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SCSI 1013 DISCRETE STRUCTURE




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## CHAPTER 4 – PART 2

# TREE

Semester 1 – 2017/2018

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## Introduction

**Definition 1.** A tree is a **connected undirected graph with no simple circuits.**

**Theorem 1.** An undirected graph is a tree if and only if there is a unique simple path between any two of its vertices.

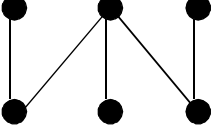
**Theorem 2 .** A tree with **m-vertices** has **m-1** edges

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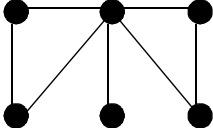
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## Which graphs are trees?

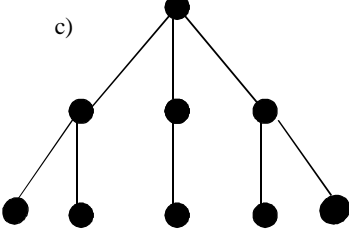
a)




b)



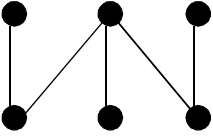
c)



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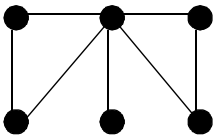
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## Solution



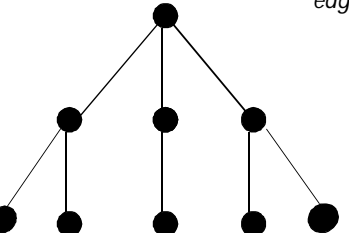
tree

vertices = 6  
edges = 5



Not a tree

vertices = 6  
edges = 7



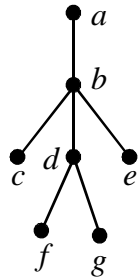
tree

vertices = 9  
edges = 8

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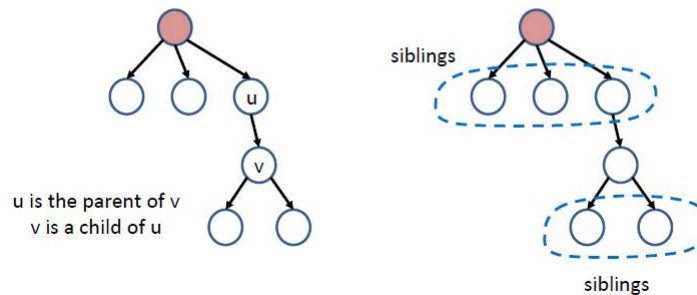
## Rooted tree

**Definition 2.** A **rooted tree** is a tree in which one vertex has been designed as the **root** and every edge is directed away from the root.



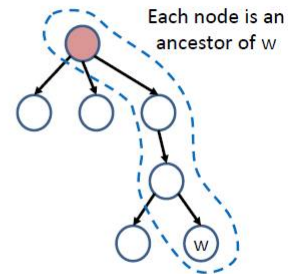
## Rooted Tree - Terminologies

- Each edge is from a **parent** to a **child**
- Vertices with the same parent are **siblings**



## Rooted Tree - Terminologies

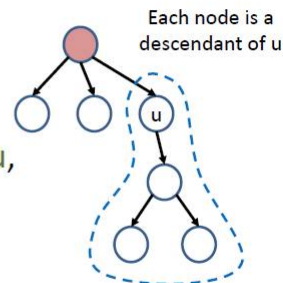
- The **ancestors** of a vertex  $w$  include all the nodes in the path from the root to  $w$
- The **proper ancestors** of a vertex  $w$  are the ancestors of  $w$ , but excluding  $w$



The whole part forms a path from root to  $w$

## Rooted Tree - Terminologies

- The **descendants** of a vertex  $u$  include all the nodes that have  $u$  as its ancestor
- The **proper descendants** of a vertex  $u$  are the descendants of  $u$ , but excluding  $u$
- The **subtree** rooted at  $u$  includes all the descendants of  $u$ , and all edges that connect between them



The whole part is the subtree rooted at  $u$

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# Rooted Tree - Terminologies

