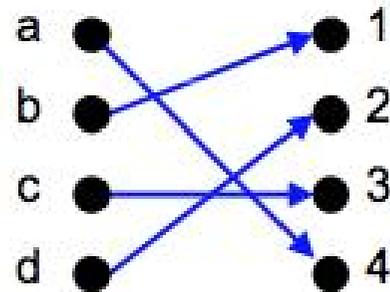# CHAPTER 2

FUNCTIONS

FUNCTION

- Let **X** and **Y** are nonempty sets.

- A **function (f)** from **X** to **Y** is a relation from **X** to **Y** having a properties:

  o The domain of f is **X**

  o If (x,y), (x,y') $\in f$, then $y = y'$

Relations vs Functions

- Not all relations are functions
- But consider the following function:


- All functions are relations!

Relations vs Functions

When to use which?

- A function is used when you need to obtain a SINGLE result for any element in the domain
  - Example: sin, cos, tan

- A relation is when there are multiple mappings between the domain and the co-domain
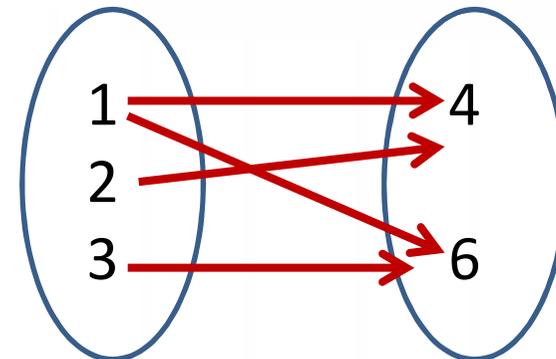 - Example: students enrolled in multiple courses

Relations vs Functions

Example

Function

Relation

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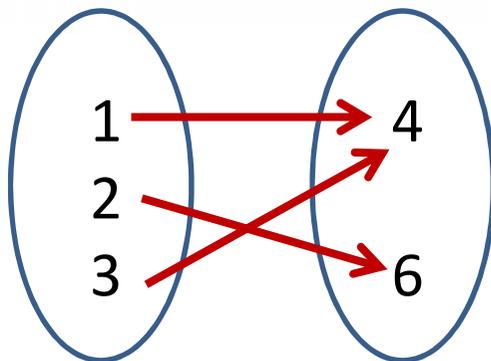
Domain, Co-domain, Range

- A function from **X** to **Y** is denoted, $f : \textbf{X} \rightarrow \textbf{Y}$

- The domain of f is the set **X**.

- The set **Y** is called the co-domain or target of f.

- The set $\{\, y \mid (x,y) \in f \,\}$ is called the range.

Given the relation, f = { (1,a), (2,b), (3,a) } from **X** = { 1, 2, 3 } to **Y** = { a, b, c } is a function from **X** to **Y**. State the domain and range.

Solution:

