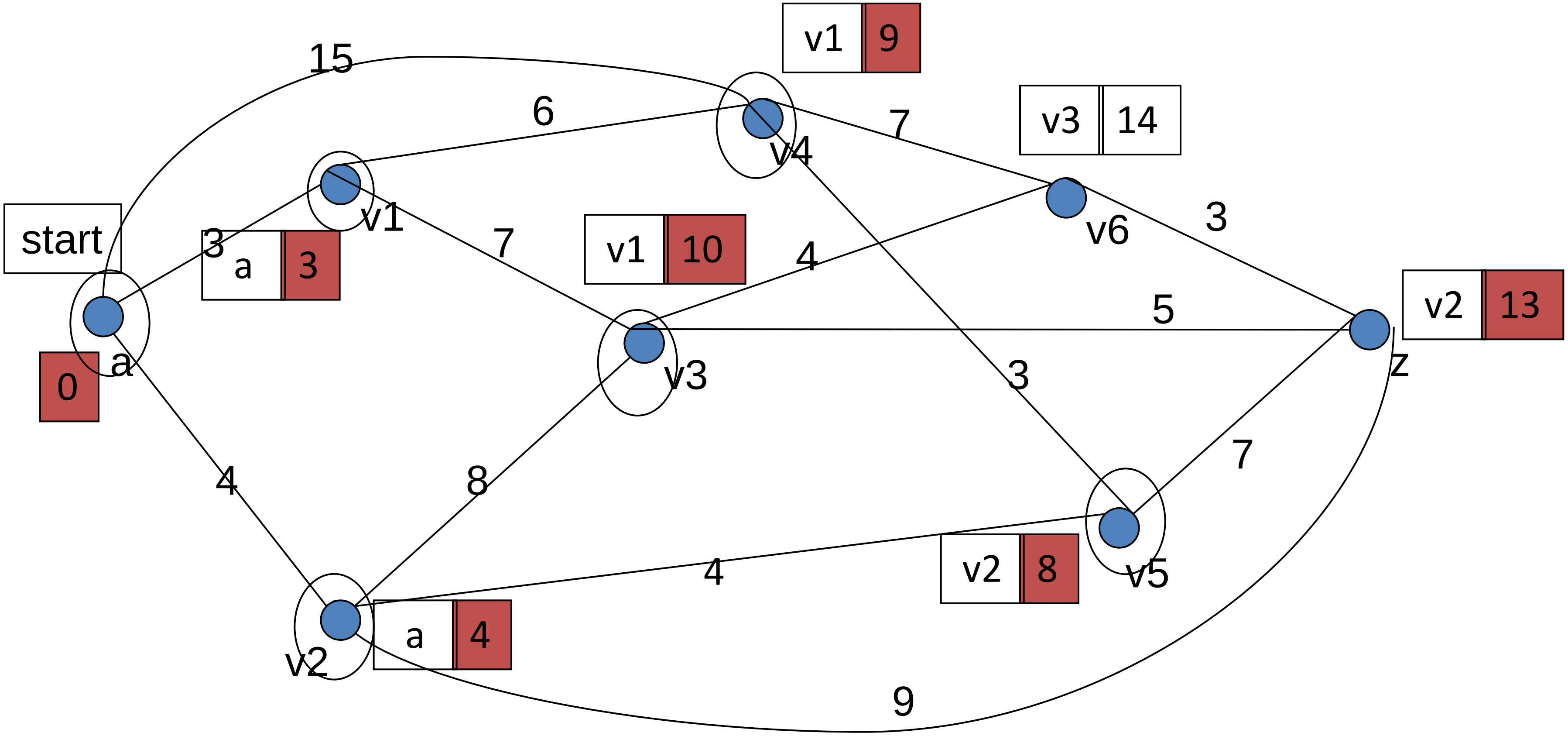
choose *v3*
because
L(*v3*)=10 = min{L(u)|u ∈ N}



