UTM UNIVERSITI TEKNOLOGI MALAYSIA

SCSI1013: Discrete Structures

# CHAPTER 2

**[Part 2]**

# FUNCTIONS

Semester (1) : 2015/2016



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## FUNCTION

- Let **X** and **Y** are nonempty sets.
- A **function (*f*)** from **X** to **Y** is a relation from **X** to **Y** having a properties:
  - The domain of $f$ is **X**
  - If $(x,y), (x,y') \in f$, then $y = y'$

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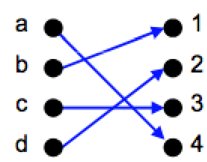
# Relations vs Functions

- Not all relations are functions
- But consider the following function:




- All functions are relations!



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# Relations vs Functions

## When to use which?

- A function is used when you need to obtain a SINGLE result for any element in the domain
  - Example: sin, cos, tan
- A relation is when there are multiple mappings between the domain and the co-domain
  - Example: students enrolled in multiple courses

## Domain, Co-domain, Range

- A function from **X** to **Y** is denoted, $f$ : **X** → **Y**
- The domain of $f$ is the set **X**.
- The set **Y** is called the co-domain or target of $f$.
- The set $\{ y \mid (x,y) \in f \}$ is called the range.



## Example

Given the relation, $f = \{ (1,a), (2,b), (3,a) \}$ from **X** = { 1, 2, 3 } to **Y**= { $a$, $b$, $c$ } is a function from **X** to **Y**. State the domain and range.

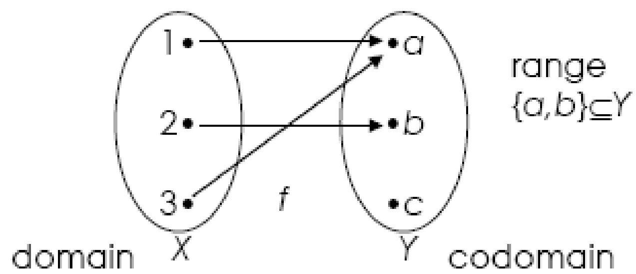
Solution:

✓The domain of $f$ is **X**

✓The range of $f$ is $\{a, b\}$

# Example (cont.)

- $f$ = { (1,$a$), (2,$b$), (3,$a$) }



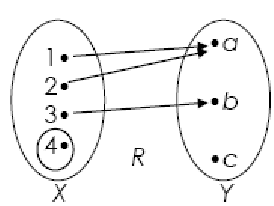



# Example

- The relation, $R$= {(1,$a$), (2,$a$), (3,$b$)} from $X$= {1, 2, 3, 4} to $Y$= {$a$, $b$, $c$} is NOT a function from $X$ to $Y$.

- The domain of $R$, { 1,2,3 } is not equal to $X$.

$R$= {(1,$a$), (2,$a$), (3,$b$)}




There is no arrow from 4

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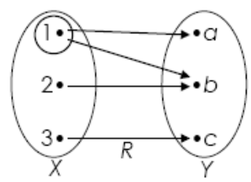
# Example

- The relation, $R = \{(1,a), (2,b), (3,c), (1,b)\}$ from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$ is NOT a funtion from X to Y
- (1,a) and (1,b) in R but a ≠ b.

$R = \{(1,a), (2,b), (3,c), (1,b)\}$

There are 2 arrows from 1



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# Notation of function: $f(x)$

- For the function, $f = \{(1,a), (2,b), (3,a)\}$
- We may write:

$$f(1)=a, f(2)=b, f(3)=a$$

- Notation $f(x)$ is used to define a function.

# Example

- Defined: $f(x) = x^2$
- $f(2) = 4$, $f(-3.5) = 12.25$, $f(0) = 0$
- $f = \{(x, x^2) \mid x \text{ is a real number}\}$

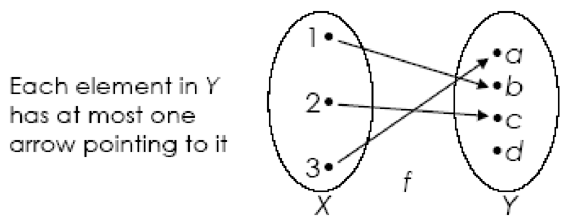


# One-to-One Function

- A function $f$ from $X$ to $Y$, is said one-to-one (or injective) if for each $y \in Y$, there is at most one $x \in X$, with $f(x)=y$.
- For all $x_1, x_2$, if $f(x_1) = f(x_2)$, then $x_1=x_2$.
  $\forall x_1 \forall x_2 ((f(x_1) = f(x_2)) \rightarrow (x_1=x_2))$