✓ The domain of f is **X**

✓ The range of f is {a, b}

example

- $f = \{ (1,a), (2,b), (3,a) \}$

domain X f Y codomain
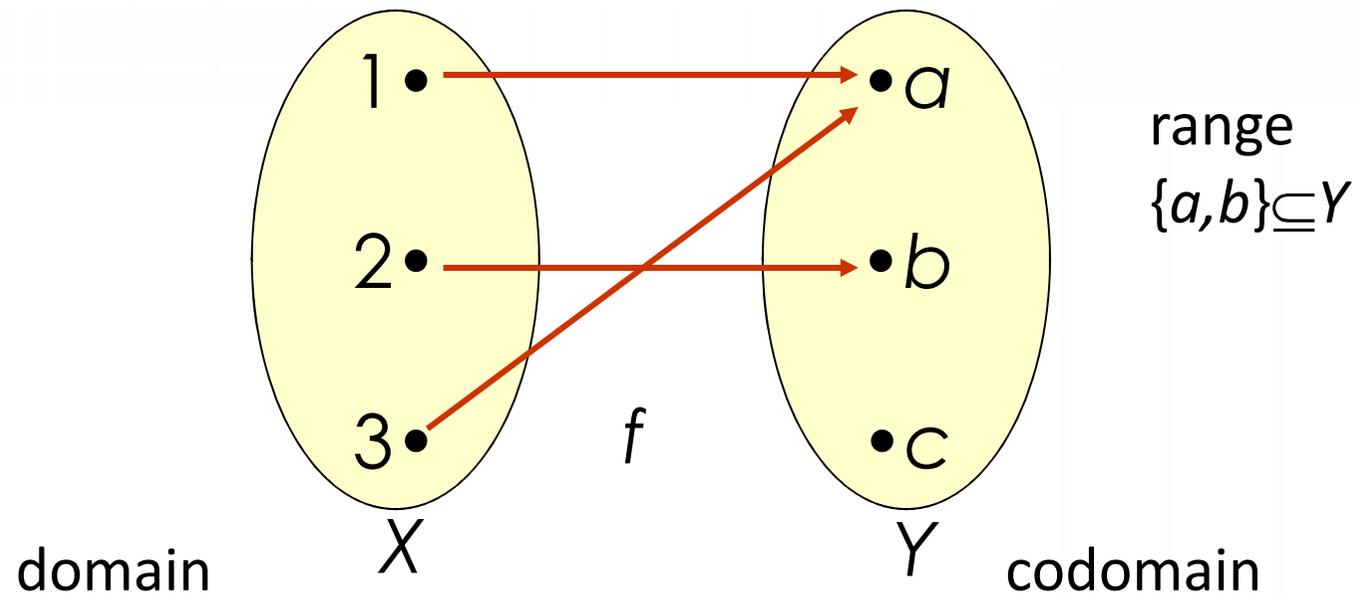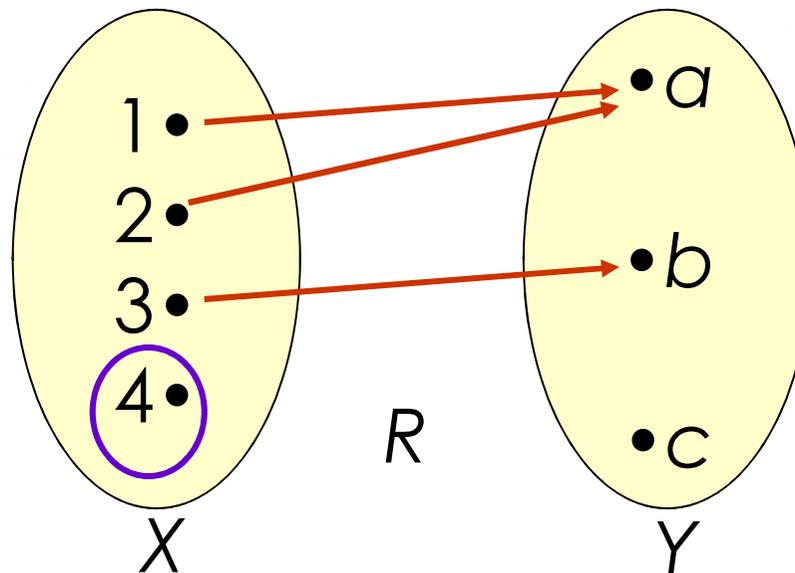
range
$\{a,b\} \subseteq Y$

example

- The relation, $R = \{(1,a), (2,a), (3,b)\}$
  from $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c\}$ is **NOT a function from X to Y.**

- The domain of R, $\{1, 2, 3\}$ is not equal to X.

example

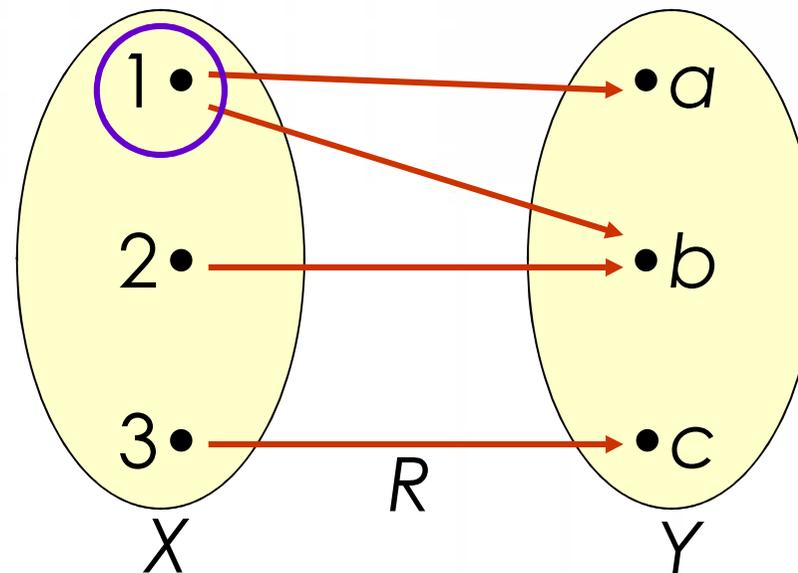$R = \{(1,a), (2,a), (3,b)\}$

There is no arrow from 4

example

- The relation, $R = \{(1,a), (2,b), (3,c), (1,b)\}$ from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$ is **NOT a function from X to Y**

- (1,a) and (1,b) in R but a ≠ b.

example

$R= \{(1,a), (2,b), (3,c), (1,b)\}$

There are 2 arrows
from 1

Notation of function: $f(x)$

- For the function, $f = \{(1,a),\ (2,b),\ (3,a)\}$

- We may write:

$$f(1)=a,\ f(2)=b,\ f(3)=a$$

- Notation $f(x)$ is used to define a function.