S={*a*, *v1*, *v2*, *v5*,*v4*}

N= {*v3*,*v6*,*z*}

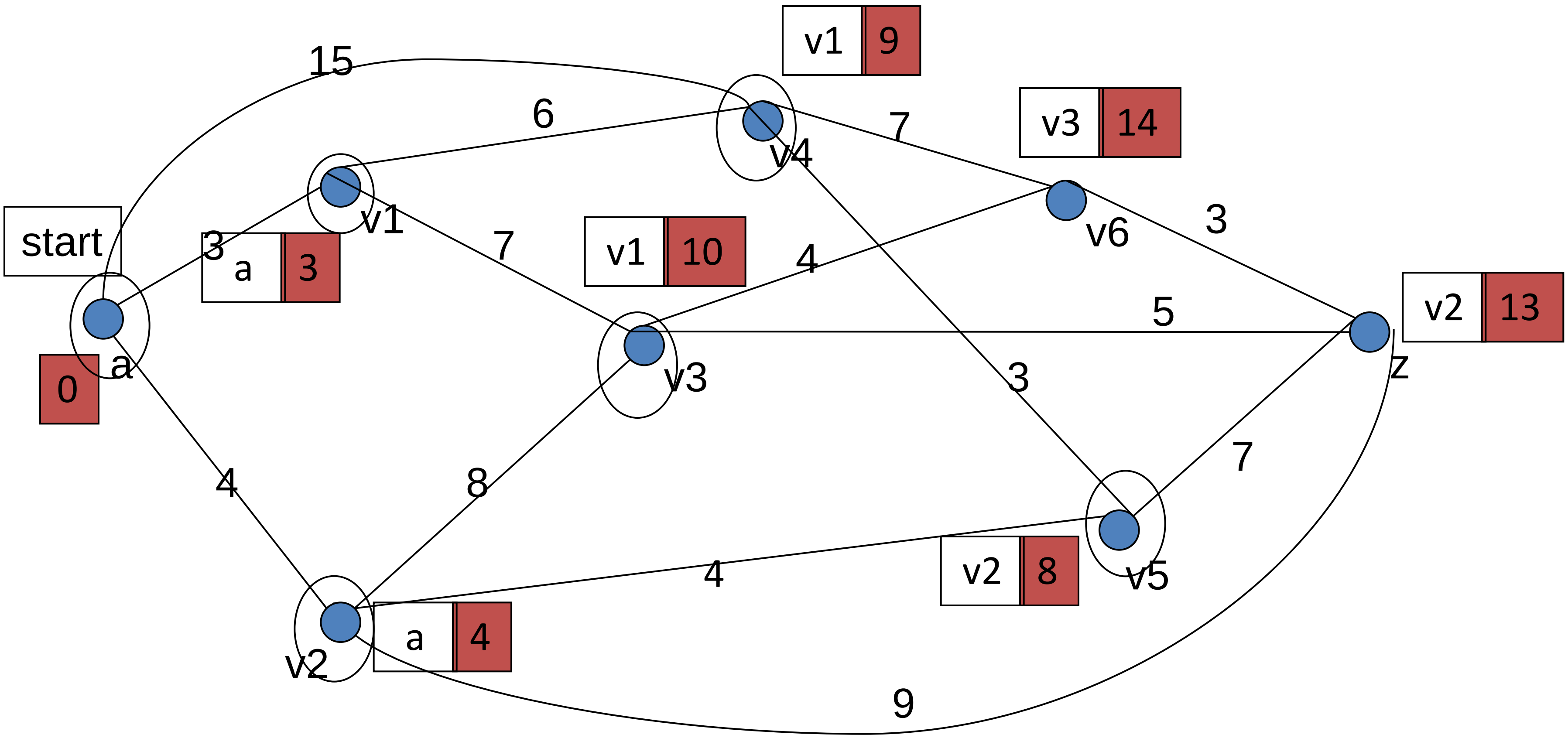
L(*v3*)+*W*[*v3*, *v6*]<*L*(*v6*)
10+4= 14 < 16
L(*v6*) = 14



