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SCSI1013: Discrete Structures


# CHAPTER 2

[Part 2]

## FUNCTIONS

Semester (1) : 2015/2016

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# FUNCTION

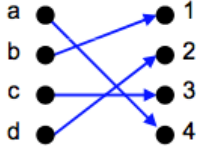
- Let **X** and **Y** are nonempty sets.
- A **function** ( $f$ ) from **X** to **Y** is a relation from **X** to **Y** having a properties:
  - The domain of  $f$  is **X**
  - If  $(x, y), (x, y') \in f$ , then  $y = y'$

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## Relations vs Functions

- Not all relations are functions
- But consider the following function:



```

graph LR
  a((a)) --> 1((1))
  b((b)) --> 2((2))
  c((c)) --> 3((3))
  d((d)) --> 4((4))
  
```

- All functions are relations!

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## Relations vs Functions

### When to use which?

- A function is used when you need to obtain a **SINGLE** result for any element in the domain
  - Example: sin, cos, tan
- A relation is when there are multiple mappings between the domain and the co-domain
  - Example: students enrolled in multiple courses


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## Domain, Co-domain, Range

- A function from  $\mathbf{X}$  to  $\mathbf{Y}$  is denoted,  $f: \mathbf{X} \rightarrow \mathbf{Y}$
- The **domain** of  $f$  is the set  $\mathbf{X}$ .
- The set  $\mathbf{Y}$  is called the **co-domain** or target of  $f$ .
- The set  $\{y \mid (x,y) \in f\}$  is called the **range**.

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## Example

Given the relation,  $f = \{ (1,a), (2,b), (3,a) \}$  from  $\mathbf{X} = \{ 1, 2, 3 \}$  to  $\mathbf{Y} = \{ a, b, c \}$  is a function from  $\mathbf{X}$  to  $\mathbf{Y}$ . State the domain and range.

Solution:

- ✓ The domain of  $f$  is  $\mathbf{X}$
- ✓ The range of  $f$  is  $\{a, b\}$

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## Example (cont.)

- $f = \{ (1,a), (2,b), (3,a) \}$

domain  $X$        $f$        $Y$  codomain

range  $\{a,b\} \subseteq Y$

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## Example

- The relation,  $R = \{ (1,a), (2,a), (3,b) \}$  from  $X = \{1, 2, 3, 4\}$  to  $Y = \{a, b, c\}$  is NOT a function from  $X$  to  $Y$ .
- The domain of  $R$ ,  $\{1,2,3\}$  is not equal to  $X$ .

$R = \{ (1,a), (2,a), (3,b) \}$

There is no arrow from 4

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## Example

- The relation,  $R = \{(1,a), (2,b), (3,c), (1,b)\}$  from  $X = \{1, 2, 3\}$  to  $Y = \{a, b, c\}$  is NOT a function from  $X$  to  $Y$
- $(1,a)$  and  $(1,b)$  in  $R$  but  $a \neq b$ .

$R = \{(1,a), (2,b), (3,c), (1,b)\}$

There are 2 arrows from 1

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## Notation of function: $f(x)$

- For the function,  $f = \{(1,a), (2,b), (3,a)\}$
- We may write:  

$$f(1)=a, f(2)=b, f(3)=a$$
- Notation  $f(x)$  is used to define a function.

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## Example

- Defined:  $f(x) = x^2$
- $f(2) = 4, f(-3.5) = 12.25, f(0) = 0$
- $f = \{(x, x^2) \mid x \text{ is a real number}\}$

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## One-to-One Function

- A function  $f$  from  $X$  to  $Y$ , is said one-to-one (or injective) if for each  $y \in Y$ , there is at most one  $x \in X$ , with  $f(x) = y$ .
- For all  $x_1, x_2$ , if  $f(x_1) = f(x_2)$ , then  $x_1 = x_2$ .  

$$\forall x_1 \forall x_2 ((f(x_1) = f(x_2)) \rightarrow (x_1 = x_2))$$

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## Example

- The function,  $f = \{ (1,b), (3,a), (2,c) \}$  from  $X = \{ 1, 2, 3 \}$  to  $Y = \{ a, b, c, d \}$  is **one-to-one**.

Each element in  $Y$  has at most one arrow pointing to it

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## Example (cont.)

- The function,  $f = \{ (1,a), (2,b), (3,a) \}$  from  $X = \{ 1, 2, 3 \}$  to  $Y = \{ a, b, c \}$  is **NOT** one-to-one.
- $f(1) = a = f(3)$

$f = \{ (1,a), (2,b), (3,a) \}$

$a$  has 2 arrows pointing to it

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## Example


Show that the function,

$$f(n) = 2n+1$$

on the set of positive integers is one-to-one.