Example

○ Defined: $f(x) = x^2$

○ $f(2) = 4, \ f(-3.5) = 12.25, \ f(0) = 0$

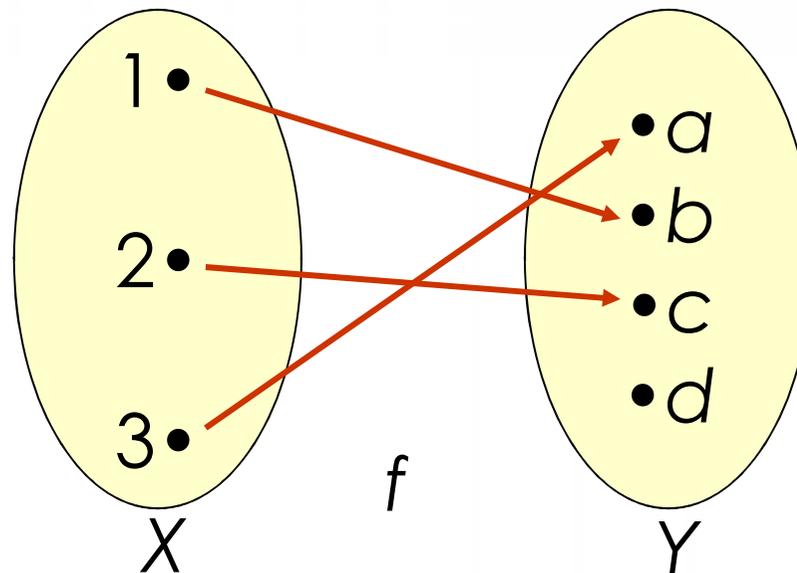○ $f = \{(x, x^2) \mid x \text{ is a real number}\}$

One-to-One Function

- A function f from X to Y, is said one-to-one (or injective) if for each $y \in Y$, there is at most one $x \in X$, with $f(x)=y$.

- For all x_1, x_2, if $f(x_1) = f(x_2)$, then $x_1=x_2$.

$$\forall x_1 \forall x_2 ((f(x_1) = f(x_2)) \rightarrow (x_1=x_2))$$

example

- The function, $f = \{ (1,b), (3,a), (2,c) \}$ from $X = \{ 1, 2, 3 \}$ to $Y = \{ a, b, c, d \}$ is one-to-one.

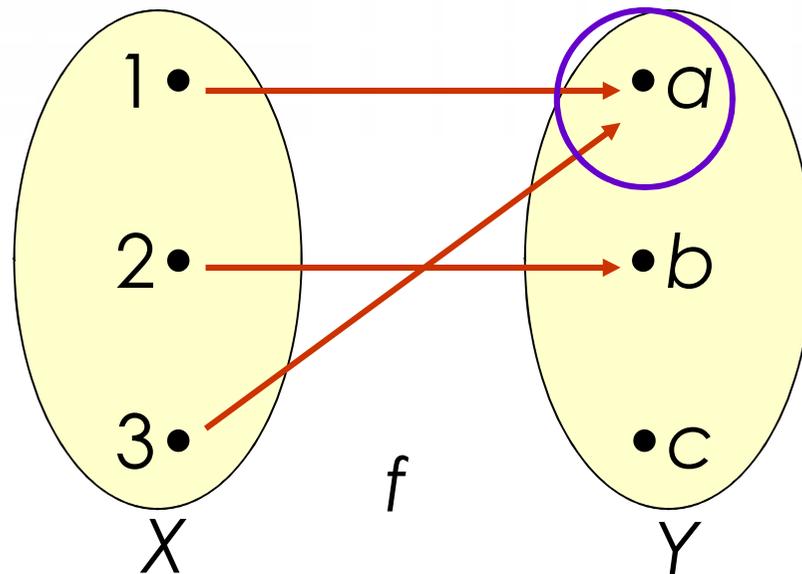Each element in *Y* has at most one arrow pointing to it

example

- The function, $f = \{ (1,a), (2,b), (3,a) \}$ from $X = \{ 1, 2, 3 \}$ to $Y = \{ a, b, c\}$ is **NOT one-to-one.**

- $f(1) = a = f(3)$

example

$f = \{ (1,a), (2,b), (3,a) \}$

a has 2 arrows pointing to it

NOT one-to-one function

example

Show that the function,

$$f(n) = 2n+1$$

on the set of positive integers is one-to-one.

example

- For all positive integer, n_1 and n_2 if $f(n_1) = f(n_2)$, then $n_1 = n_2$.

- Let, $f(n_1) = f(n_2)$, $f(n) = 2n+1$

  then $2n_1 + 1 = 2n_2 + 1$ (-1)

  $2n_1 = 2n_2$ ($\div 2$)

  $n_1 = n_2$

- This shows that f is one-to-one.

example

- Show that the function,

$$f(n) = 2^n - n^2$$

from the set of positive integers to the set of integers is NOT one-to-one.

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example

- Need to find 2 positive integers, n_1 and n_2 $n_1 \neq n_2$ with $f(n_1) = f(n_2)$.

- Trial and error,
$$f(2) = f(4)$$

- f is not one-to-one.