S={a, v1, v2, v5,v4,v3}

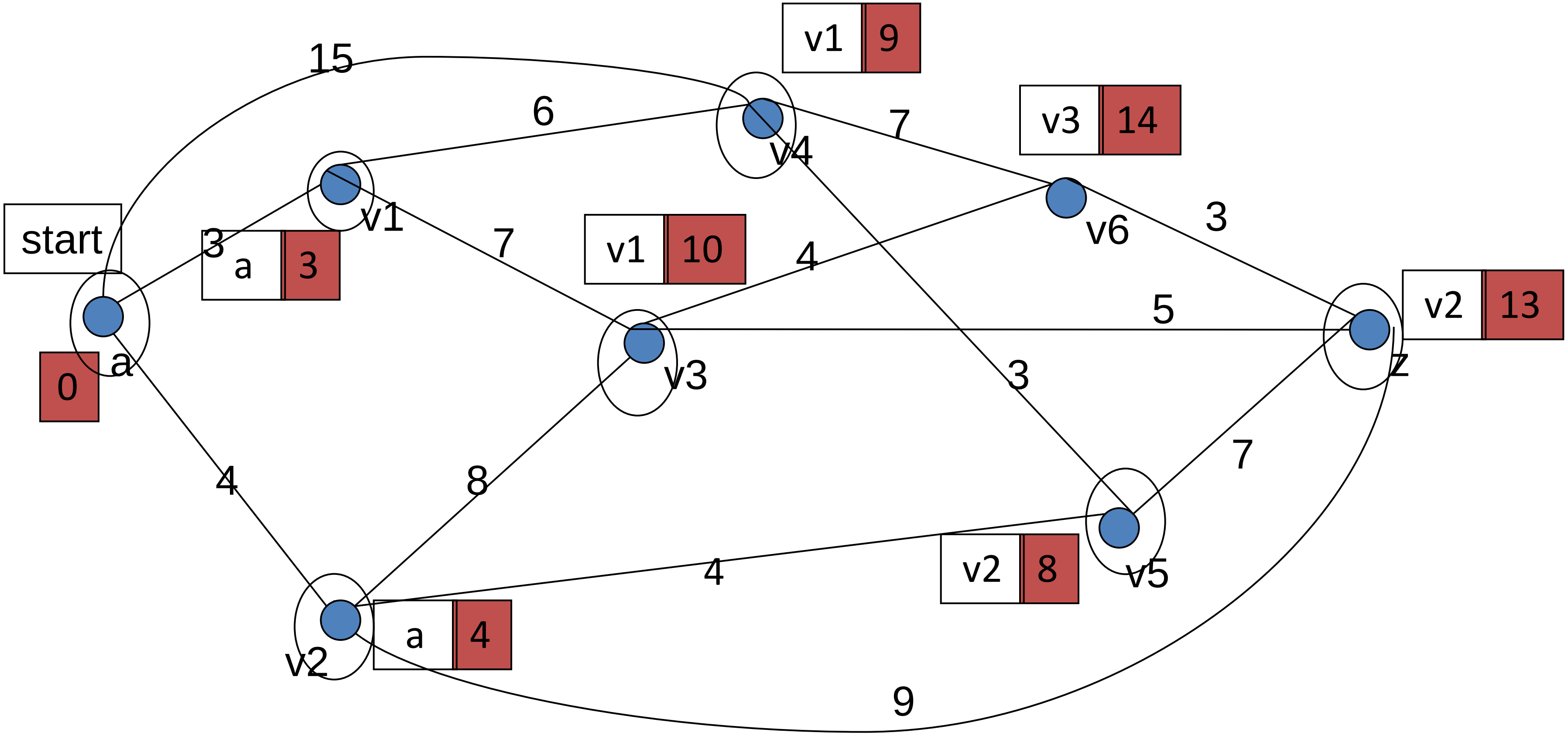
N= {v6,z}

L(*v*3)+*W*[*v*3, *z*] <*L*(*z*)
10+5= 15 > 13
L(*z*) remains the same