- Vertices with no children are called **leaves** ;  
Otherwise, they are called **internal nodes**
- If every internal node has no more than **m** children, the tree is called an **m-ary tree**
  - Further, if every internal node has exactly **m** children, the tree is a **full m-ary tree**

All internal nodes are colored

The tree is ternary (3-ary), but not full

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# Examples

$T_1$

full binary tree

$T_2$

full 3-ary tree


$T_3$

full 5-ary tree

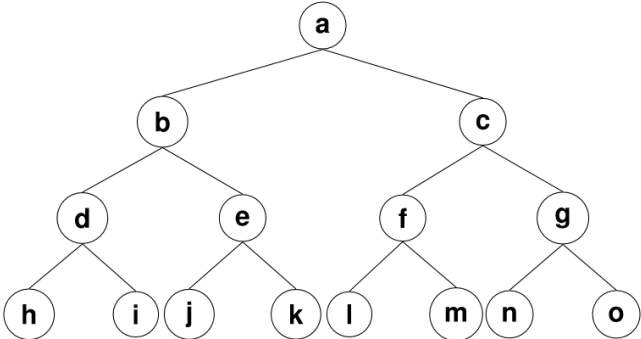
$T_4$

not full 3-ary tree

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## Exercise



Find:

- Ancestors of  $g$
- Descendants of  $g$
- Parent of  $e$
- Children of  $e$
- Sibling of  $h$

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## Properties of Trees

- Theorem 2 : A tree with  $n$  nodes has  $n-1$  edges
- Theorem 3 : A full  $m$ -ary tree with  $i$  internal vertices contains  $n = mi + 1$  vertices.

**Corollary:** A full  $m$ -ary tree with  $n$  vertices contains  $(n - 1) / m$  internal vertices, and hence  $n - (n - 1) / m = ((m - 1)n + 1) / m$  leaves

Corollary is a result in which the (usually short) proof relies heavily on a given theorem

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## Properties of Trees

Theorem 4 – A full  $m$ -ary tree with:

1.  $n$  vertices has  $i = \frac{(n-1)}{m}$  internal vertices and

$$l = \frac{(m-1)n + 1}{m} \text{ leaves}$$

## Properties of Trees

Theorem 4 – A full  $m$ -ary tree with:

2.  $i$  internal vertices has  $n = mi + 1$  vertices and

$$l = (m-1)i + 1 \text{ leaves}$$

Cont.

## Properties of Trees

Theorem – A full  $m$ -ary tree with

3.  $l$  leaves has  $n = \frac{(ml - 1)}{(m - 1)}$  vertices and

$$i = \frac{(l - 1)}{(m - 1)} \text{ internal vertices}$$

Cont.

## Example

Ex : Peter starts out a chain mail. Each person receiving the mail is asked to send it to four other people. Some people do this, and some don't

Now, there are 100 people who received the letter but did not send it out

Assuming no one receives more than one mail.  
How many people have sent the letter ?

17



## Solution

- The chain letter can be represented using **4-ary** tree. The internal vertices correspond to people who sent out the letter, and the leaves correspond to people who did not send it out. Since 100 people did not send out the letter, the number of leaves in this rooted tree is,  $l=100$ . The number of people have seen the letter is  $n=(4 \times 100 - 1) / (4 - 1) = 133$ . The number of internal vertices is  $133 - 100 = 33$ , people sent the letter.

## Exercise

- How many matches are played in a tennis tournament of 27 players?

## Exercise

Suppose 1000 people enter a chess tournament. Use a rooted tree model of the tournament to determine how many games must be played to determine a champion, if a player is eliminated after one loss and games are played until only one entrant has not lost. (Assume there are no ties.)

## Properties of Trees

- The **level** of a vertex  $v$  in a rooted tree is the length of the unique path from the root to this vertex.
- The level of the root is defined to be zero.
- The **height** of a rooted tree is the maximum of the levels of vertices.