Onto Function

- If f is a function from **X** to **Y** and the range of f is **Y**, f is said to be onto **Y** (or an onto function or a surjective function)

- For every $y \in$ **Y**, there exists at least one $x \in$ **X** such that $f(x) = y$

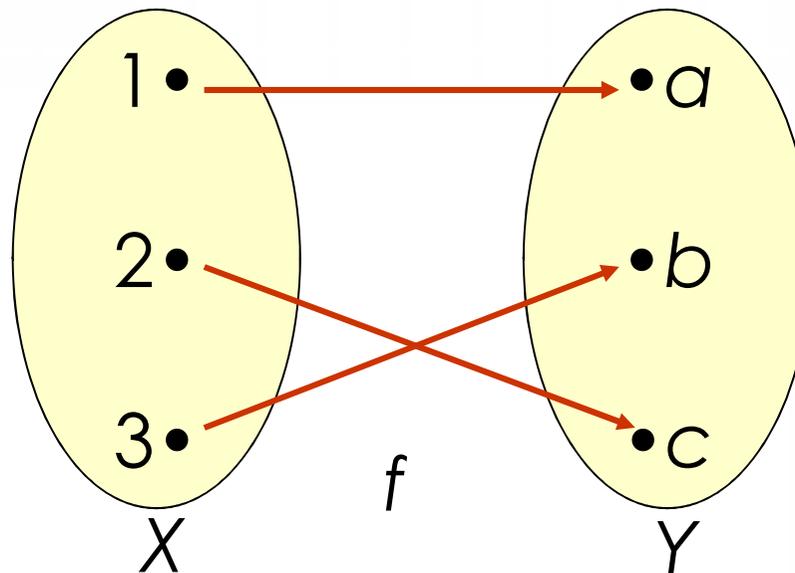$$\forall\, y \in \mathbf{Y}\ \exists\, x \in \mathbf{X}\ (f(x) = y)$$

example

- The function, $f = \{ (1,a), (2,c), (3,b) \}$
  from $X = \{ 1, 2, 3 \}$ to $Y = \{ a, b, c \}$
  is one-to-one and onto Y.

example

- $f = \{ (1,a), (2,c), (3,b) \}$
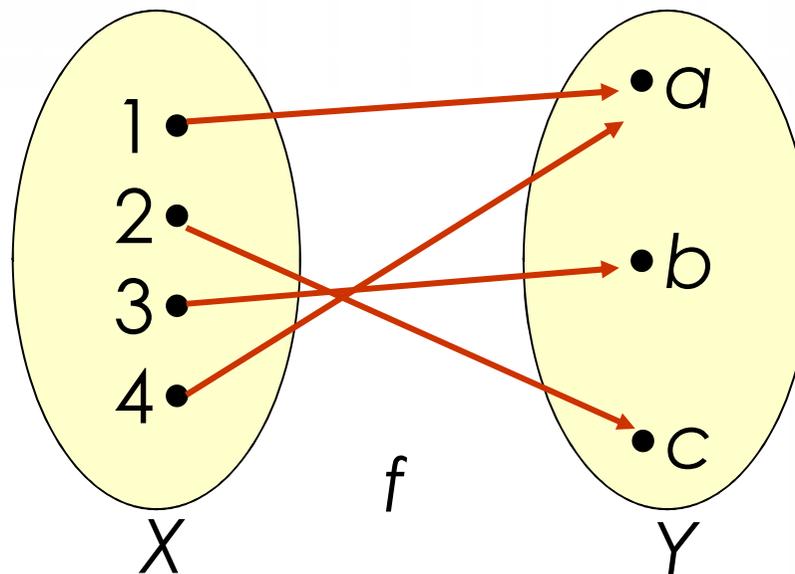
One-to-one
Each element
in Y has at
most one
arrow

Onto
Each element
in Y has at
least one
arrow pointing
to it

example

- $f = \{ (1,a),\ (2,c),\ (3,b),\ (4,a) \}$

Onto

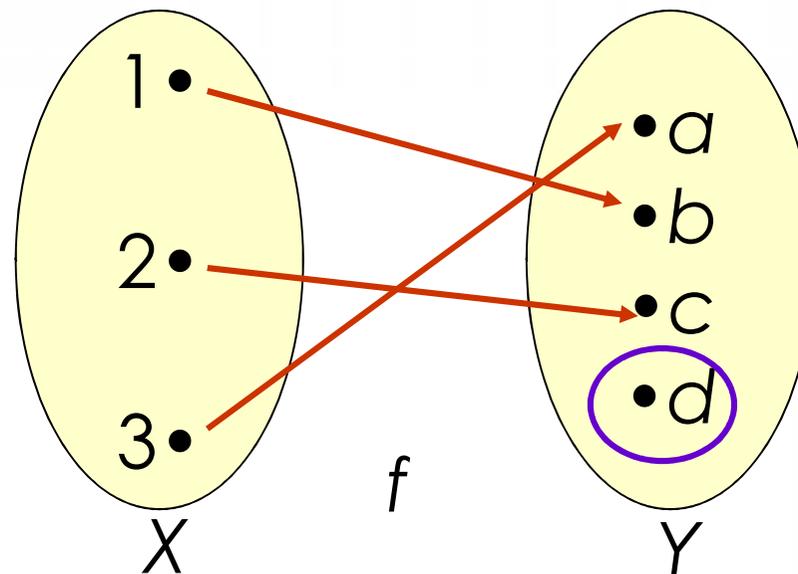Each element in Y has at least one arrow pointing to it

