CHAPTER 2

[Part 2]

FUNCTIONS

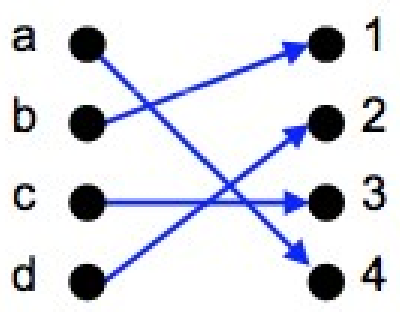
Semester (1) : 2019/2020

FUNCTION

- Let **X** and **Y** are nonempty sets.
- A **function (*f*)** from **X** to **Y** is a relation from **X** to **Y** having a properties:
  - The domain of f is **X**
  - If $(x,y), (x,y') \in f$, then $y = y'$

Relations vs Functions

- Not all relations are functions
- But consider the following function:



- All functions are relations!

Relations vs Functions

When to use which?

- A function is used when you need to obtain a SINGLE result for any element in the domain
  - Example: sin, cos, tan
- A relation is when there are multiple mappings between the domain and the co-domain
  - Example: students enrolled in multiple courses

Domain, Co-domain, Range

- A function from **X** to **Y** is denoted, f : **X** → **Y**
- The domain of f is the set **X**.
- The set **Y** is called the co-domain or target of f.
- The set $\{ y \mid (x,y) \in f \}$ is called the range.

Example

Given the relation, $f = \{ (1,a), (2,b), (3,a) \}$ from **X** = { 1, 2, 3 } to **Y**= { *a*, *b*, *c* } is a function from **X** to **Y**. State the domain and range.

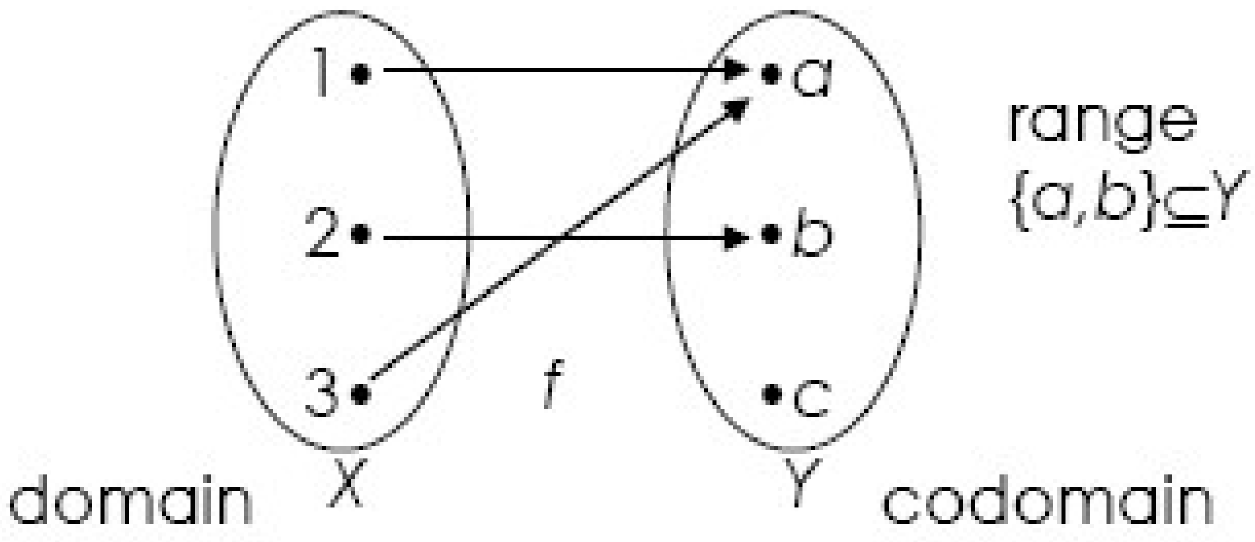
Solution:

✓The domain of f is **X**

✓The range of f is $\{a, b\}$

Example (cont.)

- $f = \{ (1,a), (2,b), (3,a) \}$

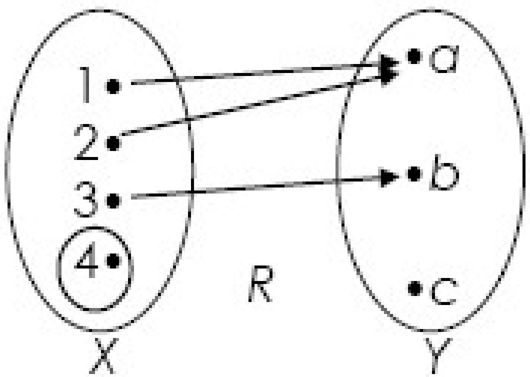


Example

- The relation, $R = \{(1,a), (2,a), (3,b)\}$ from $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c\}$ is NOT a function from X to Y.