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## Example

This height of the tree is 3

Level 0

Level 1

Level 2

Level 3

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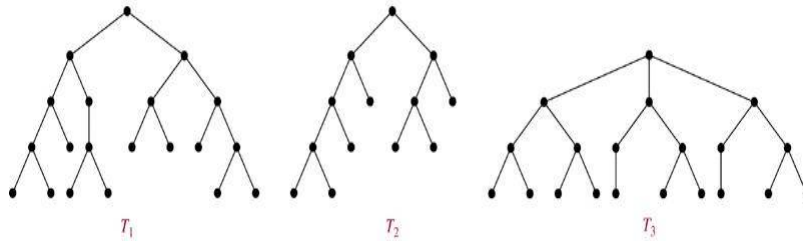
## Properties of Trees

- **Definition:** A rooted  $m$ -ary tree of height  $h$  is **balanced** if all leaves are at levels  $h$  or  $h-1$ .
- **Theorem.** There are at most  $m^h$  leaves in an  $m$ -ary tree of height  $h$ .

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## Example

Which of the rooted trees shown below are balanced?



**Sol.**  $T_1, T_3$

## Tree Traversal

### Universal Address Systems


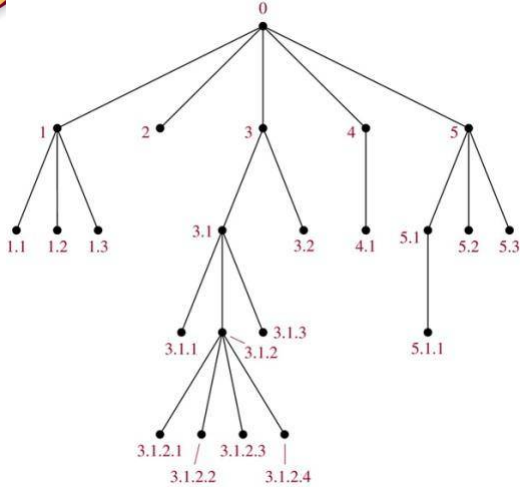
**Label vertices:**

1. root  $\rightarrow 0$ , its  $k$  children  $\rightarrow 1, 2, \dots, k$  (from left to right)
2. For each vertex  $v$  at level  $n$  with label  $A$ , its  $r$  children  $\rightarrow A.1, A.2, \dots, A.r$  (from left to right).

We can **totally order** the vertices using the lexicographic ordering of their labels in the universal address system.


$$x_1.x_2.\dots.x_n < y_1.y_2.\dots.y_m$$

if there is an  $i$ ,  $0 \leq i < n$ , with  $x_1 = y_1, x_2 = y_2, \dots, x_{i-1} = y_{i-1}$ , and  $x_i < y_i$ ; or if  $n < m$  and  $x_i = y_i$  for  $i = 1, 2, \dots, n$ .

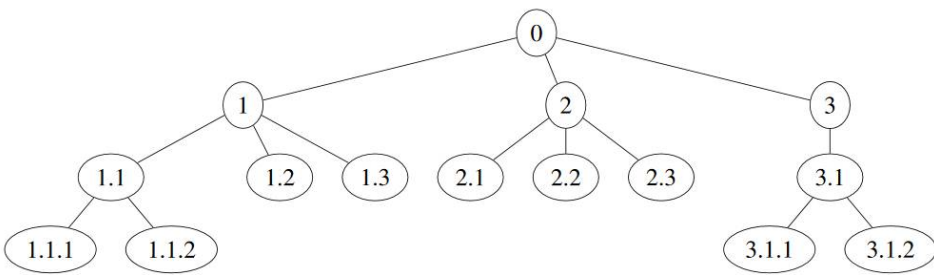



The lexicographic ordering is:  
 $0 < 1 < 1.1 < 1.2 < 1.3 < 2 < 3 < 3.1 < 3.1.1 < 3.1.2 < 3.1.2.1 < 3.1.2.2 < 3.1.2.3 < 3.1.2.4 < 3.1.3 < 3.2 < 4 < 4.1 < 5 < 5.1 < 5.1.1 < 5.2 < 5.3$

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## Exercise



Find the lexicographic ordering of the above tree.

