CHAPTER 3

COUNTING METHODS

[Part 1]

Basic Counting Principles

- Counting principle is all about choices we might make given many possibilities.
- It is used to find the number of possible outcomes.
- It provides a basis of computing probabilities of discrete events.

Basic Counting Principles

Some sample of counting problems:

- Problem 1 - How many ways are there to seat *n* couples at a round table, such that each couple sits together?
- Problem 2- How many ways are there to express a positive integer *n* as a sum of positive integers?

Basic Counting Principles

- Problem 3 - There are three boxes containing books. The first box contains 15 mathematics books by different authors, the second box contains 12 chemistry books by different authors, and the third box contains 10 computer science books by different authors. A student wants to take a book from one of the three boxes. In how many ways can the student do this?

Basic Counting Principles

Problem 4 - The department will award a free computer to either a CS student or a CS professor. How many different choices are there, if there are 530 students and 15 professors?

Problem 5 - A scholarship is available, and the student to receive this scholarship must be chosen from the Mathematics, Computer Science, or the Engineering Department. How many different choices are there for this student if there are 38 qualified students from the Mathematics Department, 45 qualified students from the Computer Science Department and 27 qualified students from the Engineering Department?

Basic Counting Principles

- There are a number of basic principles that we can use to solve such problems:
  1) **Addition Principle**
  2) **Multiplication Principle**

Addition Principle

- Suppose that tasks T_1, T_2,… T_k can be done in n_1, n_2,… n_k ways, respectively.

- If all these tasks are **independent** of each other, then the number of ways to do one of these tasks is $n_1 + n_2 + \ldots + n_k$

Basic Counting Principles

- If a task can be done in n_1 ways and a second task in n_2 ways, and if these two tasks cannot be done at the same time, then there are $n_1 + n_2$ ways to do either task.

Example 1

We want to find the number of integers between 5 and 50 that end with 1 or 7.

Solution:

- Let T denote this task. We divide T into the following tasks.
  - T_1: find all integers between 5 and 50 that end with 1.
  - T_2: find all integers between 5 and 50 that end with 7.

- T_1: find all integers between 5 and 50 that end with 1.

  ➔11, 21, 31, 41 => 4 ways

- T_2: find all integers between 5 and 50 that end with 7.

  ➔7, 17, 27, 37, 47 => 5 ways

.: $n_1 + n_2 = 4 + 5 = 9$ ways.

Example 2

We want to find the number of integers between 4 and 100 that end with 3 or 5.

Solution:

Let *T* denote this task. We divide *T* into the following tasks.

- T_1: find all integers between 4 and 100 that end with 3.

- T_2: find all integers between 4 and 100 that end with 5.

- T_1: find all integers between 4 and 100 that end with 3.

  => 13, 23, 33, 43, 53, 63, 73, 83, 93

  => 9 integers

- T_2: find all integers between 4 and 100 that end with 5.

  => 5, 15, 25, 35, 45, 55, 65, 75, 85, 95

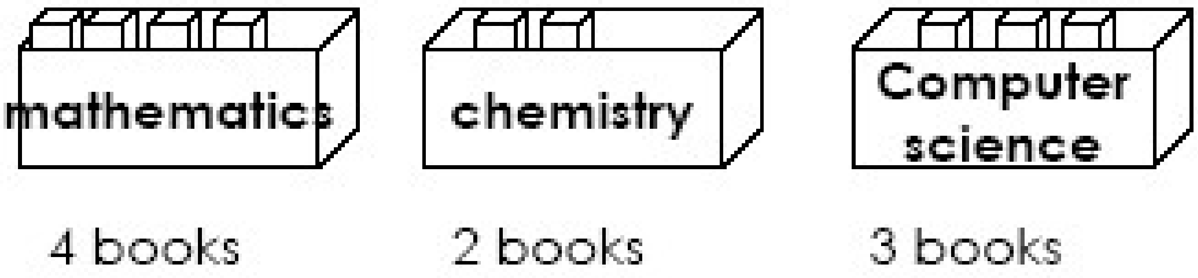
  => 10 integers

- Task T_1 can be done in 9 ways.
- Task T_2 can be done in 10 ways.
- The number of ways to do one of these tasks is,

$$9+10=19$$

- Task T can be completed in 19 ways.