- The domain of R, $\{ 1,2,3 \}$ is not equal to X.

$R = \{(1,a), (2,a), (3,b)\}$



There is no arrow from 4

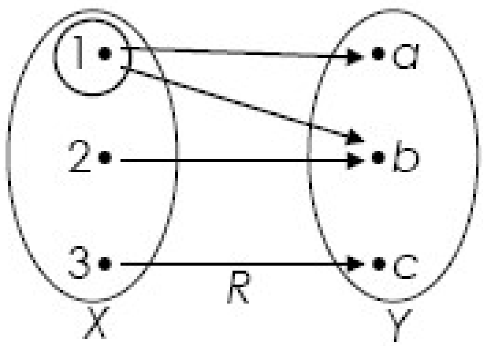
Example

- The relation, $R = \{(1,a), (2,b), (3,c), (1,b)\}$ from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$ is NOT a funtion from X to Y

- (1,a) and (1,b) in R but a ≠ b.

$R = \{(1,a), (2,b), (3,c), (1,b)\}$

There are 2 arrows from 1



Notation of function: $f(x)$

- For the function, f = {(1,a), (2,b), (3,a)}

- We may write:

$$f(1)=a, f(2)=b, f(3)=a$$

- Notation $f(x)$ is used to define a function.

Example

- Defined: $f(x) = x^2$
- $f(2) = 4,\ f(-3.5) = 12.25,\ f(0) = 0$
- $f = \{(x, x^2) \mid x \text{ is a real number}\}$

One-to-One Function

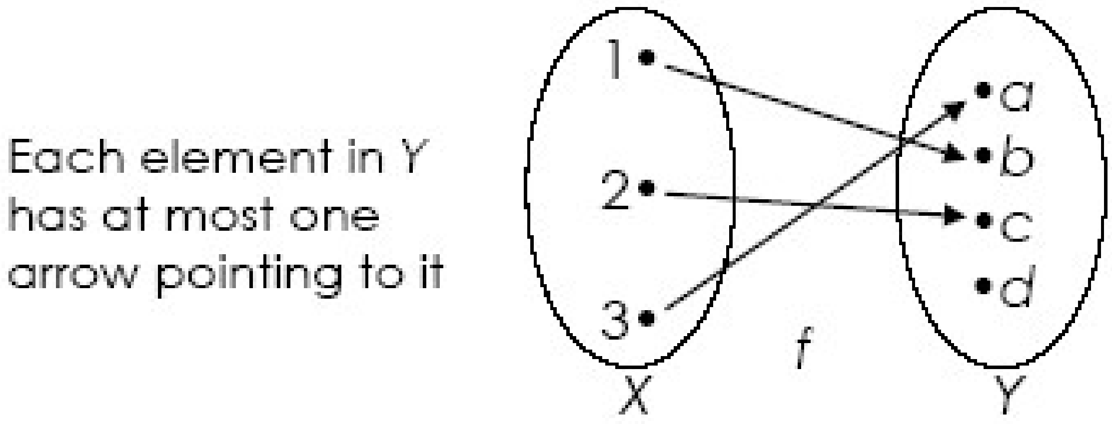
- A function f from X to Y, is said one-to-one (or injective) if for each $y \in Y$, there is at most one $x \in X$, with $f(x)=y$.

- For all x_1, x_2, if $f(x_1) = f(x_2)$, then $x_1=x_2$.