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## Tree Traversal

- **Preorder**: root, left-subtree, right subtree
- **Inorder** – left subtree, root, right sub-tree
- **Post-order** : left subtree, right sub-tree, root

## Preorder Traversal

```

Procedure preorder( $T$ : ordered rooted tree)
 $r :=$  root of  $T$ 
list  $r$ 
for each child  $c$  of  $r$  from left to right
begin
     $T(c) :=$  subtree with  $c$  as its root
    preorder( $T(c)$ )
end

```

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## Preorder Traversal - Example

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## Exercise

Determine the order in which a preorder traversal visits the vertices of the given ordered rooted tree.

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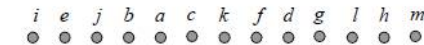
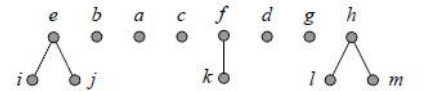
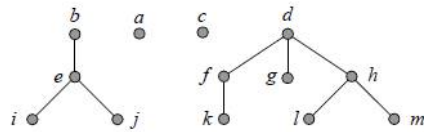
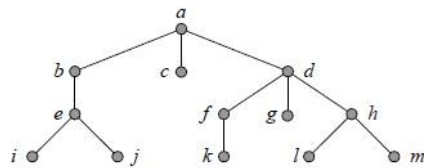
# Inorder Traversal

```


Procedure inorder(T: ordered rooted tree)
r := root of T
If r is a leaf then list r
else
begin
    l := first child of r from left to right
    T(l) := subtree with l as its root
    inorder(T(l))
    list r
    for each child c of r except for l from left to right
        T(c) := subtree with c as its root
        inorder(T(c))
end
    
```



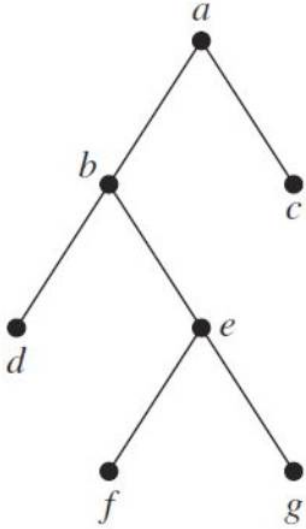
# Inorder Traversal -Example





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## Exercise




```

graph TD
  a((a)) --- b((b))
  a --- c((c))
  b --- d((d))
  b --- e((e))
  e --- f((f))
  e --- g((g))
  
```

Determine the order in which a inorder traversal visits the vertices of the given ordered rooted tree.

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## Postorder Traversal

**Procedure** *postorder*( $T$ : ordered rooted tree)

$r :=$  root of  $T$

**for** each child  $c$  of  $r$  from left to right

**begin**

$T(c) :=$  subtree with  $c$  as its root

*postorder*( $T(c)$ )

**end**

list  $r$

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## Postorder Traversal - Example

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## Exercise

Determine the order in which a postorder traversal visits the vertices of the given ordered rooted tree.

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## Exercise

```

graph TD
    a((a)) --- b((b))
    a --- c((c))
    b --- d((d))
    b --- e((e))
    c --- f((f))
    c --- g((g))
    c --- h((h))
    e --- i((i))
    e --- j((j))
    h --- k((k))
    h --- l((l))
    j --- m((m))
    j --- n((n))
    j --- o((o))
    l --- p((p))
        
```

Determine the order of preorder, inorder and postorder of the given rooted tree.

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## Exercise

```

graph TD
    a((a)) --- b((b))
    a --- c((c))
    a --- d((d))
    b --- e((e))
    b --- f((f))
    c --- g((g))
    c --- h((h))
    d --- i((i))
    d --- j((j))
    d --- k((k))
    f --- l((l))
    f --- m((m))
    f --- n((n))
    h --- o((o))
    h --- p((p))
    j --- q((q))
    j --- r((r))
    q --- s((s))
    q --- t((t))
    t --- u((u))
        
```

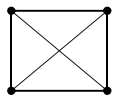
Determine the order of preorder, inorder and postorder of the given rooted tree.