Example 3



- A student wants to take a book from one of the three boxes.
- In how many ways can the students do this?

- T_1: choose a mathematics book
  - 4 ways
- T_2: choose a chemistry book
  - 2 ways
- T_3: choose a computer science book.
  - 3 ways
- The number of ways to do one of these tasks is,
  - 4+2+3=9

Exercise

- There are 8 male students and 21 female students in Discrete Structure class. Among all of them, 7 students are Chinese and the rest are Malay.
  - ➢ In how many ways can we select 1 student - a boy or a girl?
  - ➢ In how many ways can we select 1 student - a Chinese or a Malay?

Exercise - Solution

- In how many ways can we select 1 student - a boy or a girl?
  - To select a boy
    - 8 ways
  - To select a girl
    - 21 ways
  - The number of ways
    - **8+21=29 ways**

- In how many ways can we select 1 student - a Chinese or a Malay?
  - To select a Chinese
    - 7 ways
  - To select a Malay
    - 22 ways
  - The number of ways
    - **7+22=29 ways**

Multiplication Principle

- A task T can be completed in k successive steps.

  Step 1 can be completed in n_1 different ways.

  Step 2 can be completed in n_2 different ways.

  Step *k* can be completed in n_k different ways.

- Then the task T can be completed in $n_1.n_2.... n_k$ different ways.

Suppose that a procedure can be broken down into two successive tasks. If there are n_1 ways to do the first task and n_2 ways to do the second task after the first task has been done, then there are $n_1 n_2$ ways to do the procedure.

Example 4

- Morgan is a lead actor in a new movie. She needs to shoot a scene in the morning in studio A and an afternoon scene in studio C. She looks at the map and finds that there is no direct route from studio A to studio C. Studio B is located between studios A and C. Morgan's friends Brad and Jennifer are shooting a movie in studio B. There are three roads, say A_1, A_2, and A_3, from studio A to studio B and four roads, say B_1, B_2, B_3, and B_4, from studio B to studio C. In how many ways can Morgan go from studio A to studio C and have lunch with Brad and Jennifer at Studio B?

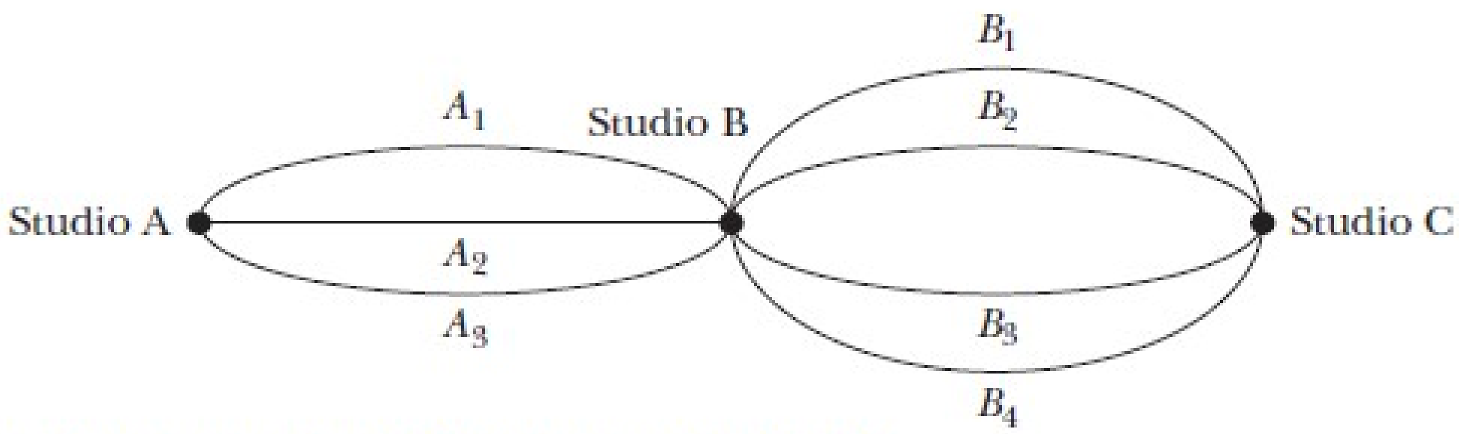


FIGURE 7.1 Routes from studio A to studio C

- There are 3 ways to go from studio *A* to studio *B* and 4 ways to go from studio *B* to studio *C*.
  - T1 = 3
  - T2 = 4
- The number of ways to go from studio *A* to studio *C* via studio *B* is $3 \times 4 = 12$.

