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Analysis of Variance (ANOVA)

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Outline

- Introduction
- One-Way ANOVA with Equal Sample Sizes
- One-Way ANOVA with Unequal Sample Sizes
- Two-Way ANOVA

Not covered!

Introduction

- ANOVA is a method of testing the equality of three or more population means by analyzing sample variances.
- The purpose of ANOVA is to test for significant differences between Means.
- Elementary concepts provide a brief introduction to the basics of statistical significance testing.

One-Way ANOVA with Equal Sample Sizes

- We assume that the populations have normal distribution and same variance (or standard deviation), and the samples are random and independent of each other.

Notation:

n = size of each sample

k = number of samples

$s_{\bar{x}}^2$ = variance sample means

s_p^2 = pooled variance obtained by calculating the mean of the sample variances

One-Way ANOVA with Equal Sample Sizes

- Test statistic:

$$F = \frac{\text{variance between sample}}{\text{variance within sample}} = \frac{ns_{\bar{x}}^2}{s_p^2}$$

- Variance between sample
  - Also called variation due to treatment
  - An estimate of the common population variance σ^2 that is based on the variability among the sample **means**
- Variance within sample
  - Also called variation due to error
  - An estimate of the common population variance σ^2 based on the sample **variances**

One-Way ANOVA with Equal Sample Sizes

- The critical value of F:
  - numerator degrees of freedom = $k - 1$
  - denominator degrees of freedom = $k(n - 1)$
  - with k is the number of samples and n is the number of data in a sample

Example 1

Table below lists the head injury to car crash test dummies for four different types of cars. Use a 0.05 significance level to test the null hypothesis that the different types of car have the same mean.

Subcompact Cars	Compact Cars	Midsize Cars	Full-Size Cars
681	643	469	384
428	655	727	656
917	442	525	602
898	514	454	687
420	525	259	360

Example 1

- Step 1: define hypothesis
  - $H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4$
  - H_1 : at least one mean is different

Example 1

- Step 2: For each sample, find $n, \bar{x}$ and s

- Sample 1 (Subcompact Cars)

$$n = 5$$

$$\bar{x} = \frac{681 + 428 + 917 + 898 + 420}{5} = 668.8$$

$$s = \sqrt{\frac{(681 - 668.8)^2 + (428 - 668.8)^2 + (917 - 668.8)^2 + (898 - 668.8)^2 + (420 - 668.8)^2}{5 - 1}}$$

$$= 242.0$$

Example 1

- Sample 2 (Compact Cars)

$$n = 5$$

$$\bar{x} = \frac{643 + 655 + 442 + 514 + 525}{5} = 555.8$$

$$s = \sqrt{\frac{(643 - 555.8)^2 + (655 - 555.8)^2 + (442 - 555.8)^2 + (514 - 555.8)^2 + (525 - 555.8)^2}{5 - 1}}$$

$$= 91.0$$

- Sample 3 (Midsize Cars)

$$n = 5$$

$$\bar{x} = \frac{469 + 727 + 525 + 454 + 259}{5} = 486.8$$

$$s = \sqrt{\frac{(469 - 486.8)^2 + (727 - 486.8)^2 + (525 - 486.8)^2 + (454 - 486.8)^2 + (259 - 486.8)^2}{5 - 1}}$$

$$= 167.7$$

Example 1

- Sample 3 (Full-Size Cars)

$$n = 5$$

$$\bar{x} = \frac{384 + 656 + 602 + 687 + 360}{5} = 537.8$$

$$s = \sqrt{\frac{(384 - 537.8)^2 + (656 - 537.8)^2 + (602 - 537.8)^2 + (687 - 537.8)^2 + (360 - 537.8)^2}{5 - 1}}$$

$$= 154.6$$

- Step 3: Find variance between samples

- Step 3a: Find mean between samples

$$\bar{\bar{x}} = \frac{668.8 + 555.8 + 486.8 + 537.8}{4} = 562.3$$

- Step 3b: Find standard deviation between samples

$$s_{\bar{x}}^2 = \sqrt{\frac{(668.8 - 562.3)^2 + (555.8 - 562.3)^2 + (486.8 - 562.3)^2 + (537.8 - 562.3)^2}{4 - 1}}$$