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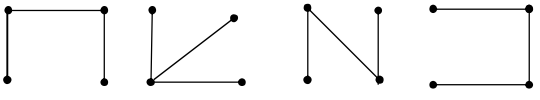
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## Spanning Trees

- A spanning tree is a simple graph that is a subgraph of  $G$  and **contains every vertex of  $G$**  and is a **tree**.



A connected undirected graph



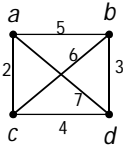
Four spanning trees of the graph

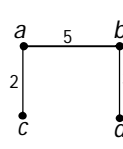
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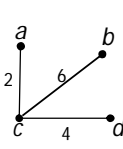
## Minimum Spanning Tree (MST)

- A Minimum Spanning Tree is a spanning tree on a weighted graph that has minimum total weight.
- Example

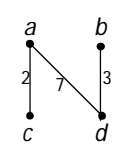




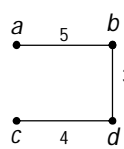
$T_1 = 10$



$T_2 = 12$




$T_3 = 12$

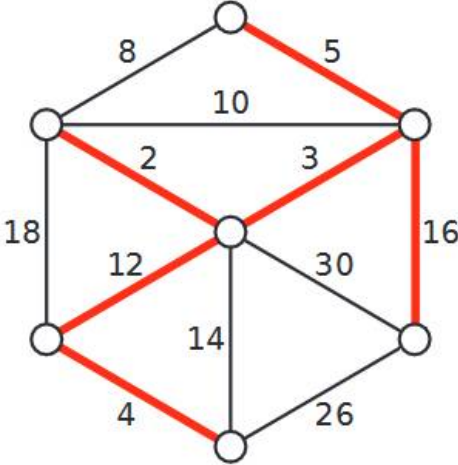


$T_4 = 12$

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## Minimum Spanning Tree (MST)



A weighted graph and its minimum spanning tree.

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## Muddy City Problem

Once upon a time there was a city that had no roads. Getting around the city was particularly difficult after rainstorms because the ground became very muddy. Cars got stuck in the mud and people got their boots dirty. The mayor of the city decided that some of the streets must be paved, but didn't want to spend more money than necessary because the city also wanted to build a swimming pool.

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## Muddy City Problem

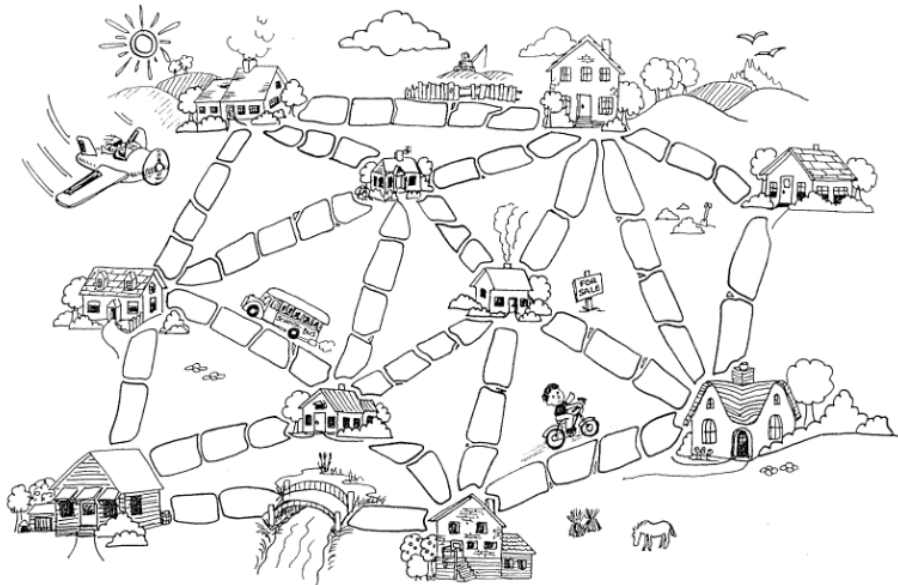
The mayor therefore specified two conditions:

1. Enough streets must be paved so that it is possible for everyone to travel from their house to anyone else's house only along paved roads, and
2. The paving should cost as little as possible.

