# DISCRETE STRUCTURES CHAPTER 2 PART 1

# RELATIONS

# Definition

- Let *A* and *B* be two sets.
- A binary relation, or simply a relation *R* from set *A* to set *B* is a subset of the cartesian product *A* x *B*

$$a \in A, b \in B, (a,b) \in A \times B \text{ and } R \subseteq A \times B$$

- If $(a,b) \in R$, we say *a* is related to *b* by *R* write as *a R b* ( $a\,R\,b \leftrightarrow (a,b) \in R$ )

# Example 1

*Let A= { 1, 2, 3, 4 } and B= { p, q, r }*

$R = \{ (1, q), (2, r), (3, q), (4, p) \}$

$R \subseteq A \times B$

*R is the relation from A to B*

$1Rq$ ( *1* is related to *q*)

$3 \not R p$ ( *1* is not related to *p*)

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# Relations

- Binary relations: $xRy$
  On sets $x \in X \quad y \in Y \qquad R \subseteq X \times Y$
- Example:
  "less than" relation from $A=\{0,1,2\}$ to $B=\{1,2,3\}$

Use traditional notation

0 < 1, 0 < 2, 0 < 3, 1 < 2, 1 < 3, 2 < 3

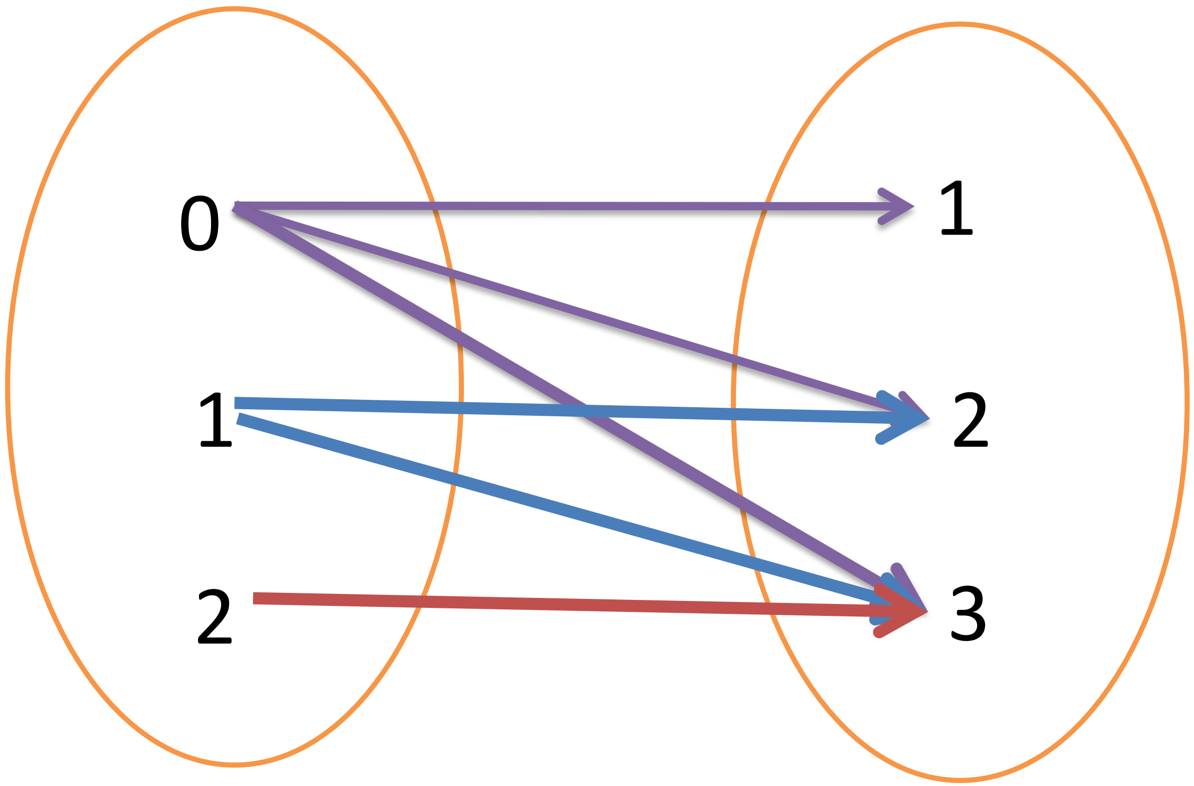
Or use set notation

$A \times B=\{(0,1),(0,2),(0,3),(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)\}$

$R=\{(0,1),(0,2),(0,3), (1,2),(1,3),(2,3)\}$

Or use Arrow Diagrams

# Arrow Diagram




$R=\{(0,1),(0,2),(0,3),\ (1,2),(1,3),(2,3)\}$

# Example 2

$A$ = {New Delhi, Ottawa, London, Paris, Washington}
$B$ = {Canada, England, India, France, United States}

Let $x \in A$, $y \in B$.

Define the relation between $x$ and $y$ by "$x$ is the capital of $y$"

$R$ = {(New Delhi, India), (Ottawa, Canada), (London, England), (Paris, France), (Washington, United States)}

# Definition

- If *R* is a relation from set *A* into itself, we say that *R* is a relation on *A*.