  $\forall x_1 \forall x_2 ((f(x_1) = f(x_2)) \rightarrow (x_1=x_2))$

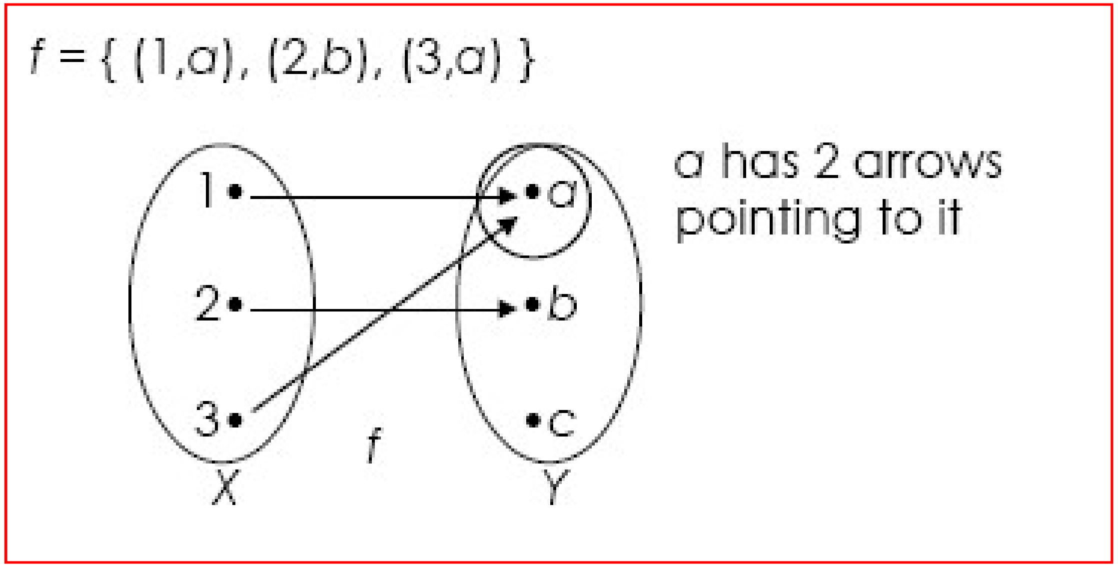
Example

- The function, f = { (1,*b*), (3,*a*), (2,*c*) } from **X** = { 1, 2, 3 } to **Y** = { *a, b, c, d* } is one-to-one.



Example (cont.)

- The function, f = { (1,*a*), (2,*b*), (3,*a*) } from **X** = { 1, 2, 3 } to **Y** = { *a, b, c*} is NOT one-to-one.
- $f(1) = a = f(3)$



Example

Show that the function,

$$f(n) = 2n+1$$

on the set of positive integers is one-to-one.

Solution

- For all positive integer, n_1 and n_2 if $f(n_1) = f(n_2)$, then $n_1 = n_2$.

- Let, $f(n_1) = f(n_2)$, $f(n) = 2n+1$

  then $2n_1 + 1 = 2n_2 + 1$ $\quad (-1)$

  $2n_1 = 2n_2$ $\quad (\div 2)$

  $n_1 = n_2$

- This shows that f is one-to-one.

Example

- Show that the function,

  $$f(n) = 2^n - n^2$$

  from the set of positive integers to the set of integers is NOT one-to-one.

Solution

- Need to find 2 positive integers, n_1 and n_2
  $n_1 \neq n_2$ with $f(n_1) = f(n_2)$.

- trial and error,

  $$f(2) = f(4)$$

- f is not one-to-one.