Here is the layout of the city. The number of paving stones between each house represents the cost of paving that route. Find the best route that connects all the houses, but uses as few counters (paving stones) as possible.


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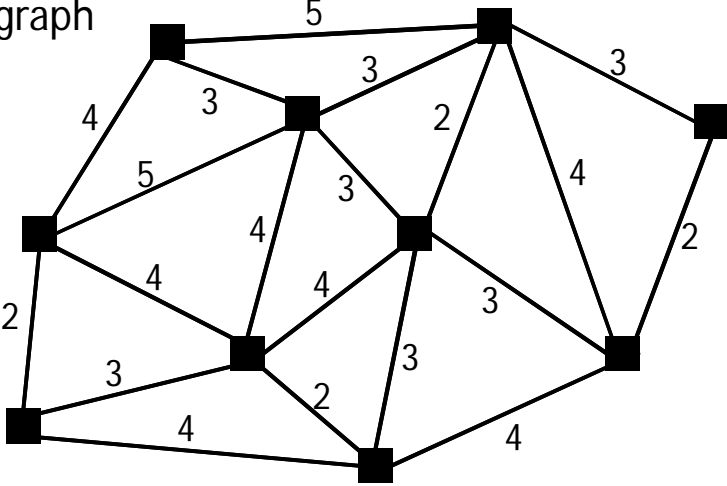
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# Muddy City Problem

The graph

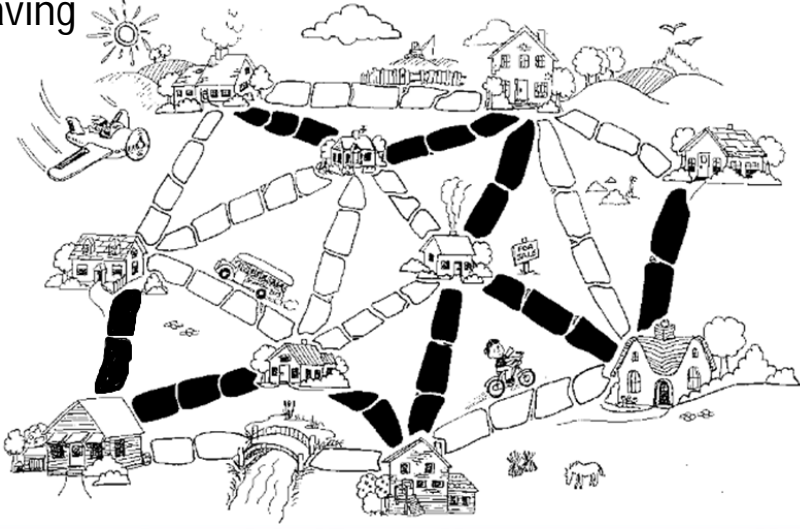


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# Muddy City Problem

The paving



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## Application of MST: Examples

- In the design of electronic circuitry, it is often necessary to make a set of pins electrically equivalent by wiring them together.
- Running cable TV to a set of houses. What's the least amount of cable needed to still connect all the houses?



## Finding MST

- Kruskal's algorithm: start with no nodes or edges in the spanning tree and repeatedly add the cheapest edge that does not create a cycle





## Kruskal algorithm

Procedure Kruskal ( $G$ : weighted connected undirected graph with  $n$  vertices)