Example 5

- The letters *A, B, C, D*, and *E* are to be used to form strings of length 4.
- How many strings can be formed if we do not allow repetitions?
- For example:
  – *BADE, ACBD, AEBC* ..

- T_1: choose the first letter → 5 ways
- T_2: choose the second letter → 4 ways
- T_3: choose the third letter → 3 ways
- T_4: choose the fourth letter → 2 ways
- There are 5.4.3.2= 120 strings.

Example 6

- The letters *A, B, C, D*, and *E* are to be used to form strings of length 4.
- How many strings can be formed if we allow repetitions?
- For example:
  – *BABB, AABB, ACEE* ..

- T_1: choose the first letter ➔ 5 ways
- T_2: choose the second letter ➔ 5 ways
- T_3: choose the third letter ➔ 5 ways
- T_4: choose the fourth letter ➔ 5 ways
- There are 5.5.5.5= 625 strings.

Example 7

- The letters *A, B, C, D*, and *E* are to be used to form strings of length 4.
- How many strings begin with *A*, if repetitions are not allowed?
- For example:
  - *ADEC, ACBD, AEBC* ..

- T_1: choose the first letter ➔ *A* (1 way)
- T_2: choose the second letter ➔ 4 ways
- T_3: choose the third letter ➔ 3 ways
- T_4: choose the fourth letter ➔ 2 ways
- There are 1.4.3.2= 24 strings.

Exercise

- Danial, Kenny and Joseph are fighting over a turn to play a game that can only has 2 players at a time. In how many ways can we select 2 players at a time?
- Diana, Sherry, Devi and Mary are going to DSI using a motorcycle. In how many ways can we select 2 peoples to ride the motorcycle?

Exercise

- There are 8 male students and 21 female students in Discrete Structure class. Among all of them, 7 students are Chinese and the rest are Malay.
  - In how many ways can we select 2 students - a boy and a girl?
  - In how many ways can we select 2 students - a Chinese and a Malay?

Exercise - Solution

- In how many ways can we select 2 students - a boy and a girl?
  - To select a boy
    - 8 ways
  - To select a girl
    - 21 ways
  - The number of ways
    - **8•21=168 ways**

- In how many ways can we select 2 students - a Chinese and a Malay?
  - To select a Chinese
    - 7 ways
  - To select a Malay
    - 22 ways
  - The number of ways
    - **7•22=154 ways**

Exercise

- Suppose that, in order to declare a variable name in a computer language, the name must have five characters. The first character must be a letter, and the remaining characters can be letters or digits. How many different variable names are possible?

Basic Counting Principles

- The counting problem that we have considered so far involved either the addition principle or the multiplication principle.
- Sometimes, however, we need to use both of these counting principles to solve a particular problem.

Example 8

- How many 8-bit strings begin either 101 or 111?
  - $\underline{1}\ \underline{0}\ \underline{1}$ _ _ _ _ _ ➔ 2. 2. 2. 2. 2 = 32
  - $\underline{1}\ \underline{1}\ \underline{1}$ _ _ _ _ _ ➔ 2. 2. 2. 2. 2 = 32
  - 32 + 32 =64

Example

How many bit strings of length 8 either start with a 1 or end with 00?

Task 1: Construct a string of length 8 that starts with a 1.

There is one way to pick the first bit (1),
two ways to pick the second bit (0 or 1),
two ways to pick the third bit (0 or 1),
.
.
.
two ways to pick the eighth bit (0 or 1).

- **Product rule:** Task 1 can be done in $1 \cdot 2^7 = 128$ ways.

Example (cont.)