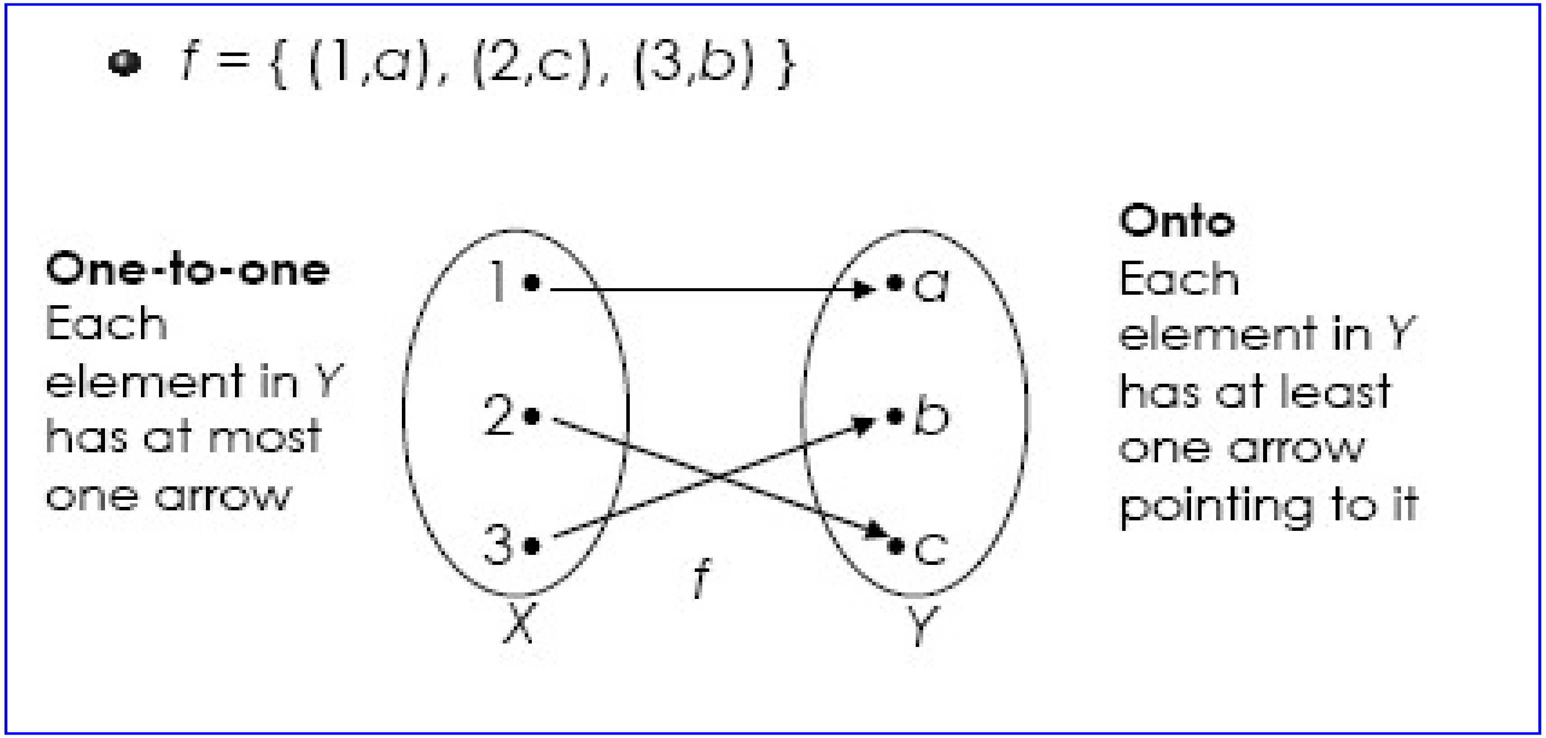
Onto Function

- If f is a function from **X** to **Y** and the range of f is **Y**, f is said to be onto **Y** (or an onto function or a surjective function)
- For every $y \in$ **Y**, there exists at least one $x \in$ **X** such that $f(x) = y$