$T :=$  empty graph

for  $i := 1$  to  $n-1$

begin

$e :=$  any edge in  $G$  with smallest weight that does not

form a simple circuit when added to  $T$

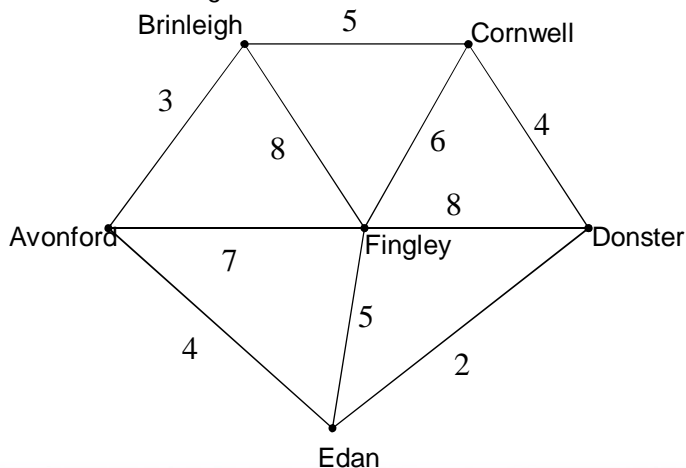
$T := T$  with  $e$  added

end ( $T$  is a minimum spanning tree of  $G$ )



## Example

A cable company want to connect five villages to their network which currently extends to the market town of Avonford. What is the minimum length of cable needed?



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## Example

We model the situation as a network, then the problem is to find the minimum connector for the network.

$\begin{array}{c}
 B \quad 5 \quad C \\
 \diagdown \quad \diagup \\
 A \quad 3 \quad 8 \quad 6 \quad D \\
 \diagup \quad \diagdown \quad \diagup \quad \diagdown \\
 E \quad 4 \quad 7 \quad 5 \quad 2 \quad F
 \end{array}$

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## Kruskal's Algorithm

List the edges in order of size:

ED	2
AB	3
AE	4
CD	4
BC	5
EF	5
CF	6
AF	7
BF	8
CF	8

$\begin{array}{c}
 B \quad 5 \quad C \\
 \diagdown \quad \diagup \\
 A \quad 3 \quad 8 \quad 6 \quad D \\
 \diagup \quad \diagdown \quad \diagup \quad \diagdown \\
 E \quad 4 \quad 7 \quad 5 \quad 2 \quad F
 \end{array}$

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## Kruskal's Algorithm

Select the shortest edge in the network

**ED 2**

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## Kruskal's Algorithm

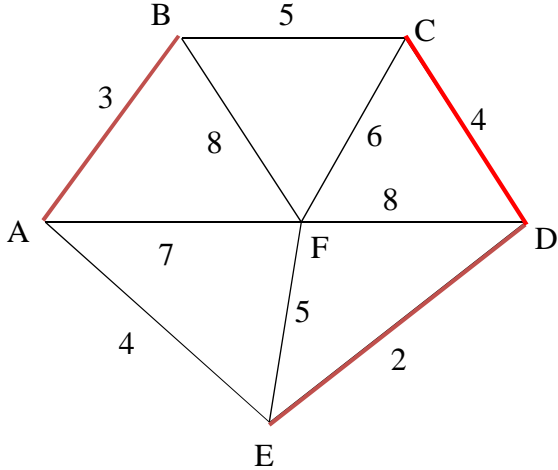
Select the next shortest edge which does not create a cycle

**ED 2**  
**AB 3**

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## Kruskal's Algorithm



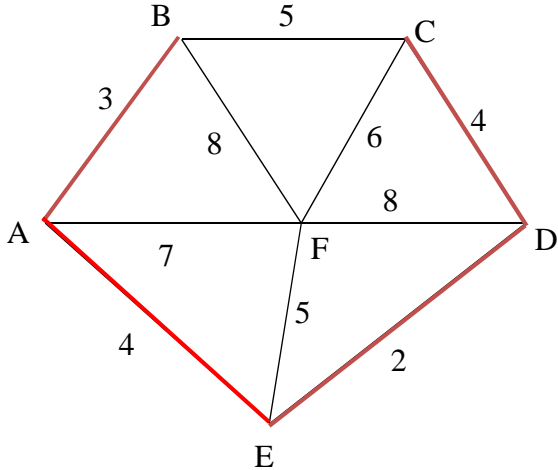
Select the next shortest edge which does not create a cycle

**ED 2**  
**AB 3**  
**CD 4 (or AE 4)**

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## Kruskal's Algorithm



Select the next shortest edge which does not create a cycle

**ED 2**  
**AB 3**  
**CD 4**  
**AE 4**

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## Kruskal's Algorithm

Select the next shortest edge which does not create a cycle

**ED 2**  
**AB 3**  
**CD 4**  
**AE 4**  
~~**BC 5**~~ – forms a cycle  
**EF 5**

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## Kruskal's Algorithm

All vertices have been connected.

