SCSI1013: Discrete Structures

CHAPTER 2 – Part 3

Recurrence Relation

Definition

- A **recurrence relation** is an equation that defines a sequence $\{a_n, a_{n+1}, \ldots\}$ based on a rule that gives the next term as a function of one or more of the previous term in the sequence for all integers n with $n \geq n_0$, ( ) $a_0, a_1, \ldots$
- A sequence is called a **solution** of a recurrence relation if it terms satisfy the recurrence relation.

Simple Recurrence Relation

The simplest form of a **recurrence relation** is the case where the next term depends only on the immediately previous term.

3, 8, 13, 18, 23, . . .

The above sequence shows a pattern, given initial condition, $a_1 = 3$:

3, 3+5, 8+5, 13+5, 18+5, . . .
a_1 , a_2, a_3, a_4, a_5, . . .

generated from an equation:

$$a_n = a_{n-1} + 5, \qquad n \geq 2$$

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n_{th} term of a sequence

Recurrence relation can be used to compute any n-*th* term of the sequence, given the initial condition, a_0:

$$a_n = a_{n-1} + 5\ ,\ n \geq 2$$

3, 8, 13, 18, 23, **28,** $a_{n-1} + 5$, . . .

$a_2 = a_1 + 5$, 3 + 5 = 8
$a_3 = a_2 + 5$, 8 + 5 = 13
$a_4 = a_3 + 5$, 13 + 5 = 18
⋮
$a_6 = a_5 + 5$, 23 + 5 = 28

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Example 1

Consider the following sequence:

3, 9, 27, 81, 243, . . .

The above sequence shows a pattern:

$3^1, 3^2, 3^3, 3^4, 3^5, \ldots$

$a_1, a_2, a_3, a_4, a_5, \ldots$

Recurrence relation is defined by:

$$a_n = 3^n, n \geq 1$$

Example 2

Given initial condition, $a_0 = 1$ and recurrence relation:

$$\boldsymbol{a_n = 1 + 2a_{n-1} \ , n \geq 1}$$

First few sequence are:

$a_1 = 1 + 2\,(1) = 3$
$a_2 = 1 + 2(3) = 7$
$a_3 = 1 + 2(7) = 15$

1, 3, 7, 15, 31, 63, ...

Example 3

Given initial condition, a_0 = 1, a_1 = 2 and recurrence relation:

$$a_n = 3(a_{n-1} + a_{n-2}),\ n \geq 2$$

First few sequence are:

$a_2 = 3(2 + 1) = 9$
$a_3 = 3(9 + 2) = 33$
$a_4 = 3(33 + 9) = 126$

1, 2, 9, 33, 126, 477, 1809, 6858, 26001,...

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Exercise

1) $a_n = 6a_{n-1} - 9a_{n-2}$, where $a_0 = 2$, $a_1 = 3$
2) $a_n = 2a_{n-1} - a_{n-2}$, where $a_0 = 5$, $a_1 = 3$

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Example 4

For a photo shoot, the staff at a company have been arranged such that there are 10 people in the front row and each row has 7 more people than in the row in front of it. Find the recurrence relation and compute number of staff in the first 5 rows.

Example 4 -Solution

Notice that the difference between the number of people in successive rows is a constant amount.

This means that the n_{th} term of this sequence can be found using:

$$a_n = a_{n-1} + 7, \quad n \geq 2 \text{ with } a_1 = 10$$

Example 4 -Solution

Number of staff in the first 5 rows:

$a_1 = 10$,
$a_2 = a_1 + 7$, $10 + 7 = 17$
$a_3 = a_2 + 7$, $17 + 7 = 24$
$a_4 = a_3 + 7$, $24 + 7 = 31$
$a_5 = a_4 + 7$, $31 + 7 = 38$

10, 17, 24, 31, 38

Example 5

Find a recurrence relation and initial condition for

1, 5, 17, 53, 161, 485, ...

Solution:
Look at the differences between terms:

4, 12, 36, 108, ...

The differences in the sequence is growing by a factor of 3

Example 5 -Solution

However the original sequence is not.

1(3)=**3**, 5(3)=**15**,17(3)=**51**, . . .

1, 5, 17, 53, 161, 485, ...

It appears that we always end up with 2 less than the next term.

So, the recurrence relation is defined by:

$\boldsymbol{a_n = 3(a_{n-1}) + 2,\ n \geq 1}$, **with initial condition,** $\boldsymbol{a_0 = 1}$

Example 6

A depositor deposits RM 10,000 in a savings account at a bank yielding 5% per year with interest compounded annually. How much money will be in the account after 30 years? Let P_n denote the amount in the account after n years.

Example 6 - Solution

Derive the following **recurrence relation**:

$$P_n = P_{n-1} + 0.05P_{n-1} = 1.05P_{n-1}$$

Where, P_n = Current balance and P_{n-1} = Previous year balance and 0.05 is the compounding interest.

Example 6 - Solution

Initial condition, P_0 = 10,000. Then,

$P_1 = 1.05P_0$
$P_2 = 1.05P_1 = (1.05)^2P_0$
$P_3 = 1.05P_2 = (1.05)^3P_0$
...
$P_n = 1.05P_{n-1} = (1.05)^nP_0$,

now we can use this formula to calculate n_{th} term without iteration

Example 6 - Solution

Let us use this formula to find P_{30} under the initial condition $P_0 = 10,000$:

$P_{30} = (1.05)^{30}(10,000) = 43,219.42$

After 30 years, the account contains RM 43,219.42.

Exercise

Consider the following sequence:

1, 5, 9, 13, 17

Find the recurrence relation that defines the above sequence.

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Exercise

A basketball is dropped onto the ground from a height of 15 feet. On each bounce, the ball reaches a maximum height 55% of its previous maximum height.

a)Write a recursive formula, a_n , that completely defines the height reached on the n_{th} bounce, where the first term in the sequence is the height reached on the ball's first bounce.

b)How high does the basketball reach after the 4_{th} bounce? Give your answer to two decimal places.