  $a \in A,\ b \in A\ (a,b) \in A \times A$ and $R \subseteq A \times A$

- Example

  Let A = (1,2,3,4,5) and R be defined by $a,b \in A$,

$$aRb \leftrightarrow b - a = 2$$

$$R = \{(1,3),(2,4),(3,5)\}$$

# Exercise 1

Write the relation $R$ as $(x,y) \in R$

(i) The relation $R$ on {1, 2, 3, 4} defined by $(x,y) \in R$ if $x^2 \geq y$

(ii) The relation $R$ on {1, 2, 3, 4, 5} defined by $(x,y) \in R$ if 3 divides $x-y$

# Domain and Range

Let *R,* a relation from *A* to *B.*

The set, **{ $a \in A$ | $(a,b) \in R$ for some $b \in B$ }** is called the **domain** of *R.*

The set, **{ $b \in B$ | $(a,b) \in R$ for some $a \in A$ }** is called **the range** of *R.*

# Example 3

Let $R$ be a relation on $X = \{1, 2, 3, 4\}$ defined by $(x,y) \in R$ *if* $x \leq y$, *and* $x,y \in X$.

Then,
$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$

The **domain and range** of $R$ are both equal to $X$.

# Example 4

Let $X$ = {2, 3, 4} and $Y$ = {3, 4, 5, 6, 7}

If we **define a relation *R* from *X* to *Y* by,**

**$(x,y) \in R$ if $y/x$** (*with zero* remainder)

We obtain,

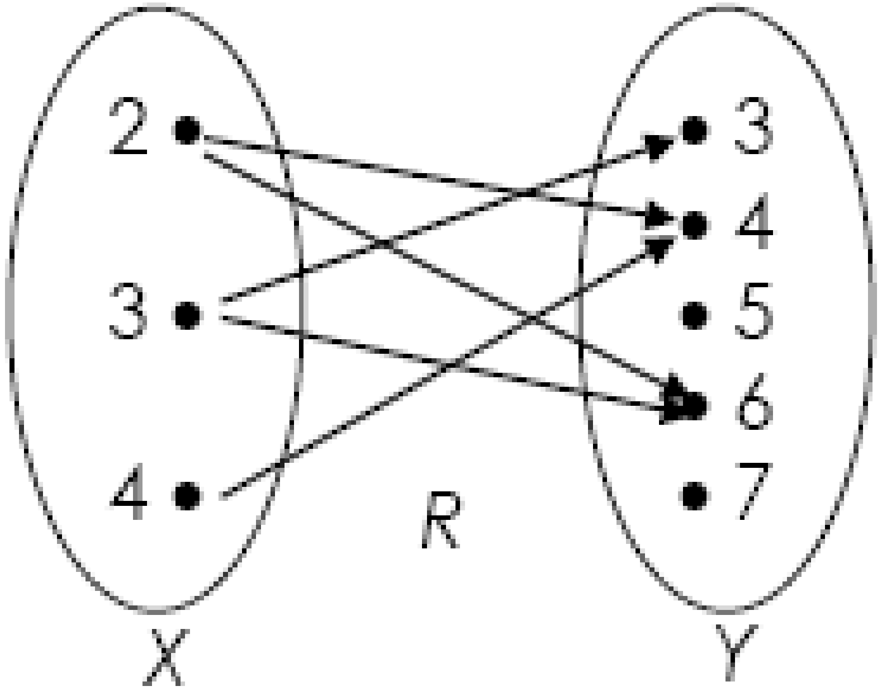
$R$ = { (2,4), (2,6), (3,3), (3,6), (4,4) }

The domain of $R$ is {2,3,4}

The **range** of R is {3,4,6}

# Example 4(cont)

$R = \{ (2,4), (2,6), (3,3), (3,6), (4,4) \}$




Arrow diagram

# Exercise 2

Find range and domain for:

(i) The relation $R$={(1,2), (2,1), (3,3), (1,1), (2,2)} on $X$={1, 2, 3}

(ii) The relation $R$ on {1, 2, 3, 4} defined by $(x,y) \in R$ if $x^2 \geq y$

# Diagraph

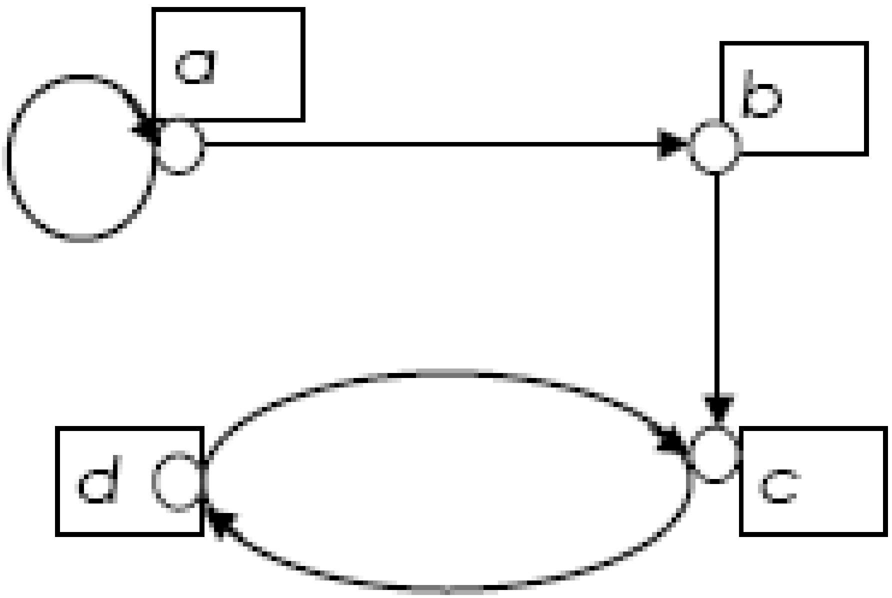
An informative way to picture a relation on a set is to draw its digraph.