The solution is

**ED 2**  
**AB 3**  
**CD 4**  
**AE 4**  
**EF 5**

Total weight of tree:  
18

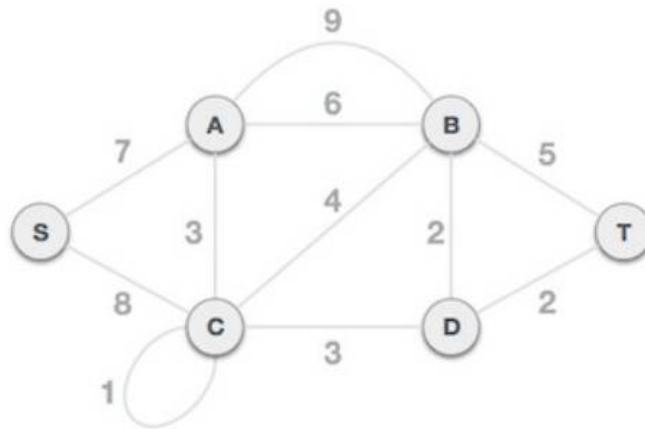
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## Kruskal's Algorithm

Important notes:

- The graph given should be a tree
  - Remove all loops (if any)
  - Remove all parallel edges
    - keep the one which has the least weight associated and remove all others

## Example



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## Example

- Remove all loops and parallel edges

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## Example

- Arrange all edges in their increasing order of weight

BD	DT	AC	CD	CB	BT	AB	SA	SC
2	2	3	3	4	5	6	7	8

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## Example

- Add the edge which has the least weightage

Next weight is 3, and associated edges are AC and CD

The least weight is 2 and edges involved are BD and DT

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## Example

Next weight is 4, and we observe that adding it will create a circuit in the graph. Thus, we ignore it.

We observe that edges with weight 5 and 6 also create circuits. We ignore them and move on.

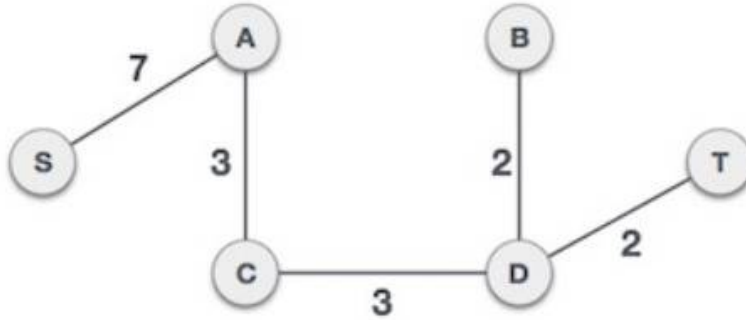
Now we are left with only one node to be added. Between the two least weighted edges available 7 and 8, we shall add the edge with weight 7.

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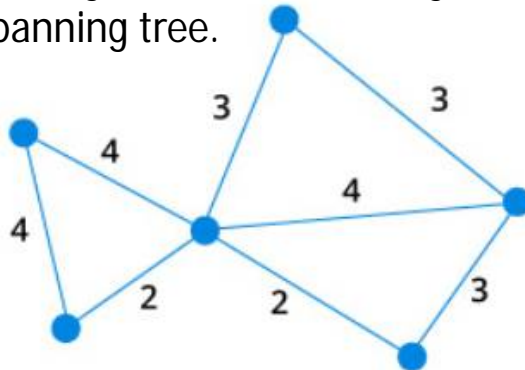
## Example

Now we have minimum spanning tree with total weight is 17.



## Exercise

Find the minimum spanning tree using Kraskal's Algorithm and give the total weight for the minimum spanning tree.



## Exercise

Find the minimum spanning tree using Kraskal's Algorithm and give the total weight for the minimum spanning tree.