$S=\{a, v1, v2, v5, v4, v3\}$
$N=\{v6, z\}$

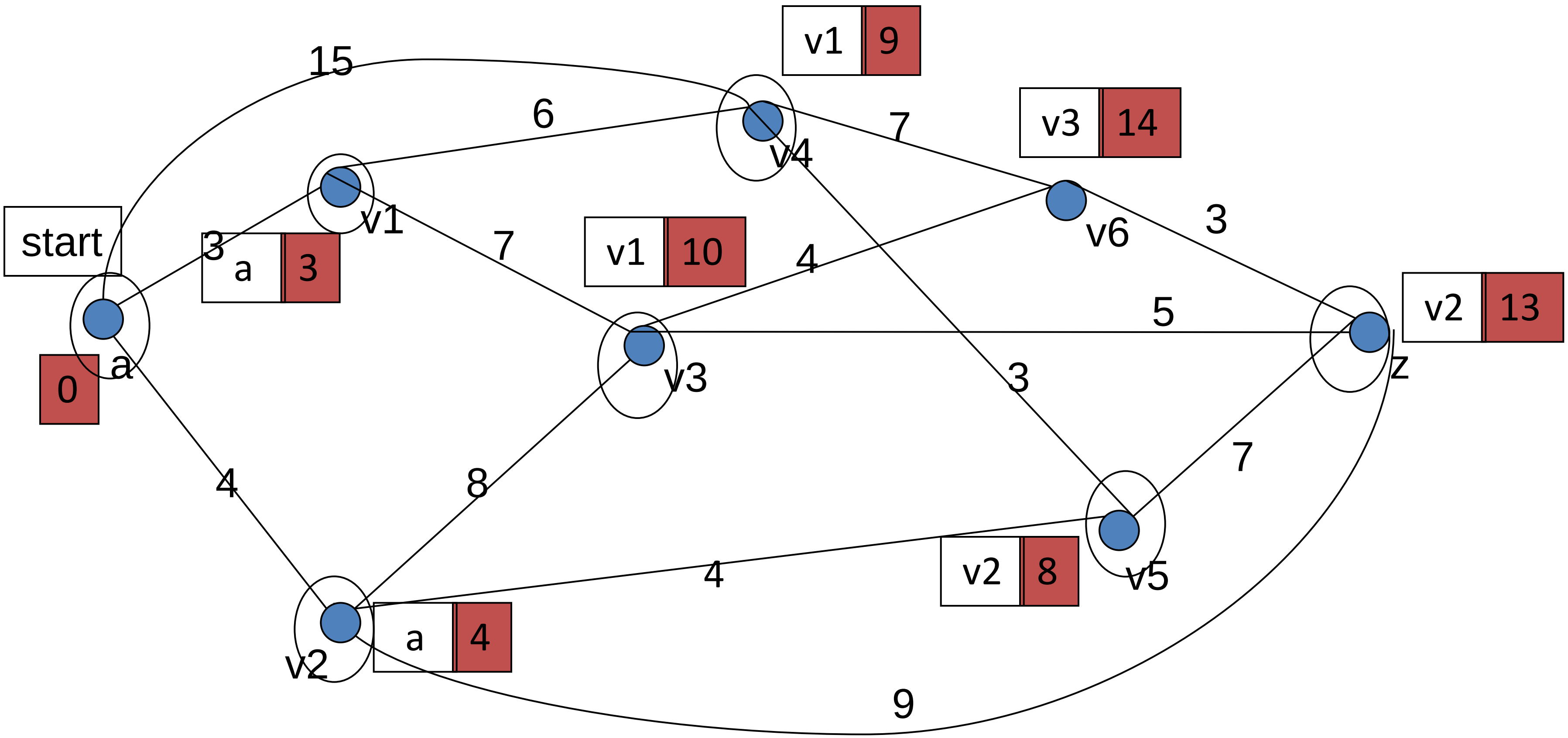
choose z
because
$L(z)=13 = \min\{L(u) \mid u \in N\}$