$$\forall\, y \in \mathbf{Y}\; \exists\, x \in \mathbf{X}\; (f(x) = y)$$

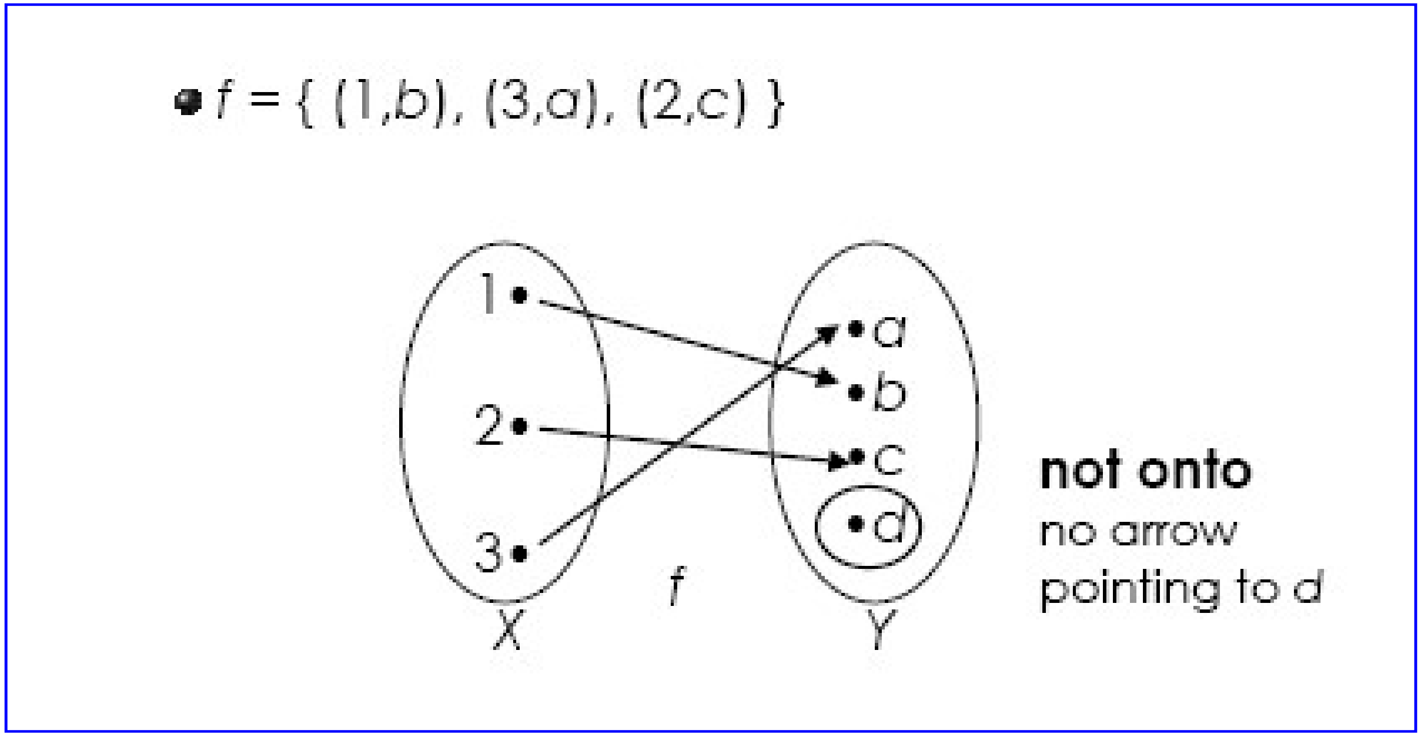
Example

- The function, f = { (1,a), (2,c), (3,b) } from **X** = { 1, 2, 3 } to **Y** = { a, b, c } is one-to-one and onto **Y**.



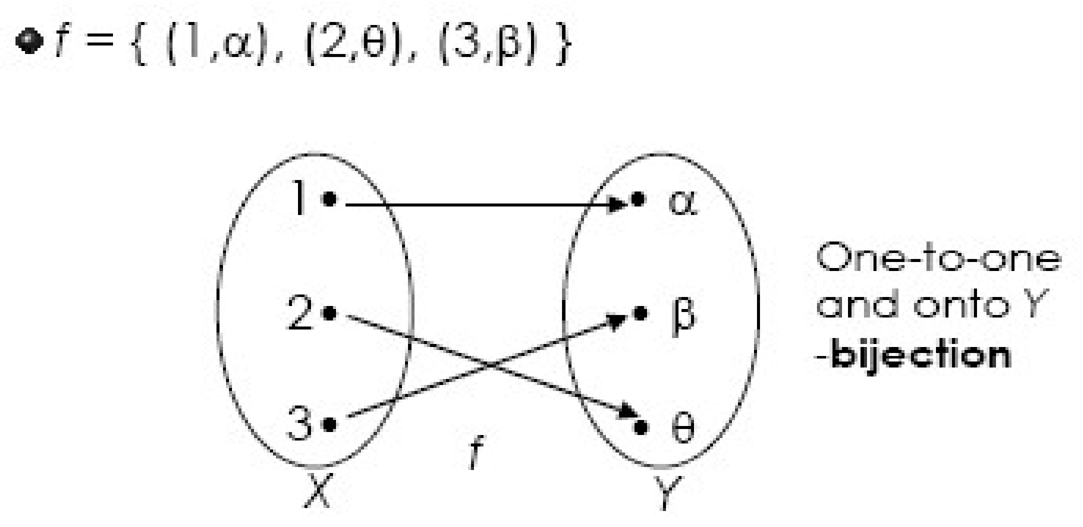
Example

- The function, $f = \{ (1,b), (3,a), (2,c) \}$ is not onto **Y** = $\{a, b, c, d\}$



Bijection Function

- A function, f is called one-to-one correspondence (or bijective/bijection) if f is both one-to-one and onto.
- Example:



Exercise

Determine which of the relations f are functions from the set **X** to the set **Y**. In case any of these relations are functions, determine if they are one-to-one, onto **Y**, and/or bijection.

a) **X** = { -2, -1, 0, 1, 2 } , **Y** = { -3, 4, 5 } and
f = { (-2,-3), (-1,-3), (0,4), (1,5), (2,-3) }

=> Function, ONTO

b) **X** = { -2, -1, 0, 1, 2 } , **Y** = { -3, 4, 5 } and
f = { (-2,-3), (1,4), (2,5)}

=> Not Function

c) **X** = **Y**= { -3, -1, 0, 2 } and
f = { (-3,-1), (-3,0), (-1,2), (0,2), (2,-1) }