- Let *R* be a relation on a finite set *A.*
- Draw dots (vertices) to represent the elements of *A.*
- If the element $(a,b) \in R$, draw an arrow (called a directed edge) from *a* to *b*

# Example 5

The relation $R$ on $A = \{a, b, c, d\}$,

$R = \{(a, a), (a, b), (c, d), (d, c), (b,c)\}$



# Example 6

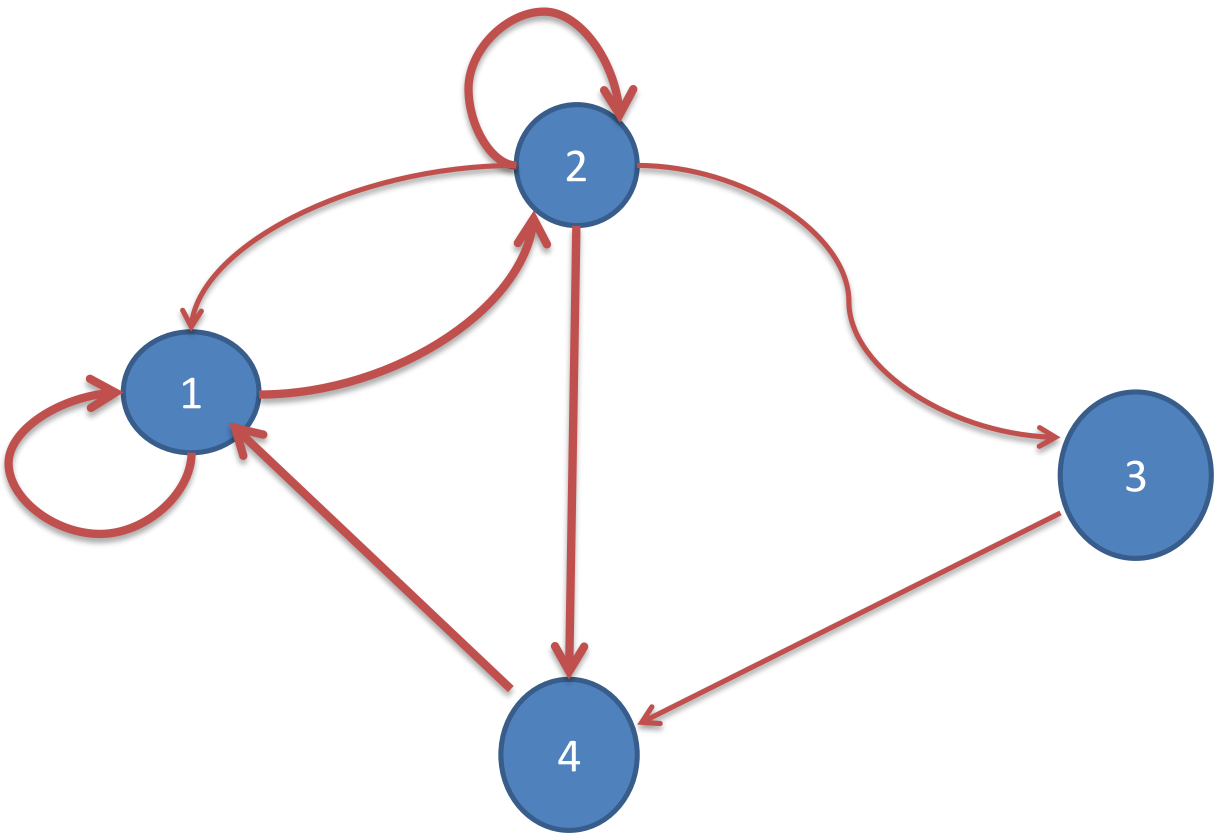
Let

$A = \{ 1,2,3,4\}$ and