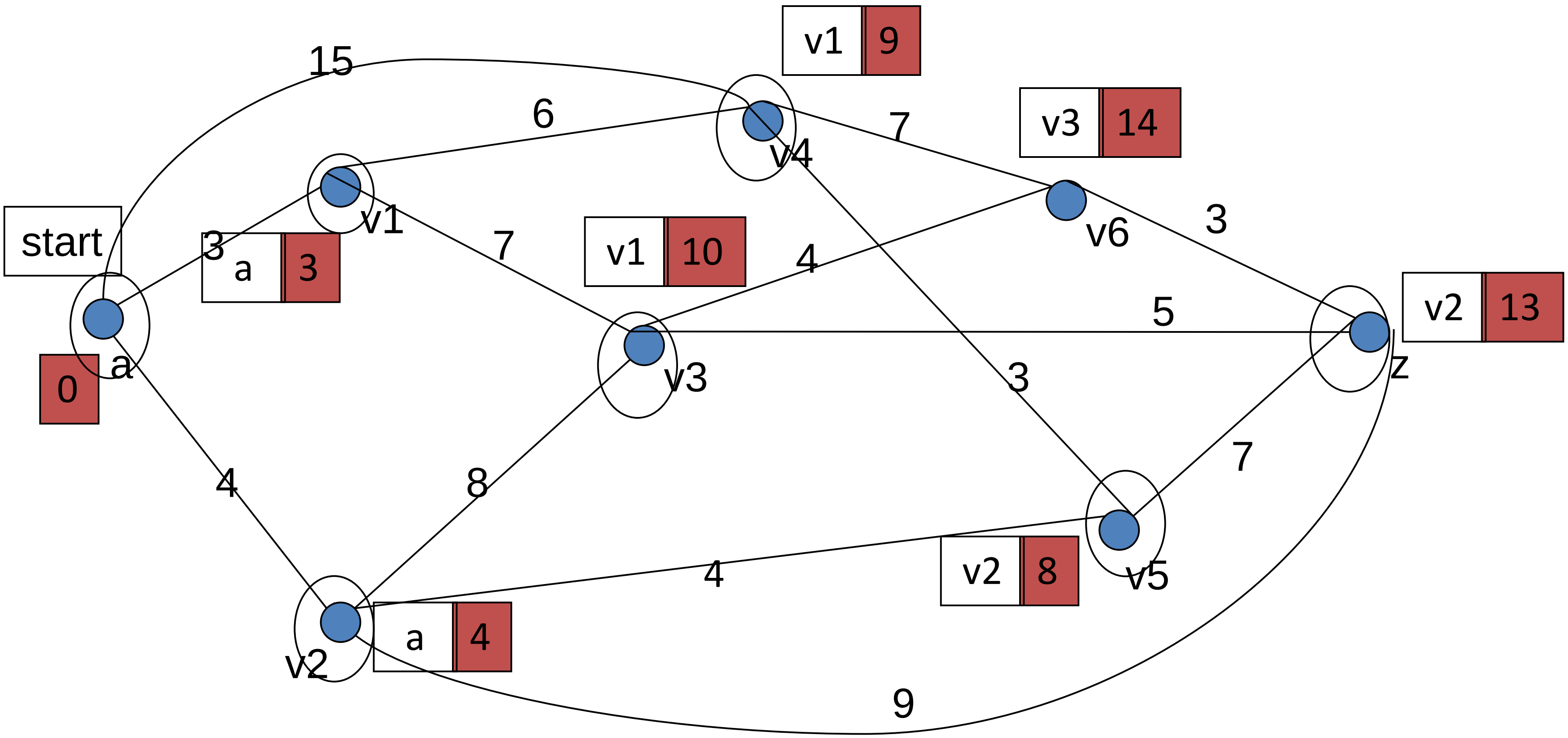
$S=\{a, v1, v2, v5, v4, v3, z\}$

$N= \{v6\}$

The loop terminates because $z \in S$



Shortest path from *a* to z is a -> v2-> z, with the length 13.



Exercise

- Use Dijkstra's algorithm to find the length of a shortest path from ***a*** to ***z***.