example

- The function, f = { (1,b), (3,a), (2,c) }
  is **not onto Y** = {a, b, c, d}

- It is onto {a, b, c}

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example

$f = \{ (1,b), (3,a), (2,c) \}$

not onto
no arrow
pointing to *d*

prepared by Razana Alwee
28
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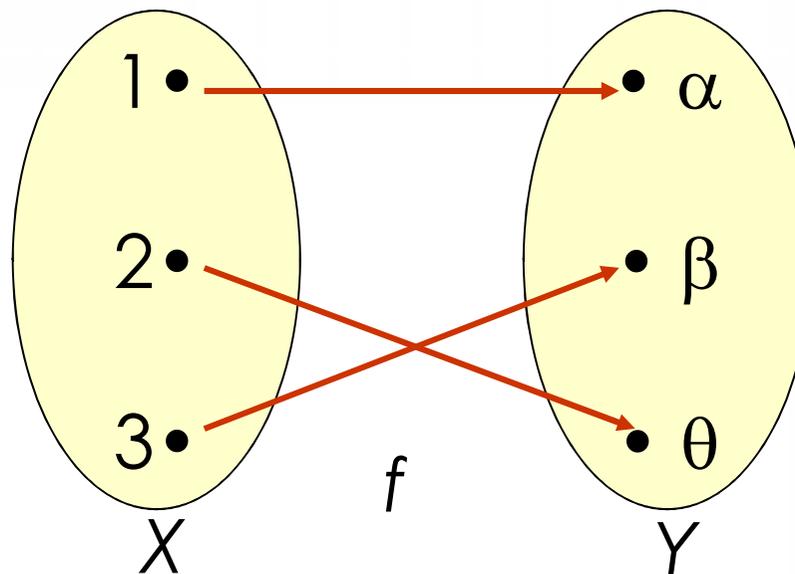- A function, f is called one-to-one correspondence (or bijective/bijection) if f is both one-to-one and onto.

example

- The function, $f = \{ (1,\alpha), (2,\theta), (3,\beta) \}$ from $X = \{ 1, 2, 3 \}$ to $Y = \{\alpha, \beta, \theta \}$ is one-to-one and onto Y.

- The function f is a bijection

example

$f = \{ (1,\alpha), (2,\theta), (3,\beta) \}$

One-to-one and onto *Y* -bijection

Determine which of the relations f are functions from the set **X** to the set **Y**. In case any of these relations are functions, determine if they are one-to-one, onto **Y**, and/or bijection.

a) X = { -2, -1, 0, 1, 2 } , **Y** = { -3, 4, 5 } and
f = { (-2,-3), (-1,-3), (0,4), (1,5), (2,-3) }

b) **X** = { -2, -1, 0, 1, 2 } , **Y** = { -3, 4, 5 } and
f = { (-2,-3), (1,4), (2,5)}

c) **X** = **Y**= { -3, -1, 0, 2 } and
f = { (-3,-1), (-3,0), (-1,2), (0,2), (2,-1) }
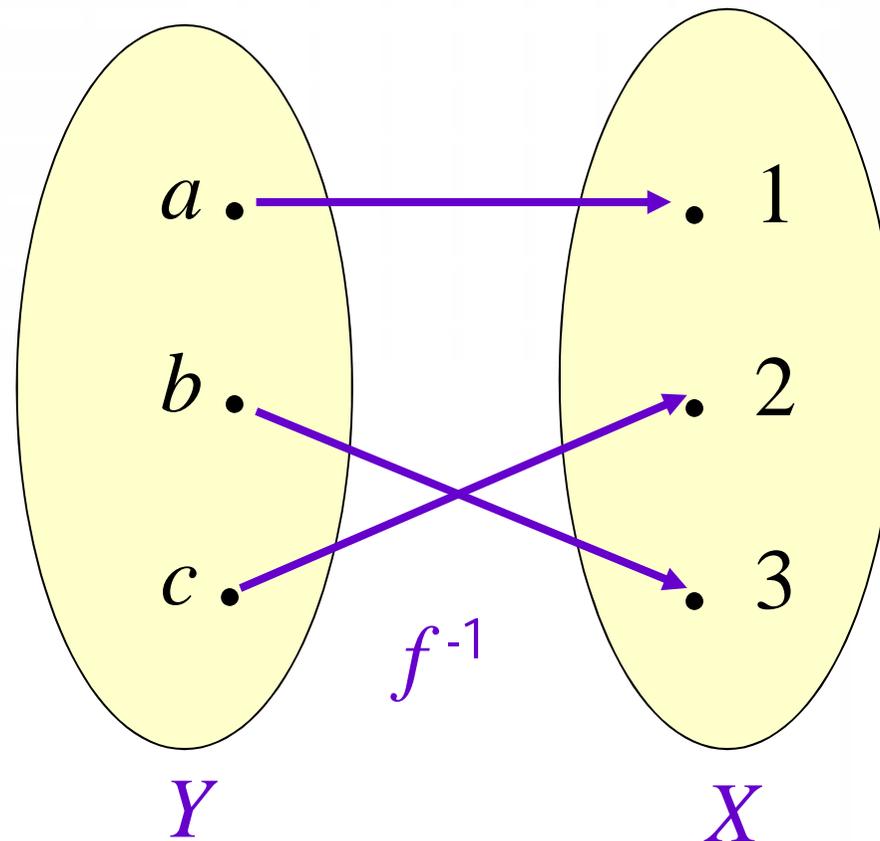
Inverse Function