$R = \{ (1,1), (1,2), (2,1), (2,2), (2,3), (2,4),$
$(3,4) , (4,1)\}$

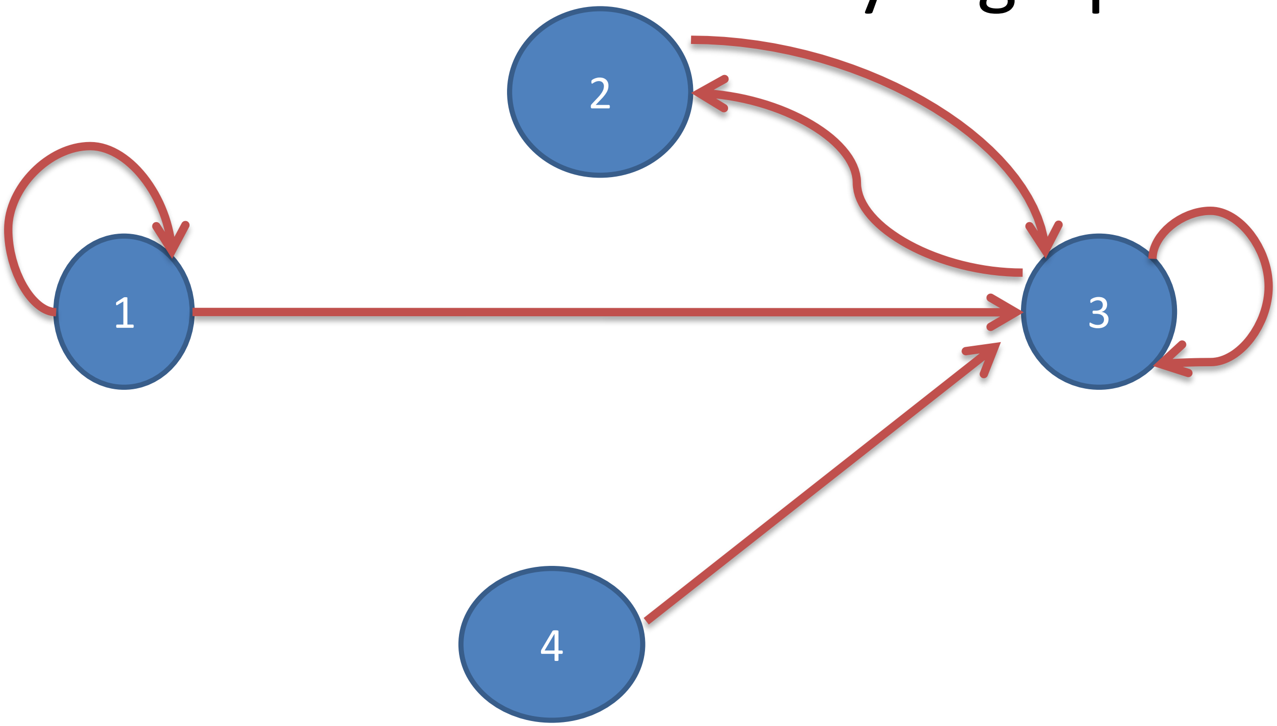
Draw the digraph of $R$

# Digraph



# Example 7

Find the relation determined by digraph below




Since $a_i$ $R$ $a_j$ if and only if there is an edge from $a_i$ to $a_j$, so

R = { (1,1), (1,3), (2,3), (3,2), (3,3), (4,3)}

# Exercise 3

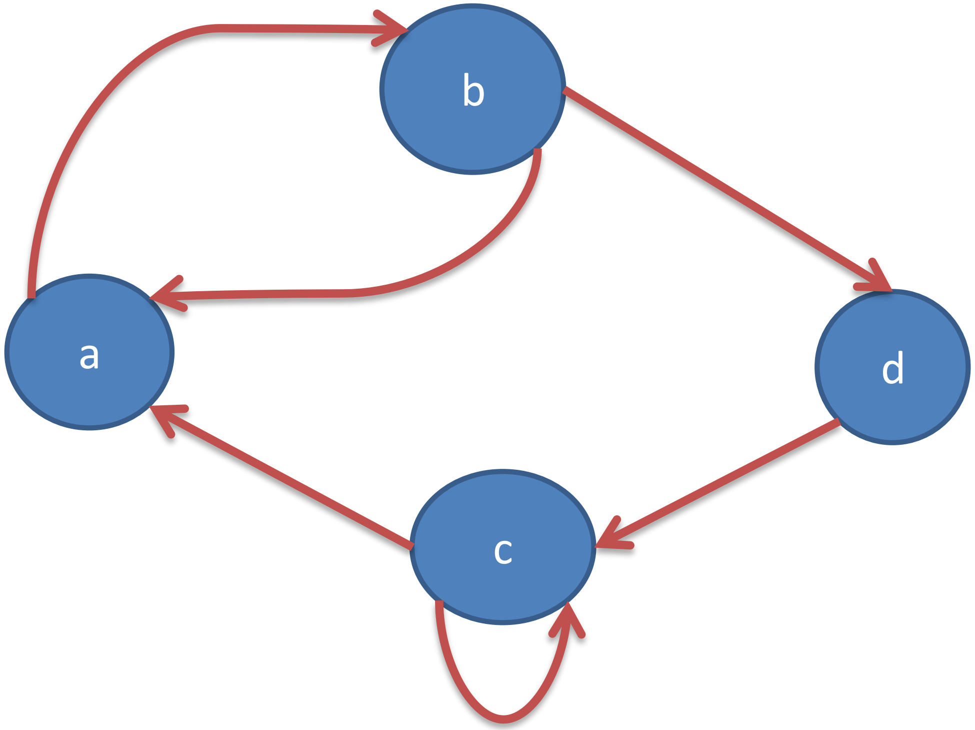
Draw the diagraph of the relation.