## Example

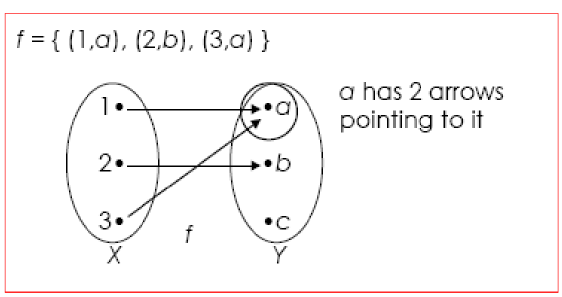
- The function, $f$ = { (1,*b*), (3,*a*), (2,*c*) } from **X** = { 1, 2, 3 } to **Y** = { *a, b, c, d* } is one-to-one.






## Example (cont.)

- The function, $f$ = { (1,*a*), (2,*b*), (3,*a*) } from **X** = { 1, 2, 3 } to **Y** = { *a, b, c*} is NOT one-to-one.
- $f(1) = a = f(3)$



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## Example

Show that the function,

$$f(n) = 2n+1$$

on the set of positive integers is one-to-one.



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## Solution

- For all positive integer, $n_1$ and $n_2$ if $f(n_1) = f(n_2)$, then $n_1 = n_2$.
- Let, $f(n_1) = f(n_2)$, $f(n) = 2n+1$

  then $2n_1 + 1 = 2n_2 + 1$ ( -1)

  $2n_1 = 2n_2$ (÷2)

  $n_1 = n_2$
- This shows that $f$ is one-to-one.

# Example

- Show that the function,

  $f(n) = 2^n - n^2$

  from the set of positive integers to the set of integers is NOT one-to-one.



# Solution

- Need to find 2 positive integers, $n_1$ and $n_2$ $n_1 \neq n_2$ with $f(n_1) = f(n_2)$.
- trial and error,

  $f(2) = f(4)$

- $f$ is not one-to-one.

# Onto Function

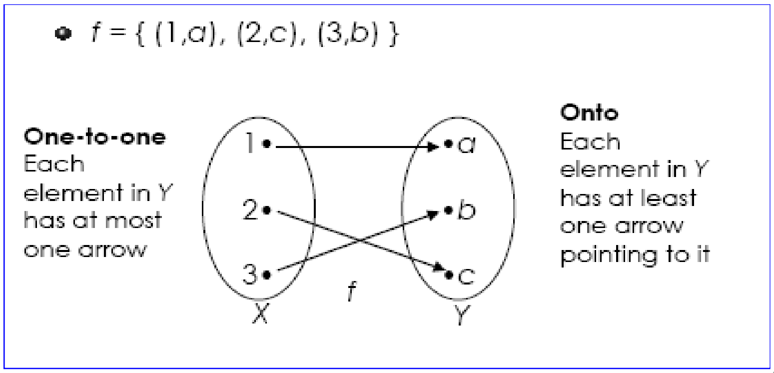
- If $f$ is a function from **X** to **Y** and the range of $f$ is **Y**, $f$ is said to be onto **Y** (or an onto function or a surjective function)
- For every $y \in$ **Y**, there exists at least one $x \in$ **X** such that $f(x) = y$

$$\forall y \in \mathbf{Y} \; \exists x \in \mathbf{X} \; ( f(x) = y )$$



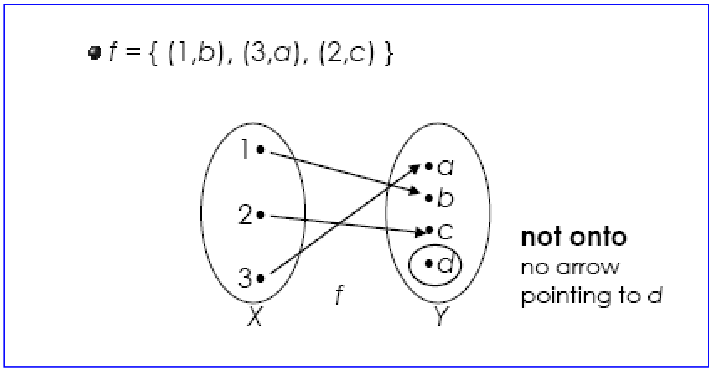
# Example

- The function, $f = \{ (1,a), (2,c), (3,b) \}$ from **X** = { 1, 2, 3 } to **Y** = { $a, b, c$ } is one-to-one and onto **Y**.



