SCSI1013: Discrete Structures

CHAPTER 1

SET THEORY
[Part 1: Set & Subset]

Introduction

- Why are we studying sets
  - The concept of set is basic to all of mathematics and mathematical applications.
  - Serves as a basis of description of higher concept and mathematical reasoning
  - Set is fundamental in many areas of Computer Science.
    - For us, sets are useful to understand the principles of *counting and probability* theory

Sets

- The concept of set is basic to all of mathematics and mathematical applications.
- A set is a well-defined collection of distinct objects.
- These objects are called members or elements of the set.

Sets

- Well-defined means that we can tell for certain whether an object is a member of the collection or not.
- If a set is finite and not too large, we can describe it by listing the elements in it.

Example

- A is a set of all positive integers less than 10, $A=\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- B is a set of first 5 positive odd integers, $B=\{1, 3, 5, 7, 9\}$
- C is a set of vowels, $C=\{a, e, i, o, u\}$

Defining Sets

- This can be done by:
  - Listing ALL elements of the set within braces.
  - Listing enough elements to show the pattern then an ellipsis.
  - Use set builder notation to define “rules” for determining membership in the set.

Example

1. Listing ALL elements. $A = \{1, 2, 3, 4\}$ explicitly
2. Demonstrating a pattern. $\mathbb{N} = \{1, 2, 3, \ldots\}$ implicitly
3. Using set builder notation. $P = \{x | x \in \mathbb{R} \text{ and } x \notin \mathbb{C}\}$ implicitly

Sets

- A set is determined by its elements and not by any particular order in which the element might be listed.
- **Example**,

  A={1, 2, 3, 4},

  A might just as well be specified as

  {2, 3, 4, 1} or {4, 1, 3, 2}

Sets

- The elements making up a set are assumed to be distinct, we may have duplicates in our list, only one occurrence of each element is in the set.
- **Example**

  {*a, b, c, a, c*} → {*a, b, c*}

  {1, 3, 3, 5, 1} → {1, 3, 5}

Sets

- Use uppercase letters *A, B, C* … to denote sets, lowercase denote the elements of set.
- The symbol $\in$ stands for 'belongs to'
- The symbol $\notin$ stands for 'does not belong to'

Example

- $X=\{a, b, c, d, e\}$, $b \in X$ and $m \notin X$
- $A=\{\{1\}, \{2\}, 3, 4\}$, $\{2\} \in A$ and $1 \notin A$

Sets

- If a set is a large finite set or an infinite set, we can describe it by listing a property necessary for memberships
- Let *S* be a set, the notation,

  $A = \{x \mid x \in S, P(x)\}$ or $A = \{x \in S \mid P(x)\}$

  means that *A* is the set of all elements x of *S* such that x satisfies the property *P*.

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Example

- Let A={1, 2, 3, 4, 5, 6}, we can also write A as,

  $A=\{x \mid x \in Z,\ 0 < x < 7\}$

  if Z denotes the set of integers.

- Let $B=\{x \mid x \in Z, x>0\}$, B={1, 2, 3, 4, …}

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Example

The set of natural numbers: $\mathbb{N} = \{1, 2, 3, \ldots\}$
The set of integers: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
The set of positive integers: $\mathbb{Z}^{+} = \{1, 2, 3, \ldots\}$

The set of Rational Numbers (fractions): $1/2, 2/3, 5/7, etc \in \mathbb{Q}$

More formally: $\mathbb{Q} = \{ a/b \mid a, b \in \mathbb{R},\ b \neq 0 \}$

The set of Irrational Numbers: $\sqrt{2}, \pi$, or e are irrational

The Real numbers $= \mathbb{R} =$ the union of the rational numbers with the irrational numbers

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Some Symbols Used With Set Builder Notation

The standard form of notation for this is called "set builder notation".

For instance, $\{x \mid x$ is an odd positive integer$\}$ represents the set {1, 3, 5, 7, 9, …}

$\{x \mid x$ is an odd positive integer$\}$ is read as

"the set consisting of all x such that x is an odd positive integer".

The vertical bar, " | ", stands for "such that"

Other "short-hand" notation used in working with sets

"$\forall$" stands for "for every"
"$\exists$" stands for "there exists"
"$\bigcup$" stands for "union"
"$\bigcap$" stands for "intersection"
"$\subseteq$" stands for "is a subset of"
"$\subset$" stands for "is a (proper) subset of"
"$\not\subset$" stands for "is a not a (proper) subset of"
"$\varnothing$" stands for the "empty set"
"$\in$" stands for "is an element of"
"$\notin$" stands for "is not an element of"
"$\times$" stands for "cartesian cross product"
"$=$" stands for "is equal to"

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Subset

- If every element of A is an element of B, we say that A is a subset of B and write $A \subseteq B$.

  $A=B$, if $A \subseteq B$ and $B \subseteq A$

- The empty set ($\varnothing$) is a subset of every set.

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Example

A={1, 2, 3}

Subset of A,

$\varnothing$, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3} {1, 2, 3}

Note:
A is a subset of A

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Exercise

Answer true or false

a) $\{x\} \subseteq \{x\}$
b) $\{x\} \in \{x\}$
c) $\{x\} \in \{x, \{x\}\}$
d) $\{x\} \subseteq \{x, \{x\}\}$

Proper Subsets

If A⊆B and B contains an element that is not in A, then we say "A is a **proper subset** of B": A⊂B or B⊃A.

Formally: A⊆B means ∀x [x∈A → x∈B].

For all sets: A⊆A.