(i) $R = \{(a,c),(b,b),(b,c)\}$

(ii) The relation *R* on {1, 2, 3, 4} defined by $(x,y) \in R$ if $x^2 \geq y$

# Exercise 4

Write the relation as a set of ordered pair.



# Matrices of Relations

A matrix is a convenient way to represent a relation *R* from *A* to *B.*

- Label the rows with the elements of *A* (in some arbitrary order)
- Label the columns with the elements of *B* (in some arbitrary order)

# Matrices of Relations

- Let $M_R = [m_{ij}]_{n \times p}$ be the Boolean n x p matrix

$$M_R = \begin{bmatrix} m_{11} & m_{12} & .. & .. & m_{1p} \\ m_{21} & m_{22} & .. & .. & m_{2p} \\ \vdots & \vdots & .. & .. & \vdots \\ \vdots & \vdots & .. & .. & \vdots \\ m_{n1} & m_{n2} & .. & .. & m_{np} \end{bmatrix}$$

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{otherwise} \end{cases}$$

# Example 8

- The relation,
  $R = \{ (1,b),(1,d),(2,c),(3,c),(3,b),(4,a) \}$
  from, $X = \{ 1, 2, 3, 4 \}$ to $Y = \{ a, b, c, d \}$

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array} \text{ or } M_R = \begin{array}{c} \\ 2 \\ 3 \\ 4 \\ 1 \end{array} \begin{array}{c} \begin{array}{cccc} d & b & a & c \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

# Example 9

The matrix of the relation $R$ from { 2, 3, 4 } to { 5, 6, 7, 8 } defined by

$x \, R \, y$ if $x$ divides $y$

$$
\begin{array}{c} \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c} \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}
$$

# Example

Let A={ a, b, c, d }

Let R be a relation on A.

R = { (a,a),(b,b),(c,c),(d,d),(b,c),(c,b) }

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

# Example 10

An airline services the five cities $c_1$, $c_2$, $c_3$, $c_4$ and $c_5$. Table below gives the cost (in dollars) of going from $c_i$ to $c_j$. Thus the cost of going from $c_1$ to $c_3$ is RM100, while the cost of going from $c_4$ to $c_2$ is RM200

| To<br>from | $c_1$ | $c_2$ | $c_3$ | $c_4$ | $c_5$ |
|---|---|---|---|---|---|
| $c_1$ | | 140 | 100 | 150 | 200 |
| $c_2$ | 190 | | 200 | 160 | 220 |
| $c_3$ | 110 | 180 | | 190 | 250 |
| $c_4$ | 190 | 200 | 120 | | 150 |
| $c_5$ | 200 | 100 | 200 | 150 | |

If the relation $R$ on the set of cities $A = \{c_1, c_2, c_3, c_4, c_5\} : c_i \; R \; c_j$ if and only if the cost of going from $c_i$ to $c_j$ is defined and less than or equal to RM180.

i) Find $R$.

ii) Matrices of relations for $R$

# Solution

R = $\{(c_1,c_2), (c_1,c_3), (c_1,c_4), (c_2,c_4),(c_3, c_1), (c_3,c_2), (c_4,c_3), (c_4,c_5), (c_5,c_2), (c_5,c_4)\}$

$$M_R = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

# Exercise 5

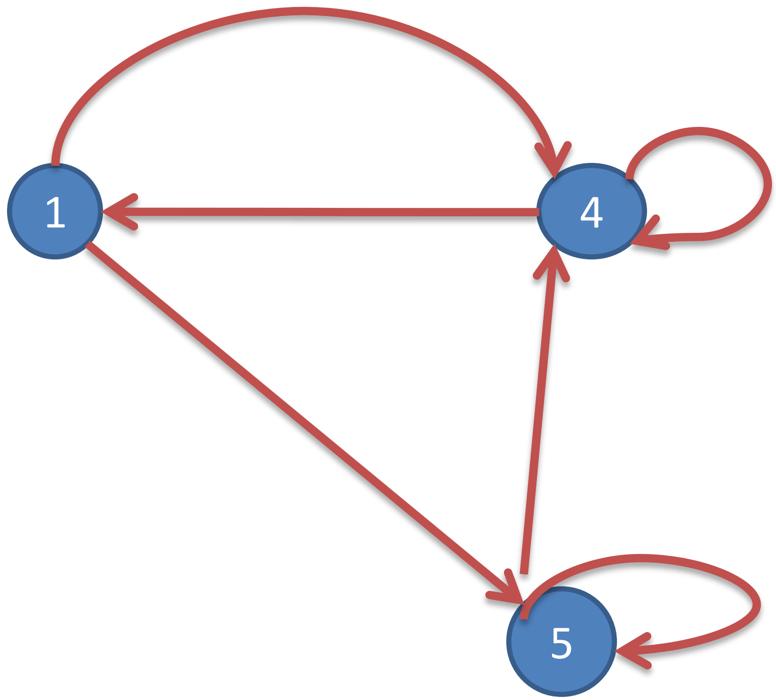
Let $A$ = {1, 2, 3,4} and $R$ is a relation from $A$ to $A$.