# Example

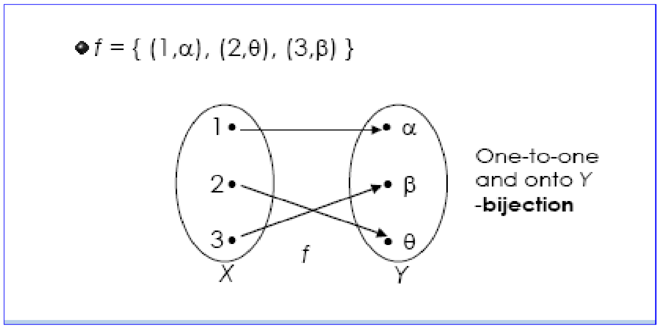
- The function, $f = \{ (1,b), (3,a), (2,c) \}$ is not onto **Y** = $\{a, b, c, d\}$






# Bijection Function

- A function, $f$ is called one-to-one correspondence (or bijective/bijection) if $f$ is both one-to-one and onto.
- Example:



# Exercise

Determine which of the relations *f* are functions from the set **X** to the set **Y**. In case any of these relations are functions, determine if they are one-to-one, onto **Y**, and/or bijection.

**a)** **X** = { -2, -1, 0, 1, 2 } , **Y** = { -3, 4, 5 } and
*f* = { (-2,-3), (-1,-3), (0,4), (1,5), (2,-3) }

b) **X** = { -2, -1, 0, 1, 2 } , **Y** = { -3, 4, 5 } and
*f* = { (-2,-3), (1,4), (2,5)}

c) **X** = **Y**= { -3, -1, 0, 2 } and
*f* = { (-3,-1), (-3,0), (-1,2), (0,2), (2,-1) }



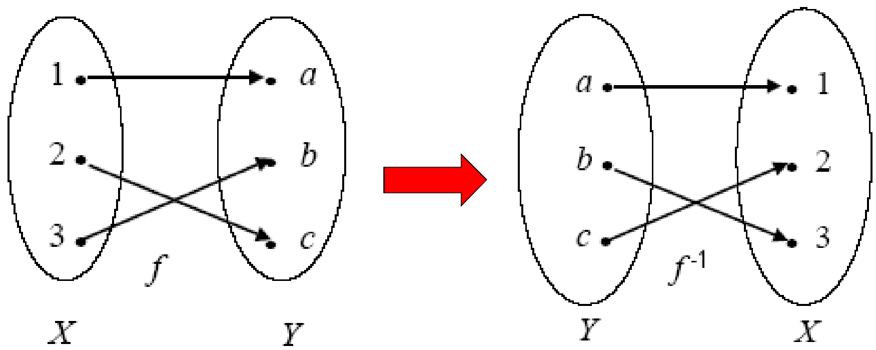
# Inverse Function

- Let *f*: **X**→**Y** be a function.
- The inverse relation $f^{-1} \subseteq$ **Y**×**X** is a function from **Y** to **X**, if and only if *f* is both one-to-one and onto **Y**.

# Example

$f = \{(1,a),(2,c),(3,b)\}$

$f^{-1} = \{(a,1),(c,2),(b,3)\}$






# Example

- The function, $f(x) = 9x + 5$ for all $x \in R$ ($R$ is the set of real numbers).
- This function is both one-to-one and onto.
- Hence, $f^{-1}$ exists.
- Prove:

Let $(y, x) \in f^{-1}$, $f^{-1}(y) = x$

$(x,y) \in f$, $\quad y = 9x + 5$

$x = (y-5)/9$

$f^{-1}(y) = (y-5)/9$

# Exercise

- Find each inverse function.

a) $f(x) = 4x + 2,\ x \in R$

b) $f(x) = 3 + (1/x),\ x \in R$



# Composition

- Suppose that $g$ is a function from **X** to **Y** and $f$ is a function from **Y** to **Z**.