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## Solution

- For all positive integer,  $n_1$  and  $n_2$  if  $f(n_1) = f(n_2)$ , then  $n_1 = n_2$ .
- Let,  $f(n_1) = f(n_2)$ ,  $f(n) = 2n+1$   
then  $2n_1 + 1 = 2n_2 + 1$       (-1)  
 $2n_1 = 2n_2$       (+2)  
 $n_1 = n_2$
- This shows that  $f$  is one-to-one.

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## Example

- ◆ Show that the function,  
$$f(n) = 2^n - n^2$$
from the set of positive integers to the set of integers is NOT one-to-one.


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## Solution

- ◆ Need to find 2 positive integers,  $n_1$  and  $n_2$   
 $n_1 \neq n_2$  with  $f(n_1) = f(n_2)$ .
- ◆ trial and error,  
$$f(2) = f(4)$$
- ◆  $f$  is not one-to-one.

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## Onto Function

- If  $f$  is a function from  $\mathbf{X}$  to  $\mathbf{Y}$  and the range of  $f$  is  $\mathbf{Y}$ ,  $f$  is said to be **onto**  $\mathbf{Y}$  (or an onto function or a surjective function)
- For every  $y \in \mathbf{Y}$ , there exists at least one  $x \in \mathbf{X}$  such that  $f(x) = y$

$$\forall y \in \mathbf{Y} \exists x \in \mathbf{X} (f(x) = y)$$

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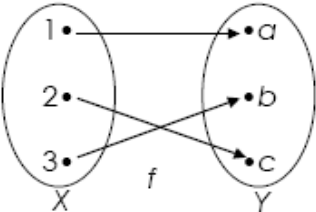


## Example

- The function,  $f = \{ (1,a), (2,c), (3,b) \}$  from  $\mathbf{X} = \{ 1, 2, 3 \}$  to  $\mathbf{Y} = \{ a, b, c \}$  is **one-to-one** and **onto**  $\mathbf{Y}$ .

•  $f = \{ (1,a), (2,c), (3,b) \}$

**One-to-one**  
Each element in  $\mathbf{Y}$  has at most one arrow



**Onto**  
Each element in  $\mathbf{Y}$  has at least one arrow pointing to it

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## Example

- The function,  $f = \{ (1,b), (3,a), (2,c) \}$  is **not onto**  $Y = \{a, b, c, d\}$

◆  $f = \{ (1,b), (3,a), (2,c) \}$

**not onto**  
no arrow pointing to  $d$

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## Bijection Function

- A function,  $f$  is called **one-to-one correspondence** (or bijective/bijection) if  $f$  is both **one-to-one** and **onto**.
- Example:

◆  $f = \{ (1,\alpha), (2,\theta), (3,\beta) \}$

**One-to-one and onto Y -bijection**

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## Exercise

Determine which of the relations  $f$  are functions from the set  $X$  to the set  $Y$ . In case any of these relations are functions, determine if they are one-to-one, onto  $Y$ , and/or bijection.

- a)  $X = \{-2, -1, 0, 1, 2\}$ ,  $Y = \{-3, 4, 5\}$  and  
 $f = \{(-2, -3), (-1, -3), (0, 4), (1, 5), (2, -3)\}$
- b)  $X = \{-2, -1, 0, 1, 2\}$ ,  $Y = \{-3, 4, 5\}$  and  
 $f = \{(-2, -3), (1, 4), (2, 5)\}$
- c)  $X = Y = \{-3, -1, 0, 2\}$  and  
 $f = \{(-3, -1), (-3, 0), (-1, 2), (0, 2), (2, -1)\}$

## Inverse Function

- Let  $f: X \rightarrow Y$  be a function.
- The **inverse** relation  $f^{-1} \subseteq Y \times X$  is a function from  $Y$  to  $X$ , if and only if  $f$  is both one-to-one and onto  $Y$ .



**Example**

$f = \{(1, a), (2, c), (3, b)\}$        $f^{-1} = \{(a, 1), (c, 2), (b, 3)\}$

$X$                        $Y$                        $Y$                        $X$

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**Example**

- The function,  $f(x) = 9x + 5$  for all  $x \in R$  ( $R$  is the set of real numbers).
- This function is both one-to-one and onto.
- Hence,  $f^{-1}$  exists.
- Prove:

Let  $(y, x) \in f^{-1}, f^{-1}(y) = x$

$(x, y) \in f, \quad y = 9x + 5$   
 $x = (y-5)/9$   
 $f^{-1}(y) = (y-5)/9$

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## Exercise

- Find each inverse function.
  - a)  $f(x) = 4x + 2, x \in R$
  - b)  $f(x) = 3 + (1/x), x \in R$

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## Composition

- Suppose that  $g$  is a function from  $X$  to  $Y$  and  $f$  is a function from  $Y$  to  $Z$ .
- The **composition** of  $f$  with  $g$ ,  

$$f \circ g$$
 is a function  

$$(f \circ g)(x) = f(g(x))$$
 from  $X$  to  $Z$ .