=> Not Function

Inverse Function

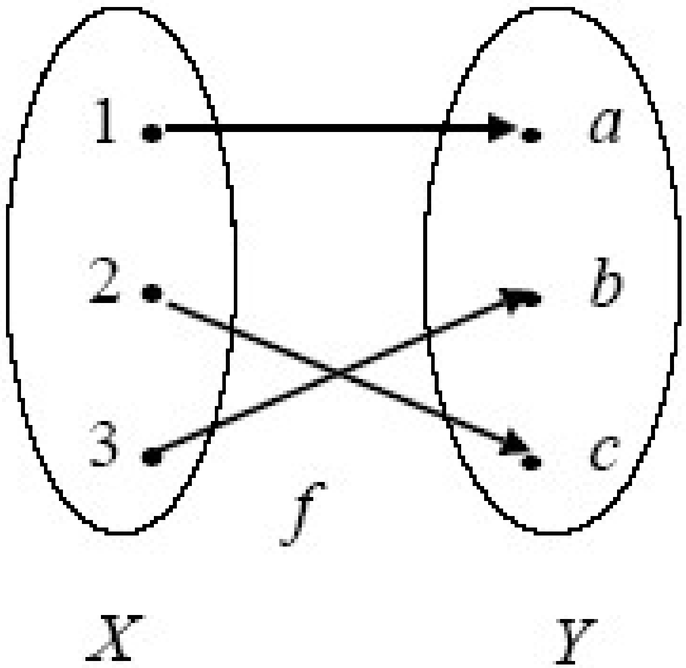
- Let f: **X**→**Y** be a function.

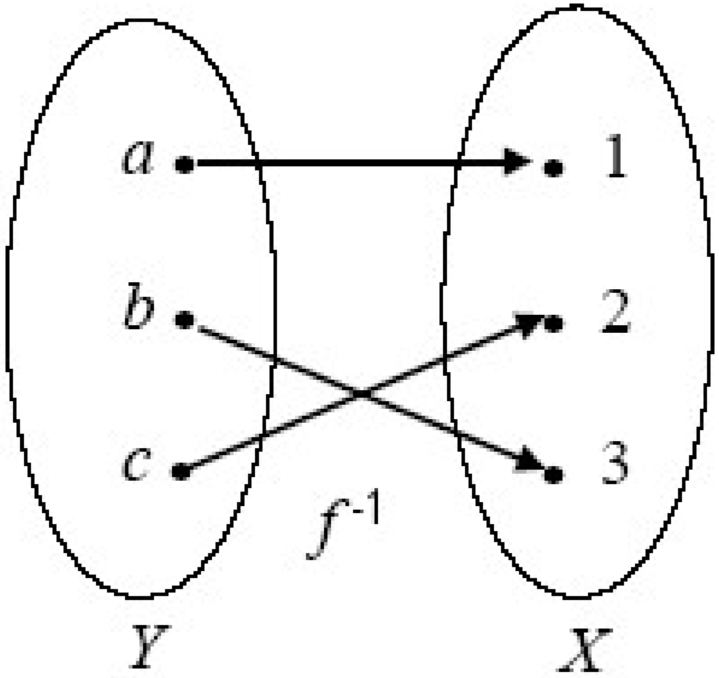
- The inverse relation $f^{-1} \subseteq$ **Y** × **X** is a function from **Y** to **X**, if and only if f is both one-to-one and onto **Y**.

Example

$f = \{(1,a),(2,c),(3,b)\}$

$f^{-1} = \{(a,1),(c,2),(b,3)\}$





Example

- The function, $f(x) = 9x + 5$ for all $x \in R$ (R is the set of real numbers).
- This function is both one-to-one and onto.
- Hence, f^{-1} exists.
- Prove:

Let $(y, x) \in f^{-1}, f^{-1}(y) = x$

$(x,y) \in f,$ $\quad y = 9x + 5$

$x = (y\text{-}5)/9$

$f^{-1}(y) = (y\text{-}5)/9$

Exercise

- Find each inverse function.

  a) $f(x) = 4x + 2, \ x \in R$

  b) $f(x) = 3 + (1/x), \ x \in R$

Composition

- Suppose that g is a function from **X** to **Y** and f is a function from **Y** to **Z**.

- The composition of f with g,

$$f \circ g$$

is a function

$$(f \circ g)(x) = f(g(x))$$

from **X** to **Z**.