- The composition of $f$ with $g$,

$$f \circ g$$

is a function

$$(f \circ g)(x) = f(g(x))$$

from **X** to **Z**.

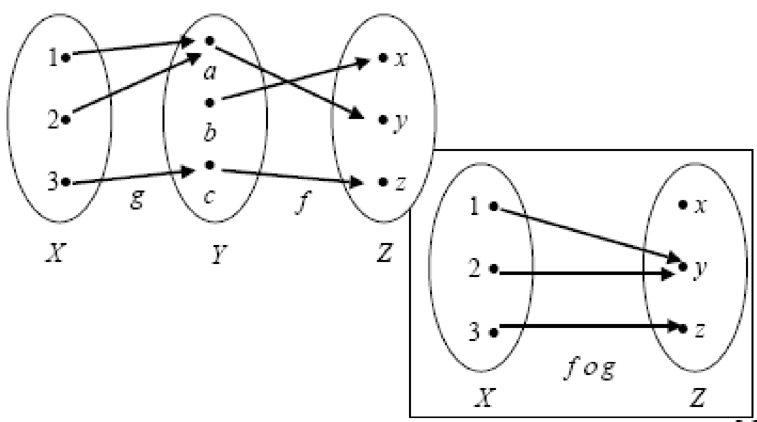
# Example

- Given, $g = \{ (1,a), (2,a), (3,c) \}$
  a function from **X** = {1, 2, 3} to **Y** = {*a*, *b*, *c*}
  and,
  $f = \{ (a,y), (b,x), (c,z) \}$
  a function from **Y** to **Z** = { *x*, *y*, *z* }.
- The composition function from **X** to **Z** is the function

  $f \circ g = \{ (1,y), (2,y), (3,z) \}$



# Example (cont.)



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# Example

$f(x) = \log_3 x$ and $g(x) = x^4$

- $f(g(x)) = \log_3 ( x^4 )$
- $g(f(x)) = (\log_3 x)^4$
- Note: $f \circ g \neq g \circ f$



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# Example

$$f(x) = \frac{1}{5}x \qquad g(x) = x^2 + 1$$

$$(g \circ f)(x) = g(f(x)) = g(\frac{x}{5})$$

$$= (\frac{x}{5})^2 + 1 = \frac{x^2}{25} + 1$$

# Example

- Composition sometimes allows us to decompose complicated functions into simpler functions.
- example $f(x) = \sqrt{\sin 2x}$

$$g(x) = \sqrt{x} \qquad h(x) = \sin x \qquad w(x) = 2x$$

$$f(x) = g(h(w(x)))$$



# Exercise

- Let $f$ and $g$ be functions from the positive integers to the positive integers defined by the equations, $f(n) = n^2$ , $g(n) = 2^n$
- Find the compositions
  a) $f \circ f$
  b) $g \circ g$
  c) $f \circ g$
  d) $g \circ f$

# Recursive Algorithms

- A recursive procedure is a procedure that invokes itself.
- A recursive algorithm is an algorithm that contains a recursive procedure.
- Recursion is a powerful, elegant and natural way to solve a large class of problems.



# Example

Factorial problem

- If $n \geq 1$,

  $n! = n\ (n-1) \Rightarrow 2 \times 1$

  and $0! = 1$
- Notice that, if $n \geq 2$, $n$ factorial can be written as,

  $n! = n(n-1)(n-2) \Rightarrow 2 \times 1 \times 0$

  $= n(n-1)!$

# Example

- n=5
- 5!

  5! = 5. 4. 3. 2. 1 = 5. 4!
  4! = 4. 3!
  3! = 3. 2!



# Recursive Algorithm for Factorial

- Input: *n*, integer ≥ 0
- Output: *n*!
- Factorial (*n*) {

```
    if (n=0)
          return 1
    return n*factorial(n-1)
}
```

# Example

- Fibonacci sequence, $f_n$

  $f_1 = 1$

  $f_2 = 1$

  $f_n = f_{n-1} + f_{n-2}$, for $n \geq 3$

  1, 1, 2, 3, 5, 8, 13, ....



Recursive algorithm for Fibonacci Sequence :

- Input: *n*
- Output: *f* (*n*)
- *f*(*n*) {

```
    if (n=1 or n=2)
        return 1
    return f (n-1) + f (n-2)
}
```

## Example



## Example- Solution