Suppose $R$ = {(1,2), (1,3), (1,4), (2,3), (2,4) , (3,4)}

- What is $R$ (represent)?
- What is matrix representation of $R$?

# Exercise 6

Let $A = \{1, 4, 5\}$ and let $R$ be given by the digraph shown below. Find $M_R$ and $R$



# In degree and out degree

If $R$ is a relation on a set $A$ and $a \in A$, then the in-degree of a (relative to relation $R$) is the number of $b \in A$ such that $(b,a) \in R$.

The out degree of $a$ is the number of $b \in A$ such that $(a,b) \in R$.

Meaning that, in terms of the digraph of R, is that the in-degree of a vertex is

**"the number of edges terminating at the vertex"**

The out-degree of a vertex is

**" the number of edges leaving the vertex"**

# Example 11

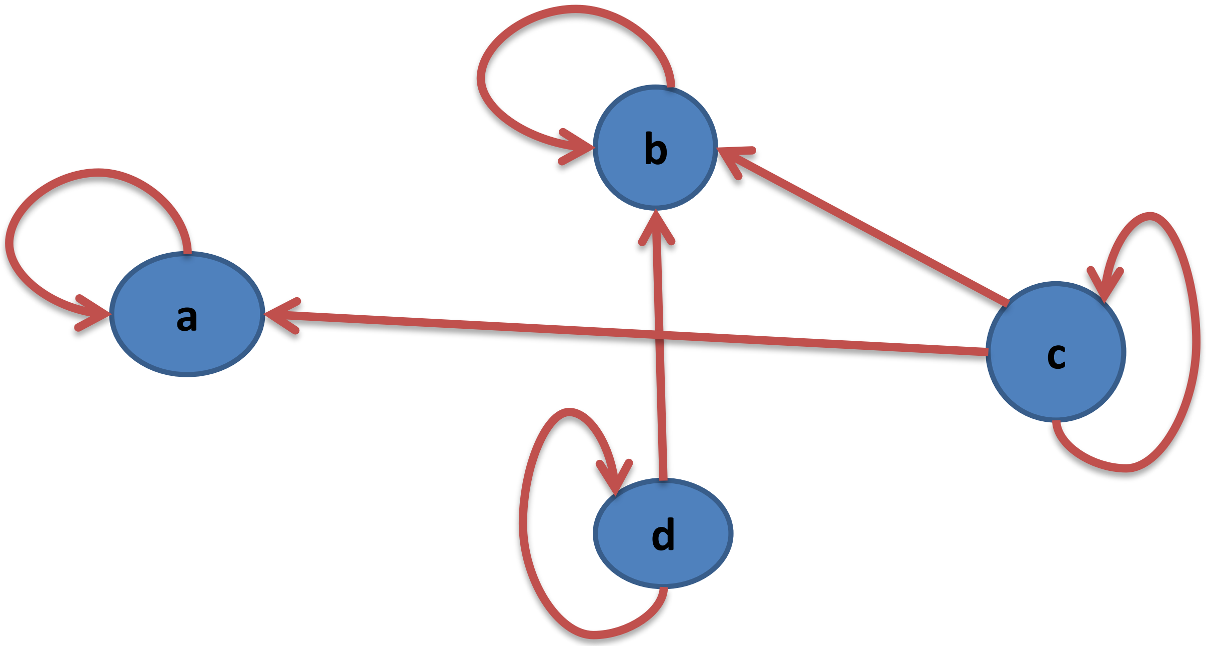
Let $A$ = {$a$, $b$, $c$, $d$}, and let $R$ be the relation on $A$ that has the matrix (given below)

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Construct the digraph of $R$, and list in-degrees and out-degrees of all vertices.