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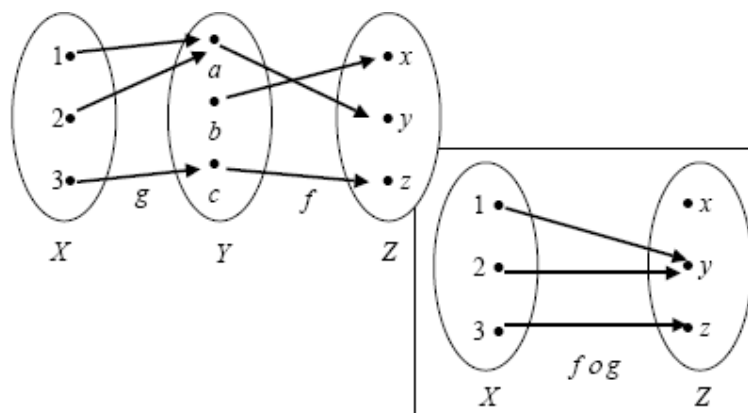
## Example


- Given,  $g = \{ (1,a), (2,a), (3,c) \}$   
 a function from  $\mathbf{X} = \{1, 2, 3\}$  to  $\mathbf{Y} = \{a, b, c\}$   
 and,  
 $f = \{ (a,y), (b,x), (c,z) \}$   
 a function from  $\mathbf{Y}$  to  $\mathbf{Z} = \{x, y, z\}$ .

- The **composition** function from  $\mathbf{X}$  to  $\mathbf{Z}$  is the function

$$f \circ g = \{ (1,y), (2,y), (3,z) \}$$

## Example (cont.)






## Example

$f(x) = \log_3 x$  and  $g(x) = x^4$

- $f(g(x)) = \log_3(x^4)$
- $g(f(x)) = (\log_3 x)^4$
- Note:  $f \circ g \neq g \circ f$

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## Example


$f(x) = \frac{1}{5}x$        $g(x) = x^2 + 1$

$$(g \circ f)(x) = g(f(x)) = g\left(\frac{x}{5}\right)$$

$$= \left(\frac{x}{5}\right)^2 + 1 = \frac{x^2}{25} + 1$$

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## Example

- Composition sometimes allows us to decompose complicated functions into simpler functions.
- example  $f(x) = \sqrt{\sin 2x}$

$$g(x) = \sqrt{x} \quad h(x) = \sin x \quad w(x) = 2x$$

$$f(x) = g(h(w(x)))$$


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## Exercise

- Let  $f$  and  $g$  be functions from the positive integers to the positive integers defined by the equations,  $f(n) = n^2$ ,  $g(n) = 2^n$
- Find the compositions
  - a)  $f \circ f$
  - b)  $g \circ g$
  - c)  $f \circ g$
  - d)  $g \circ f$

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## Recursive Algorithms

- A recursive procedure is a procedure that invokes itself.
- A recursive algorithm is an algorithm that contains a recursive procedure.
- Recursion is a powerful, elegant and natural way to solve a large class of problems.

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## Example

Factorial problem

- If  $n \geq 1$ ,  


$$n! = n(n-1) \Rightarrow 2 \times 1$$
 and  $0! = 1$
- Notice that, if  $n \geq 2$ ,  $n$  factorial can be written as,  

$$n! = n(n-1)(n-2) \Rightarrow 2 \times 1 \times 0$$

$$= n(n-1)!$$

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## Example

- $n=5$
- $5!$

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5 \cdot 4!$$
$$4! = 4 \cdot 3!$$
$$3! = 3 \cdot 2!$$


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## Recursive Algorithm for Factorial

- Input:  $n$ , integer  $\geq 0$
- Output:  $n!$
- Factorial ( $n$ ) {  
    if ( $n=0$ )  
        return 1  
    return  $n \cdot \text{factorial}(n-1)$   
}

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## Example

- Fibonacci sequence,  $f_n$ 
  - $f_1 = 1$
  - $f_2 = 1$
  - $f_n = f_{n-1} + f_{n-2}$ , for  $n \geq 3$

1, 1, 2, 3, 5, 8, 13, ....

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### Recursive algorithm for Fibonacci Sequence :

- Input:  $n$
- Output:  $f(n)$
- $f(n)$  {
  - if ( $n=1$  or  $n=2$ )
  - return 1
  - return  $f(n-1) + f(n-2)$

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Example

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Example- Solution

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