Example

- Given, $g = \{ (1,a), (2,a), (3,c) \}$

  a function from **X** = {1, 2, 3} to **Y** = {*a*, *b*, *c*}

  and,

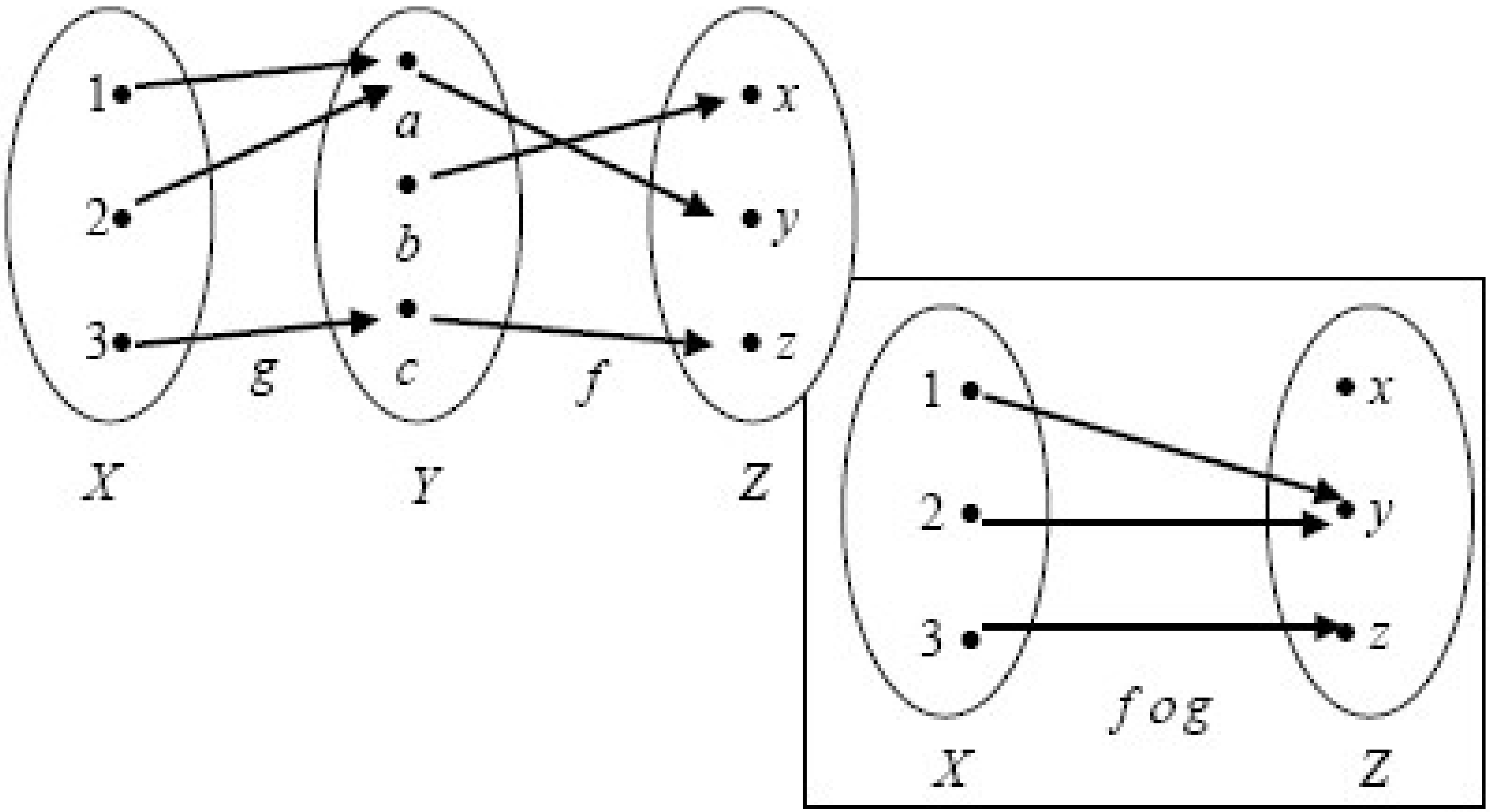
  $$f = \{ (a,y), (b,x), (c,z) \}$$

  a function from **Y** to **Z** = { *x*, *y*, *z* }.

- The composition function from **X** to **Z** is the function

  $$f \circ g = \{ (1,y), (2,y), (3,z) \}$$

Example (cont.)



Example

$f(x) = \log_3 x$ and $g(x) = x^4$

- $f(g(x)) = \log_3(x^4)$
- $g(f(x)) = (\log_3 x)^4$

- Note: $f \circ g \neq g \circ f$

Example

$$f(x) = \frac{1}{5}x \qquad g(x) = x^2 + 1$$

$$(g \circ f)(x) = g(f(x)) = g(\frac{x}{5})$$

$$= (\frac{x}{5})^2 + 1 = \frac{x^2}{25} + 1$$

Example

- Composition sometimes allows us to decompose complicated functions into simpler functions.

- example $f(x) = \sqrt{\sin 2x}$

$$g(x) = \sqrt{x} \qquad h(x) = \sin x \qquad w(x) = 2x$$

$$f(x) = g(h(w(x)))$$

Exercise

- Let f and g be functions from the positive integers to the positive integers defined by the equations, $f(n) = n^2$, $g(n) = 2^n$
- Find the compositions

  a) $f \circ f$

  b) $g \circ g$

  c) $f \circ g$

  d) $g \circ f$

Recursive Algorithms

- A recursive procedure is a procedure that invokes itself.
- A recursive algorithm is an algorithm that contains a recursive procedure.
- Recursion is a powerful, elegant and natural way to solve a large class of problems.