Task 2: Construct a string of length 8 that ends with 00.
There are two ways to pick the first bit (0 or 1),
two ways to pick the second bit (0 or 1),
.
.
.
two ways to pick the sixth bit (0 or 1),
one way to pick the seventh bit (0), and
one way to pick the eighth bit (0).
Product rule: Task 2 can be done in $2^6 = 64$ ways.

Example (cont.)

Since there are 128 ways to do Task 1 and 64 ways to do Task 2, does this mean that there are 192 bit strings either starting with 1 or ending with 00 ?
No, because here Task 1 and Task 2 can be done **at the same time**.
When we carry out Task 1 and create strings starting with 1, some of these strings end with 00.

Therefore, we sometimes do Tasks 1 and 2 at the same time, so **the sum rule does not apply**.

Example (cont)

If we want to use the sum rule in such a case, we have to subtract the cases when Tasks 1 and 2 are done at the same time.

How many cases are there, that is, how many strings start with 1 **and** end with 00?

There is one way to pick the first bit (1),

two ways for the second, …, sixth bit (0 or 1),

one way for the seventh, eighth bit (0).

Product rule: In $2^5 = 32$ cases, Tasks 1 and 2 are carried out at the same time.

Example (cont)

Since there are 128 ways to complete Task 1 and 64 ways to complete Task 2, and in 32 of these cases Tasks 1 and 2 are completed at the same time, there are

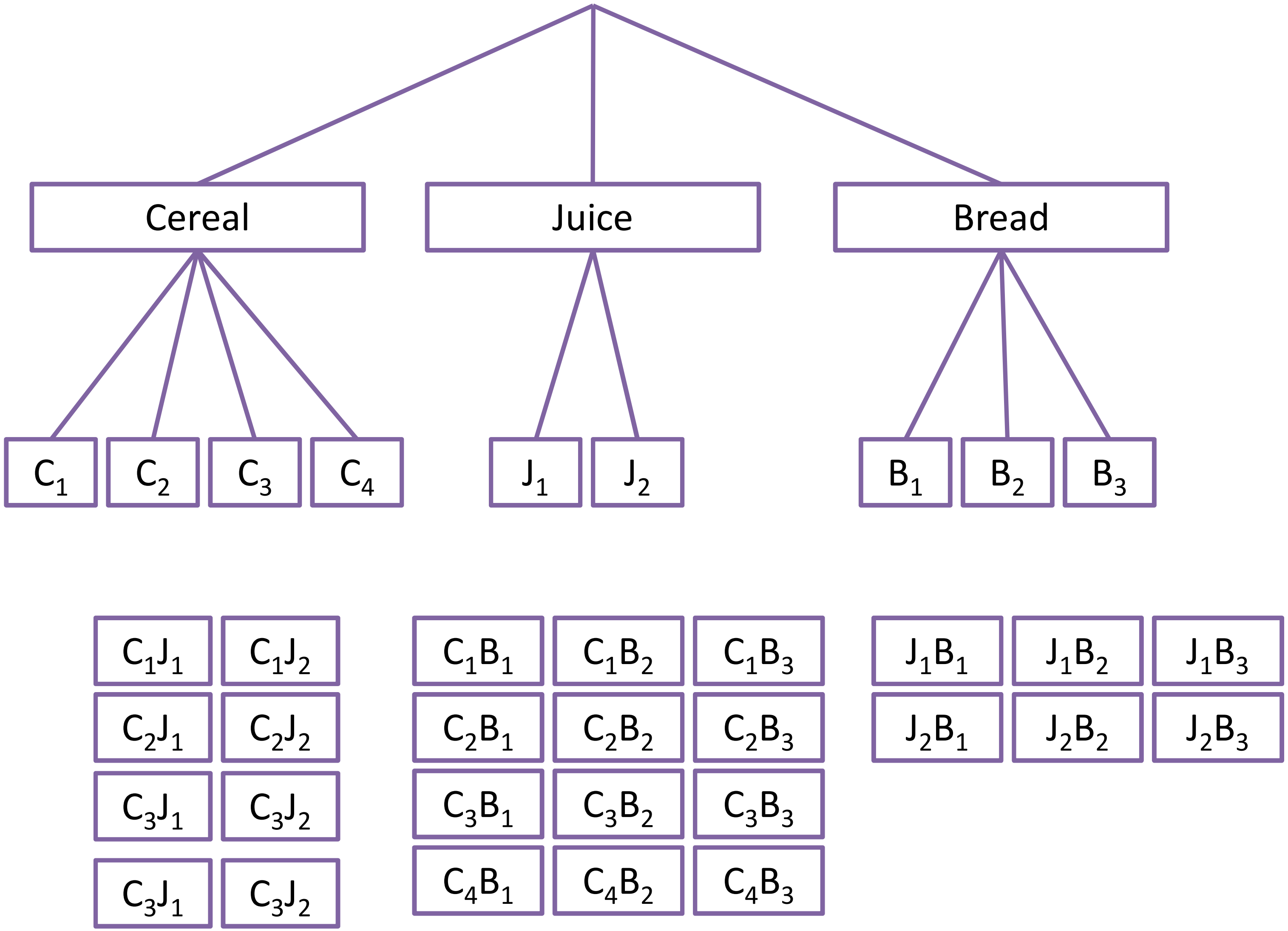
128 + 64 – 32 = 160 ways to do either task.