$$= 76.779$$

- Step 3c: Find variance between samples

$$ns_{\bar{x}}^2 = 5(76.779)^2 = 29475.1$$

Example 1

- Step 4: Find variance within samples

$$s_p^2 = \frac{(242.0)^2 + (91.0)^2 + (167.7)^2 + (154.6)^2}{4}$$

$$= 29717.4$$

- Step 5: Calculate test statistic, *F*

$$F = \frac{\text{variance between samples}}{\text{variance within samples}} = \frac{ns_{\bar{x}}^2}{s_p^2}$$

$$= \frac{29475.1}{29717.4}$$

$$= 0.992$$

Example 1

- Step 6: Calculate numerator and denominator degree of freedom
  - Numerator = k – 1 = 4 – 1 = 3
  - Denominator = k(n – 1) = 4(5 – 1) = 16

- Step 7: Find critical value of *F* with $\alpha = 0.05$ from *F*-distribution table
  - *F* critical value = 3.2389

Example 1

- Step 6: Test the claim
  - Since $F_{\text{test statistic}} < F_{\text{critical value}}$ (0.992 < 3.2389), we fail to reject the null hypothesis.
  - There is sufficient evidence to claim that the different types of cars have the same mean for head injury.

Example 2

- Table below lists the chest deceleration to car crash test dummies for four different types of cars. Use a 0.05 significance level to test the null hypothesis that the different types of car have the same mean.

Subcompact Cars	Compact Cars	Midsize Cars	Full-Size Cars
n = 5	n = 5	n = 5	n = 5
$\bar{x} = 50.4$	$\bar{x} = 53.0$	$\bar{x} = 48.8$	$\bar{x} = 46.0$
s = 6.69	s = 4.64	s = 3.35	s = 7.11

Example 2

- Mean between samples, $\bar{\bar{x}} = 49.55$
- Standard deviation between samples, $s_{\bar{x}} = 2.93$

- Variance between samples, $ns_{\bar{x}}^2 = 5(2.93)^2 = 42.92$
- Variance within samples, $s_p^2 = 32.02$

- $F_{\text{test statistic}} = 1.340$

Example 2

- Numerator = 4 – 1 =3
- Denominator = 4(5-1) = 16

- $F_{\text{critical value}}$ = 3.2389

- Since $F_{\text{test statistic}} < F_{\text{critical value}}$ (1.340 < 3.2389), we fail to reject the null hypothesis.
- There is sufficient evidence to claim that the different types of cars have the same mean for chest deceleration.

Example 3

Consider the two collections of sample data in Table 1. Note that the three samples in part A are identical to the three samples in part B, except that in part B we have added 10 to each value of sample 1 from part A.

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Example 3

	A			B		
	add 10 →					
	Sample 1	Sample 2	Sample 3	Sample 1	Sample 2	Sample 3
	7	6	4	17	6	4
	3	5	7	13	5	7
	6	5	6	16	5	6
	6	8	7	16	8	7
	$n_1 = 4$	$n_2 = 4$	$n_3 = 4$	$n_1 = 4$	$n_2 = 4$	$n_3 = 4$
	$\bar{x}_1 = 5.5$	$\bar{x}_2 = 6.0$	$\bar{x}_3 = 6.0$	$\bar{x}_1 = 15.5$	$\bar{x}_2 = 6.0$	$\bar{x}_3 = 6.0$
	$s_1^2 = 3.0$	$s_2^2 = 2.0$	$s_3^2 = 2.0$	$s_1^2 = 3.0$	$s_2^2 = 2.0$	$s_3^2 = 2.0$
Variance between samples	$ns_{\bar{x}}^2 = 4(0.0833) = 0.3332$			$ns_{\bar{x}}^2 = 4(30.0833) = 120.3332$		
Variance within samples	$s_p^2 = \dfrac{3.0 + 2.0 + 2.0}{3} = 2.3333$			$s_p^2 = \dfrac{3.0 + 2.0 + 2.0}{3} = 2.3333$		
F test statistic	$F = \dfrac{ns_{\bar{x}}^2}{s_p^2} = \dfrac{0.3332}{2.3333} = 0.1428$			$F = \dfrac{ns_{\bar{x}}^2}{s_p^2} + \dfrac{120.3332}{2.3333} = 51.5721$		