Note: If *A* is a subset of *B* and *A* does not equal B, we say that *A* is a proper subset of *B* (*A* ⊆ *B* and *A*≠*B* (*B* ⊄ *A*))

Example

- *A*={1, 2, 3}
- Proper subset of *A*,

  ∅, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}

Note:
A proper subset of a set *A* is a subset of *A* that is not equal to *A* ({1,2,3} ⊄ A)

Example

Let a set, *B*={1, 2, 3, 4, 5, 6} and the subset, *A* = {1, 2, 3}.

⇒ *A* is proper subset of *B*.

Example

A={a,b,c,d,e,f,g,h}
B={b,d,e}
C={a,b,c,d,e}
D={r,s,d,e}

- Proper subset of *A* ?? *B* and *C*

Empty Sets

The **empty set** ∅ or {} but not {∅}
is the set without elements.

Note:

- Empty set has no elements
- Empty set is a subset of any set
- There is exactly one empty set
- Properties of empty set:
  A ∪ ∅ = A, A ∩ ∅ = ∅
  A ∩ A' = ∅, A ∪ A' = U
  U' = ∅, ∅' = U

Example

∅ = $\{x \mid x$ is a real number and $x^2 = -3\}$

∅ = $\{x \mid x$ is positive integer and $x^3 < 0\}$

Equal Sets

The sets A and B are **equal** (A=B) if and only if each element of A is an element of B and vice versa.

Formally: A=B means $\forall x\ [x \in A \leftrightarrow x \in B]$.

Example

$A=\{a, b, c\}$, $B=\{b, c, a\}$, $A=B$

$C=\{1, 2, 3, 4\}$

$D=\{x \mid x$ is a positive integer and $2x < 10\}$,

$C=D$

Exercise

Determine whether each pair of sets is equal

a) {1, 2, 2, 3}, {1, 3, 2}
b) $\{x \mid x^2 + x = 2\}$, {1, -2}
c) $\{x \mid x$ is a real number and $0 < x \le 2\}$, {1,2}

Equivalent Sets

Two sets, A and B, are equivalent if there exists a one-to-one correspondence between them.

When we say sets "have the same size", we mean that they are equivalent.

Example

Set A: {A, B, C, D, E} and **Set** B: {1, 2, 3, 4, 5}

Note:
- An **equivalent set** is simply a **set** with an **equal** number of elements.
- The **sets** do not have to have the same exact elements, just the same number of elements.

Finite Sets

A set A is finite

if it is empty
or
if there is a natural number *n* such that set A is equivalent to {1, 2, 3, . . . *n*}.

Example

A= {1, 2, 3, 4}

B = {x | x is an integer, 1 ≤ x ≤ 4}

Note:
There exists a nonnegative integer n such that *A* has *n* elements (*A* is called a finite set with *n* elements)

Infinite Sets

A set *A* is infinite

if there is NOT a natural number *n* such that set *A* is equivalent to {1, 2, 3, . . . *n*}.

Infinite sets are uncountable.
Are all infinite sets equivalent?

A set is infinite if it is equivalent to a proper subset of itself!

Example

Z = {x| x is an integer}

or *Z* = {…, -3, -2, -1, 0, 1, 2, 3,…}

S={x| x is a real number and 1 ≤ x ≤ 4}

Example

C = {5, 6, 7, 8, 9, 10} ***(finite set)***

B = {x | x is an integer, 10 < x < 20} ***(finite set)***

D = {x | x is an integer, x > 0} ***(infinite set)***

Universal Set

- Sometimes we are dealing with sets all of which are subsets of a set *U*.
- This set *U* is called a universal set or a universe.
- The set *U* must be explicitly given or inferred from the context.

Universal Set

Typically we consider a set A
a part of a **universal set** $\mathcal{U}$,
which consists of all possible elements.
To be entirely correct we should say
$\forall x \in \mathcal{U}\ [x \in A \leftrightarrow x \in B]$
instead of
$\forall x\ [x \in A \leftrightarrow x \in B]$ for A=B.

Note that { x | 0<x<5} is can be ambiguous.
Compare { x | 0<x<5, x∈N } with { x | 0<x<5, x∈Q }

Example

- The sets A={1,2,3}, B={2,4,6,8} and C={5,7}
- One may choose U={1,2,3,4,5,6,7,8} as a universal set.
- Any superset of U can also be considered a universal set for these sets A, B, and C.

  For example, $U=\{x \mid x$ is a positive integer$\}$

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Cardinality of Set

- Let S be a finite set with n distinct elements, where $n \geq 0$.
- Then we write $|S|=n$ and say that the cardinality (or the number of elements) of S is n.
- **Example**

  A= {1, 2, 3}, $|A|=3$

  B= {a,b,c,d,e,f,g}, $|B|=7$

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Exercise

- If M is finite, determine the $|M|$
  - If M= {1, 2, 3, 4}
  - If M = {4, 4, 4}
  - If M= {}
  - If M= {ø,{ø}, {ø, {ø}}}

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Power Set

- The set of all subsets of a set A, denoted $P(A)$, is called the power set of A.

  $P(A)= \{X \mid X \subseteq A\}$

- If $|A|=n$, then $|P(A)| = 2^n$

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Example

- A={1,2,3}
- The power set of A,

  $P(A)$= {∅, {1}, {2}, {3}, {1,2}, {1,3}, {2,3}, {1,2,3} }

- Notice that $|A| = 3$, and $|P(A)| = 2^3 = 8$

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Exercise

- Let X= {1, 2, 2, {1}, a}
- Find:
  - $|X|$
  - Proper subset of X
  - Power set of X

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Exercise

- List the member of *P*({a, b, c, d}). Which are proper subset of {a, b, c, d}?

Summary

How to Think of Sets

The elements of a set do not have an ordering, hence {a,b,c} = {b,c,a}

The elements of a set do not have multitudes, hence {a,a,a} = {a,a} = {a}

All that matters is: "Is x an element of A or not?"

The size of A is thus the number of *different* elements