- Let $f: \mathbf{X} \rightarrow \mathbf{Y}$ be a function.

- The inverse relation $f^{-1} \subseteq \mathbf{Y} \times \mathbf{X}$ is a function from $\mathbf{Y}$ to $\mathbf{X}$, if and only if f is both one-to-one and onto $\mathbf{Y}$.

example

- $f = \{(1,a),(2,c),(3,b)\}$

- $f^{-1} = \{(a,1),(c,2),(b,3)\}$

X Y
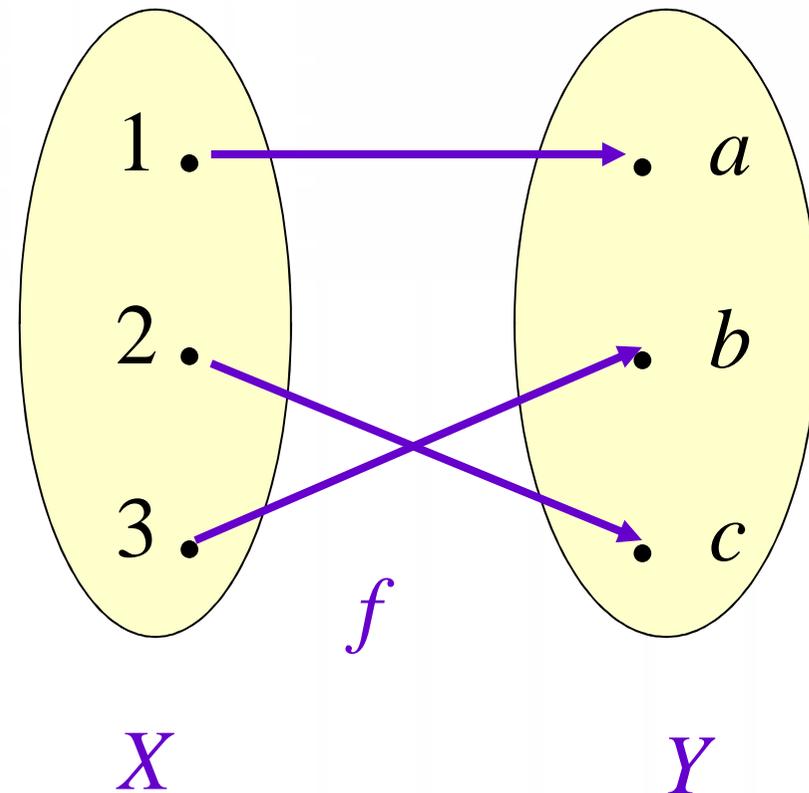
example

Example

o The function, $f(x) = 9x + 5$ for all $x \in R$ (R is the set of real numbers).

o This function is both one-to-one and onto.

o Hence, f^{-1} exists.

o Prove:

Let $(y, x) \in f^{-1}, f^{-1}(y) = x$

$(x, y) \in f,$ $y = 9x + 5$

$x = (y\text{-}5)/9$

$f^{-1}(y) = (y\text{-}5)/9$

- Find each inverse function.

     a) $f(x) = 4x + 2, \ x \in R$

     b) $f(x) = 3 + (1/x), \ x \in R$

Composition

- Suppose that g is a function from **X** to **Y** and f is a function from **Y** to **Z**.

- The composition of f with g,

$$f \, , \, g$$

is a function

$$(f \, , \, g)(x) = f(g(x))$$

from **X** to **Z**.

o Given, g = { (1,a), (2,a), (3,c) }

  a function from **X** = {1, 2, 3} to **Y** = {a, b, c}

  and,

  f = { (a,y), (b,x), (c,z) }

  a function from **Y** to **Z** = { x, y, z }.