- Adding 10, *F* value changed from = 0.1428 to *F* = 51.5721.
- Variance between samples for part A is 0.3332, but for part B it is 120.3332 (indicating that the sample means in part B are farther apart).
- Variance within sample is 2.3333 in both parts; the variance within a sample isn't affected when we add a constant to every score.
- This illustrates that the *F* test statistic is very sensitive to sample *means*, even though it is obtained through two different estimates of the common population *variance*.

Example 4

Table 2 lists the body temperatures of 5 randomly selected subjects from each of 3 different age groups. Informal examination of the 3 sample means (97.940, 98.580, 97.800) seems to suggest that the 3 samples come from populations with means that are not significantly different. Test the claim that the 3 age-group populations have the same mean body temperature, $\alpha = 0.05$.

Example 4

Table 2: Body Temperature (^oF) Categorized by Age

18-20	21-29	30 and older
98.0	99.6	98.6
98.4	98.2	98.6
97.7	99.0	97.0
98.5	98.2	97.5
97.1	97.9	97.3
$n_1=5$	$n_2=5$	$n_3=5$
$x_1=97.940$	$x_2=98.580$	$x_3=97.800$
$s_1=0.568$	$s_2=0.701$	$s_3=0.752$

Example 4

- Step 1: The claim of equal mean body temperatures can be expressed in symbolic form as $\mu_1 = \mu_2 = \mu_3$.
- Step 2: If the original claim is false, then the 3 means are not all equal. (This is difficult to express in symbols, so we will express it verbally.)
- Step 3 : Because the null hypothesis must contain the condition of equality, we have

  H_0: $\mu_1 = \mu_2 = \mu_3$

  H_1 : The 3 means are not all equal

Example 4

- Step 4: The significant level is α = 0.05.

- Step 5: Because we are testing the claim that 3 or more means are equal, we use analysis of variance with an *F* test statistic.

- Step 6: for one-way analysis of variance with equal size, the test statistic (*F* = 1.8803) is calculated after first finding the value of $ns_{\bar{x}}^2$ and s_p^2