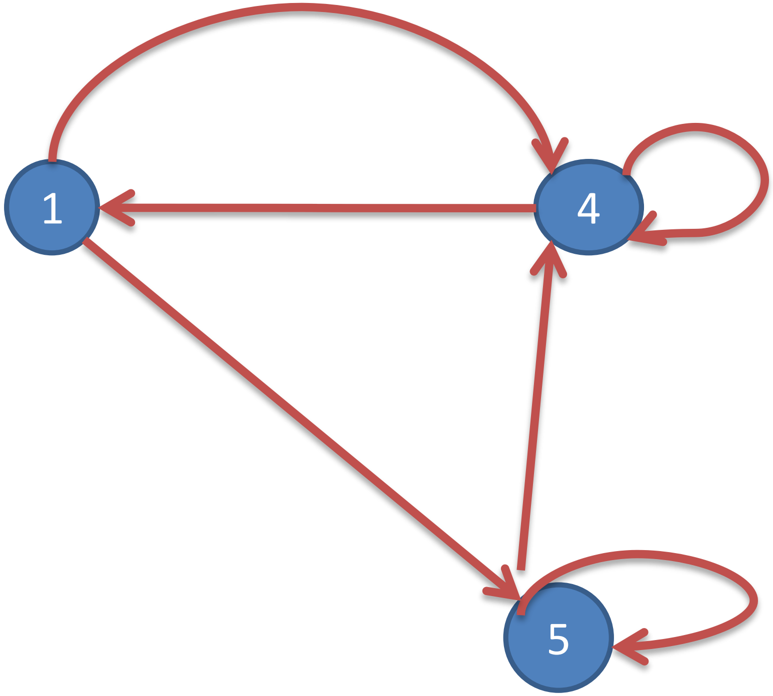
# Solution

| | a | b | c | d |
|---|---|---|---|---|
| In-degree | 2 | 3 | 1 | 1 |
| Out-degree | 1 | 1 | 3 | 2 |



# Exercise 7

Let $A = \{1, 4, 5\}$ and let $R$ be given by the digraph shown below. list in-degrees and out-degrees of all vertices.



# Reflexive Relations

- Reflexive
  - A Relation *R* on set *A* is called **reflexive** if every $a \in A$ is related to itself OR
  - A relation *R* on a set *X* is called **reflexive** if all pair $(x,x) \in R;\ \forall x : x \in X$
- Irreflexive
  - A relation *R* on a set *A* is **irreflexive** if $x \not R x$ or $(x,x) \notin R;\ \forall x : x \in X$
- Not Reflexive
  - A Relation *R* is **not reflexive** if at least one pair of $(x,x) \in R,\ \forall x : x \in X$

# Example 12

The relation $R$ on $X = \{1, 2, 3, 4\}$ defined by $(x,y) \in R$ if $x \leq y$, $x,y \in X$ is a reflexive relation.

For each element $x \in X$, $(x,x) \in R$
(1,1), (2,2), (3,3), (4,4) are each in $R$.

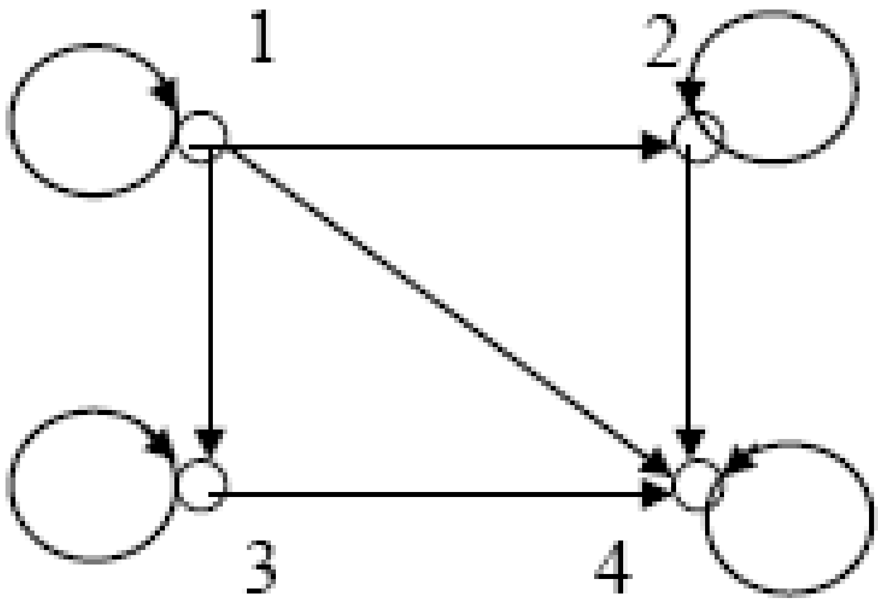
The relation $R = \{(a,a), (b,c), (c,b), (d,d)\}$ on $X=\{a, b, c, d\}$ is not reflexive.

This is because $b \in X$, but $(b,b) \notin R$ . Also $c \in X$, but $(c,c) \notin R$

# Reflexive Relations

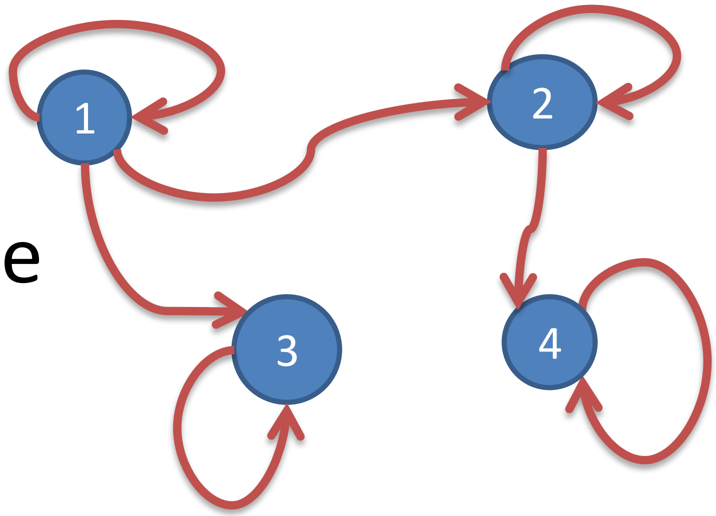
The digraph of a reflexive relation has a loop at every vertex.