o The composition function from **X** to **Z** is the function

  f o g = { (1,y), (2,y), (3,z) }
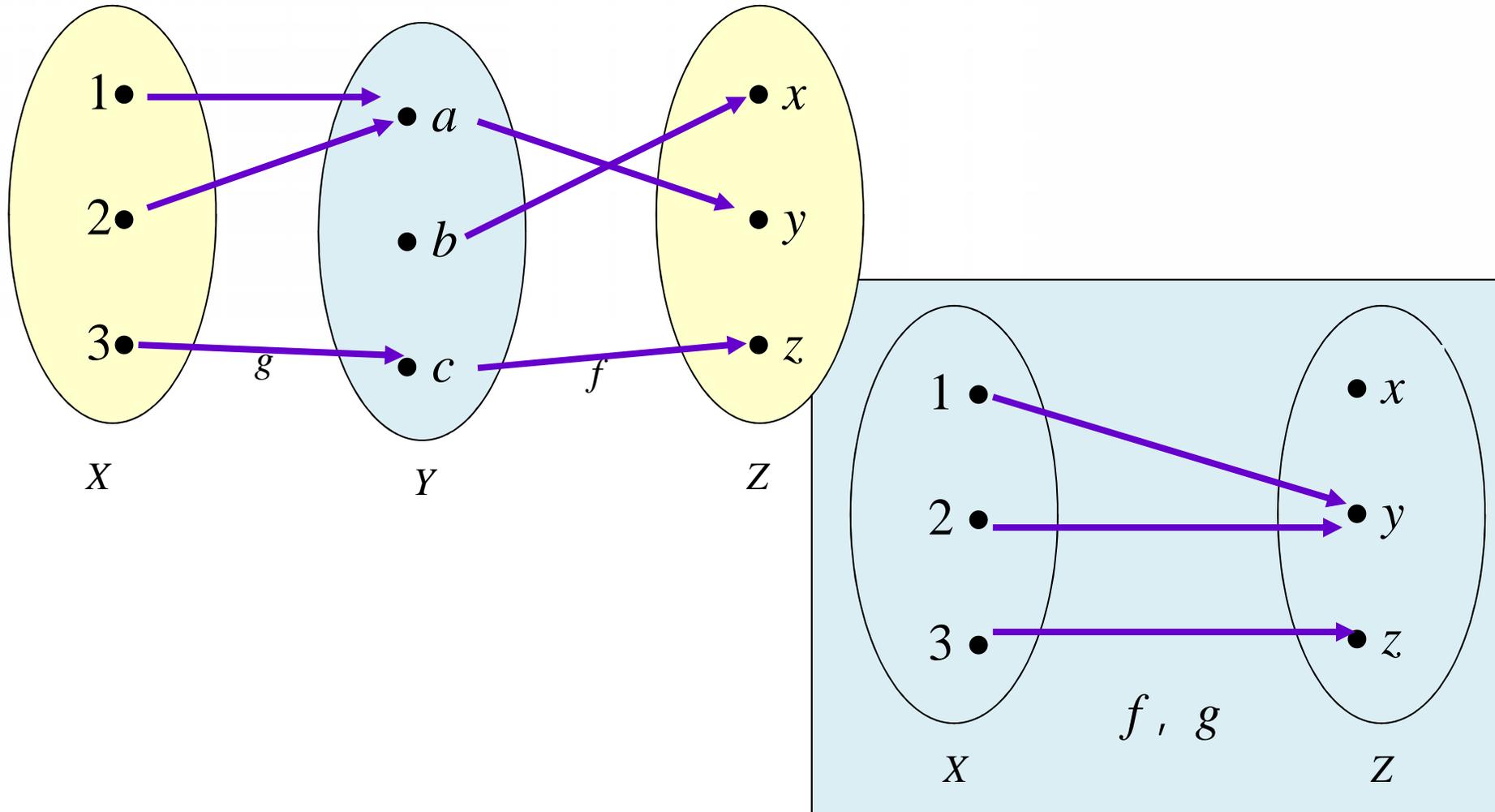
example

$$f(x) = \log_3 x \text{ and } g(x) = x^4$$

➤ $f(g(x)) = \log_3(x^4)$

➤ $g(f(x)) = (\log_3 x)^4$

➤ Note: $f \circ g \neq g \circ f$

example

$$f(x) = \frac{1}{5}x \qquad\qquad g(x) = x^2 + 1$$

$$(g,f)(x) = g(f(x)) = g(\frac{x}{5})$$

$$= (\frac{x}{5})^2 + 1 = \frac{x^2}{25} + 1$$

example

- Composition sometimes allows us to decompose complicated functions into simpler functions.

$$f(x) = \sqrt{\sin 2x}$$

- example

$$g(x) = \sqrt{x} \qquad h(x) = \sin x \qquad w(x) = 2x$$

$$f(x) = g(h(w(x)))$$

Exercise

o Let f and g be functions from the positive integers to the positive integers defined by the equations, $f(n) = n^2$, $g(n) = 2^n$

o Find the compositions

a) f , f

b) g , g

c) f , g

d) g , f

Recursive Algorithms

- A recursive procedure is a procedure that invokes itself.

- A recursive algorithm is an algorithm that contains a recursive procedure.

- Recursion is a powerful, elegant and natural way to solve a large class of problems.