Example 9

- The following items are available for breakfast.
  - 4 types of cereal
  - 2 types of juice
  - 3 types of bread

- How many ways a breakfast can be prepared if exactly 2 items are selected from 2 different groups?

- A breakfast can be prepared in either of the following 3 ways:
  - ➢A cereal & a juice
    - 4. 2 = 8
  - ➢A cereal & a bread
    - 4. 3 =12
  - ➢A juice & a bread
    - 2. 3 = 6
  - ➢8+12+6=26 ways


Cereal
Juice
Bread
C_1
C_2
C_3
C_4
J_1
J_2
B_1
B_2
B_3
C_1J_1
C_1J_2
C_2J_1
C_2J_2
C_3J_1
C_3J_2
C_3J_1
C_3J_2
C_1B_1
C_1B_2
C_1B_3
C_2B_1
C_2B_2
C_2B_3
C_3B_1
C_3B_2
C_3B_3
C_4B_1
C_4B_2
C_4B_3
J_1B_1
J_1B_2
J_1B_3
J_2B_1
J_2B_2
J_2B_3

Example 10

A six-person committee composed of Aina, Wan, Chan, Tan, Syed and Helmi is to be selected to hold as a chairperson, secretary, and treasurer.

- In how many ways can this be done?
- In how many ways can this be done if either Aina or Wan must be chairperson?
- In how many ways can this be done if Syed must hold one of the position?
- In how many ways can this be done if Tan and Helmi must hold any position?

Example 10 - Solution

- In how many ways can this be done?
  - Select the chairperson (6 ways)
  - Select the secretary (5 ways)
  - Select the treasurer (4 ways)
  - 6. 5. 4 =120

- In how many ways can this be done if either Aina or Wan must be chairperson?
  - If Aina is chairperson 5. 4 =20
  - If Wan is chairperson 5. 4 =20
  - 20+20 =40

- OR
  - ✓Select the chairperson (2 ways)
  - ✓Select the secretary (5 ways)
  - ✓Select the treasurer (4 ways)
  - ✓2. 5. 4 =40

- In how many ways can this be done if Syed must hold one of the position?
  - If Syed is chairperson 5. 4 =20
  - If Syed is secretary 5. 4 =20
  - If Syed is treasurer 5. 4 =20
  - 20+20+20 =60

- OR
  - Assign Syed for any position is 3 ways
  - Fill the highest remaining position is 5 ways
  - Fill the last position is 4 ways
  - 3. 5. 4 =60

- In how many ways can this be done if Tan and Helmi must hold any position?
  - Assign Tan - 3 ways
  - Assign Helmi - 2 ways
  - Fill the remaining position - 4 ways
  - 3. 2. 4 =24

Exercise

- How many eight-bit strings have either the second or the fourth bit 1(or both)?

- How many license plates of 2 letters from A to Z, followed by 3 digits from 0 to 9 can be made, if repetition of letters is not allowed?

Exercise

- How many 5-digit telephone numbers can be constructed using digit 0-9 if each number starts with 67 and no digit appears more than once?