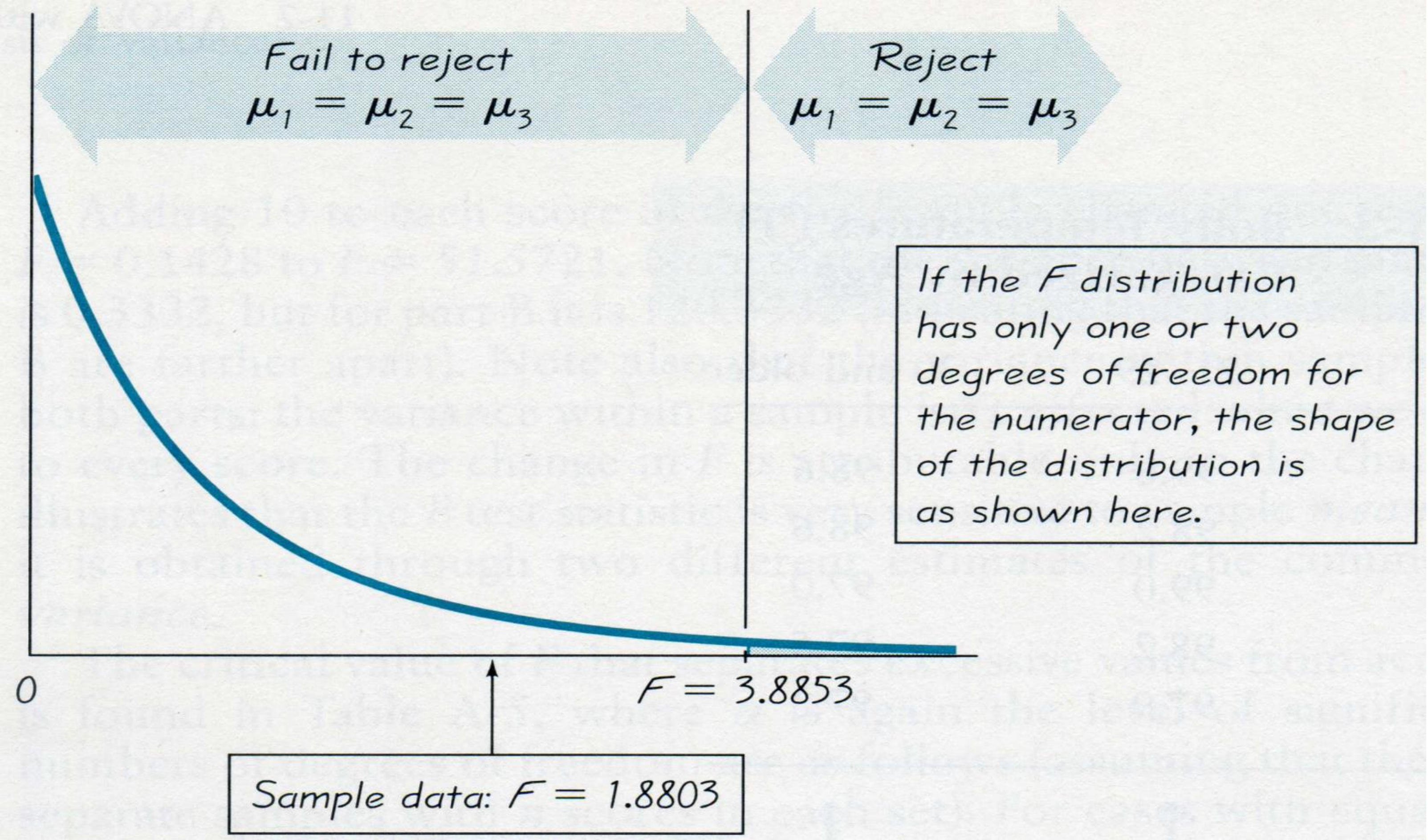
$$ns_{\bar{x}}^2 = 5(0.172933) = 0.864665$$

$$s_p^2 = \frac{0.568^2 + 0.701^2 + 0.752^2}{3} = 0.459843$$

$F_{\text{test statistic}} = 1.8803$

$F_{\text{critical value}} = 3.8853$

Example 4



Fail to reject
$\mu_1 = \mu_2 = \mu_3$
Reject
$\mu_1 = \mu_2 = \mu_3$
If the F distribution has only one or two degrees of freedom for the numerator, the shape of the distribution is as shown here.
0
F = 3.8853
Sample data: F = 1.8803

Example 4

- Step 7: Because the test statistic of F = 1.8803 does not fall in the critical region bounded by F = 3.8853, we fail to reject the null hypothesis of $\mu_1 = \mu_2 = \mu_3$.

- Step 8: There is not sufficient evidence to warrant rejection of the claim that the 3 population of different age groups have the same mean body temperature. Perhaps there really is a different, but the sample size is too small and/or the sample differences are not large enough to justify that condition. (We used small sizes of n = 5 to keep the calculation manageable.)

Exercise

- A student lives in a home with a solar electric system. At the same time each day, she collected voltage readings from a meter connected to the system and the results are listed in the accompanying table. Use a 0.05 significance level to test the claim that the mean voltage reading is the same under the three different types of day. Can we conclude that sunny days result in greater amounts of electrical energy?

Sunny Days	Cloudy Days	Rainy Days
13.5	12.7	12.1
13.0	12.5	12.2
13.2	12.6	12.3
13.9	12.7	11.9
13.8	13.0	11.6
14.0	13.0	12.2