Factorial problem

o If n 1,

$n! = n\,(n-1) => 2_x 1$

and $0! = 1$

o Notice that, if $n \geq 2$, n factorial can be

written as,

$n! = n(n-1)(n-2) => 2_x 1_x 0$

$= n(n-1)!$

example

- n=5

- 5!

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5 \cdot 4!$$

$$4! = 4 \cdot 3!$$

$$3! = 3 \cdot 2!$$

Algorithm

- Input: *n*, integer ≥ 0

- Output: *n*!

- Factorial (*n*) {

    if (*n*=0)

           return 1

    return *n**factorial(*n*-1)

    }

example

- Fibonacci sequence, f_n

$$f_1 = 1$$

$$f_2 = 1$$

$$f_n = f_{n-1} + f_{n-2}, \qquad n \geq 3$$

1, 1, 2, 3, 5, 8, 13, ...

Algorithm

- Input: n
- Output: $f(n)$
- $f(n)$ {

    if (n=1 or n=2)

         return 1

    return $f(n-1) + f(n-2)$

    }

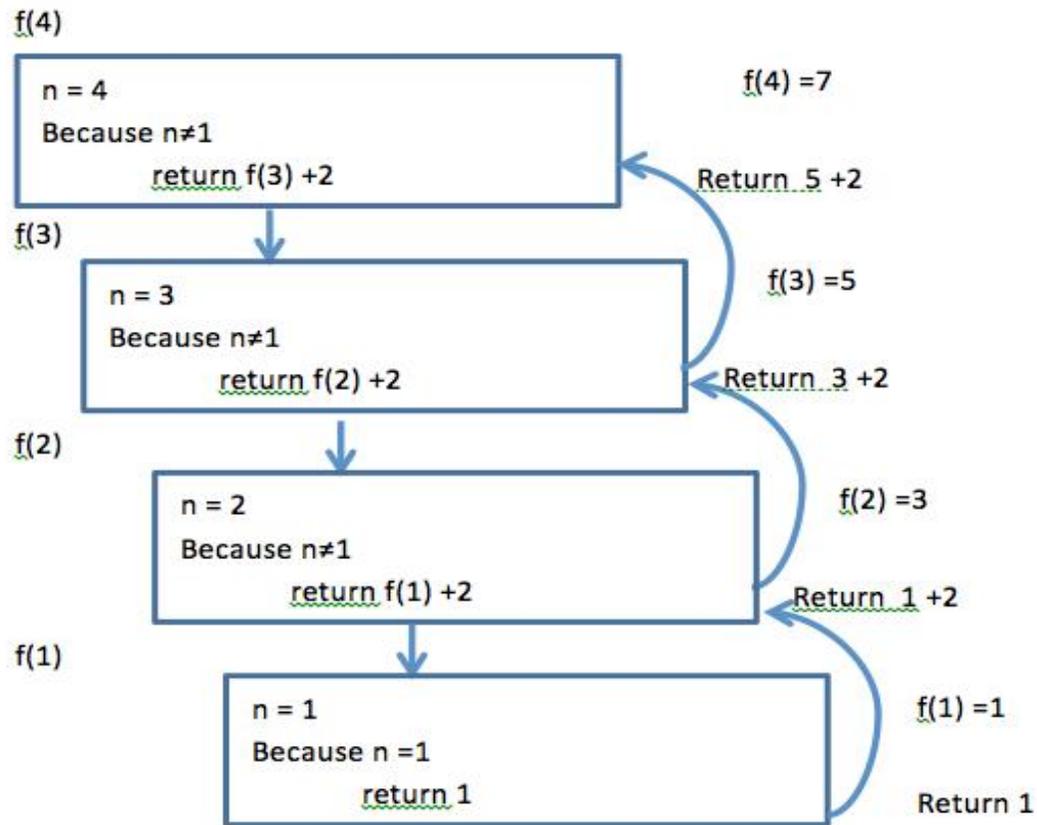Consider the following arithmetic sequence: 1, 3, 5, 7, 9,...

Suppose a_n is the term sequence. The generating rule is $a_n=a_n\text{-}1+2$, for $n \geq 1$. The relevant recursive algorithm can be written as

$f(n)$
{   if $(n = 1)$

    return 1

   return $f(n-1)+2$

}

Use the above recursive algorithm to trace $n = 4$

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Example- Solution

Trace the output if n = 4

f(4)

n = 4	f(4) =7
Because n≠1	
return f(3) +2	Return 5 +2

f(3)

n = 3	f(3) =5
Because n≠1	
return f(2) +2	Return 3 +2

f(2)

n = 2	f(2) =3
Because n≠1	
return f(1) +2	Return 1 +2

f(1)

n = 1	f(1) =1
Because n =1	
return 1	Return 1

Answer = 7

exercise

- Given

$$s_n = 3 + 6 + 9 + \ldots + 3n$$

- $$s_1 = 3 \qquad s_n = s_{n-1} + 3n \quad \text{for} \quad n \geq 2$$

- Write a recursive algorithm to compute S_n, $n \geq 1$.