Example

Factorial problem

- If $n \geq 1$,

  $n! = n\,(n-1) \Rightarrow 2 \times 1$

  and $0! = 1$

- Notice that, if $n \geq 2$, n factorial can be written as,

  $n! = n(n-1)(n-2) \Rightarrow 2 \times 1 \times 0$

  $= n(n-1)!$

Example

- n=5
- 5!

$$5! = 5.\ 4.\ 3.\ 2.\ 1 = 5.\ 4!$$
$$4! = 4.\ 3!$$
$$3! = 3.\ 2!$$

Recursive Algorithm for Factorial

- Input: n, integer ≥ 0
- Output: $n!$
- Factorial (n) {

```
    if (n=0)
            return 1
    return n*factorial(n-1)
}
```

Example

- Fibonacci sequence, f_n

$$f_1 = 1$$

$$f_2 = 1$$

$$f_n = f_{n-1} + f_{n-2}, \text{ for } n \geq 3$$

1, 1, 2, 3, 5, 8, 13, ...

Recursive algorithm for Fibonacci Sequence :

- Input: n
- Output: $f(n)$
- $f(n)$ {

```
    if (n=1 or n=2)
            return 1
    return f(n-1) + f(n-2)
}
```

Example

Consider the following arithmetic sequence: 1, 3, 5, 7, 9,...

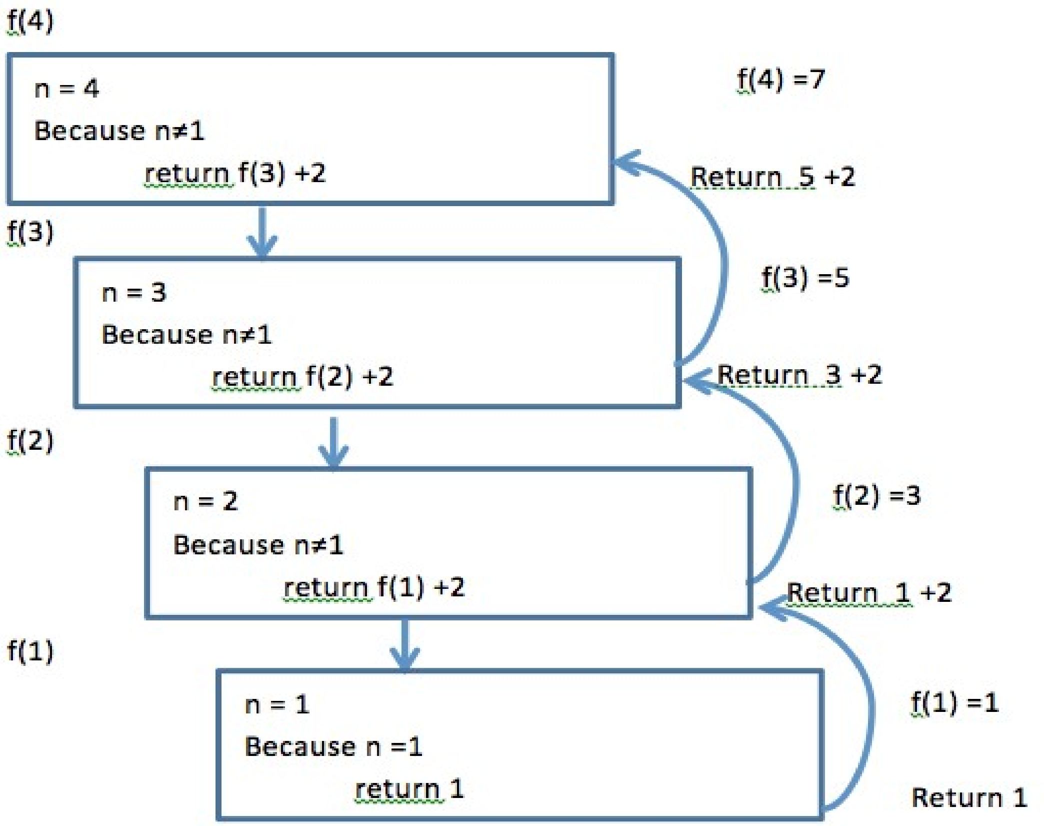
Suppose a_n is the term sequence. The generating rule is $a_n = a_{n-1} + 2$, for $n \geq 1$. The relevant recursive algorithm can be written as

```
f(n)
{   if (n = 1 )
            return 1
    return f(n – 1) +2
}
```

Use the above recursive algorithm to trace n = 4

Example- Solution

Trace the output if n = 4



Answer = 7