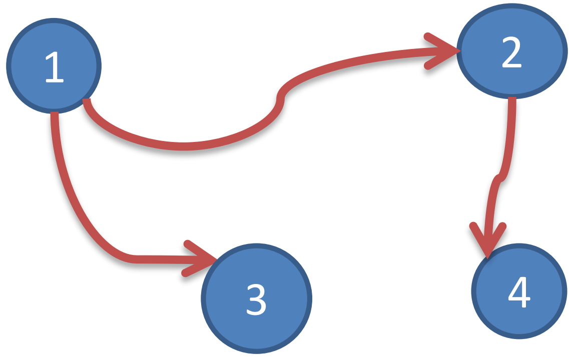
example



# Reflexive Relations

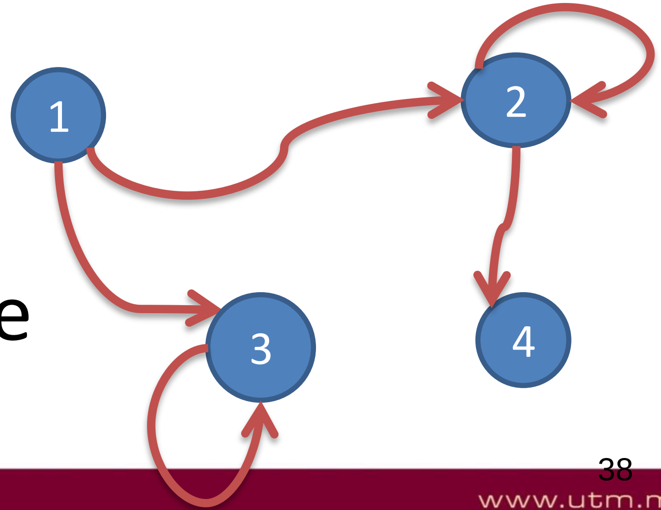
- Reflexive




- Irreflexive

- Not Reflexive



# Reflexive Relations

The relation R is reflexive if and only if the matrix of relation has 1's on the main diagonal.

example

| | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | 1 | 0 | 0 | 0 |
| $b$ | 0 | 1 | 1 | 0 |
| $c$ | 0 | 1 | 1 | 0 |
| $d$ | 0 | 0 | 0 | 1 |

# Reflexive Relations

The relation R is *irreflexive* if and only if the matrix relation have all 0's on its main diagonal

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

# Reflexive Relations

The relation *R* is not reflexive.

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \textcircled{0} & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$b \in X$

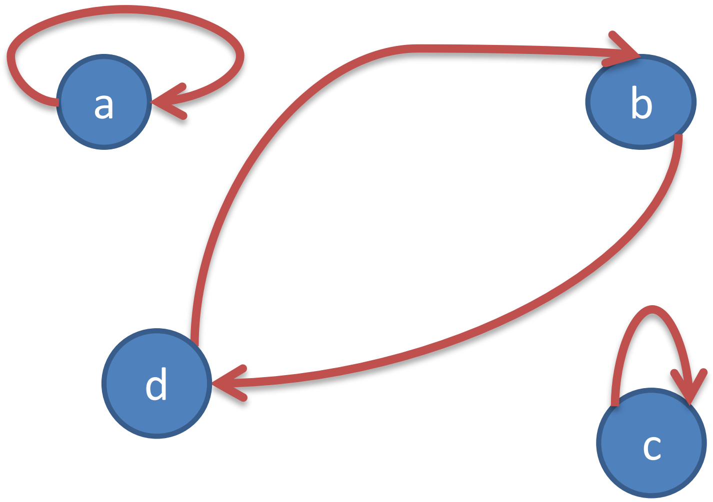
$(b,b) \notin R$

# Exercise 8

# Exercise 9

(i) Let $R$ be the relation on $X=\{1,2,3,4\}$ defined by $(x,y)\in R$ if $x\leq y$, $x,y\in X$. Determine whether $R$ is a reflexive relation.

(ii) The relation $R$ on $X=\{a,b,c,d\}$ given by the below diagraph. Is $R$ a reflexive relation?



# Exercise 10

Let A = { 1,2,3,4). Construct the matrix of relation of *R*. Then, determine whether the relation is reflexive, not reflexive or irreflexive.

(i) *R* = { (1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,4))

(ii) *R* = { (1,3), (1,1), (3,1), (1,2), (3,3), (4,4)}

(iii) *R* ={ (1,2), (1,3), (3,1), (1,1), (3,3), (3,2), (1,4), (4,2), (3,4)}

(iv) *R* ={ (1,2), (1,3), (3,2), (1,4), (4,2), (3,4)}

# Symmetric Relations

A relation $R$ on a set $X$ is called symmetric if for all $x,y \in X$, if $(x,y) \in R$, then $(y,x) \in R$.

$$\forall x,y \in X, (x,y) \in R \rightarrow (y,x) \in R$$

Let $M$ be the matrix of relation $R$.

The relation $R$ is symmetric if and only if for all $i$ and $j$, the $ij$-th entry of $M$ is equal to the $ji$-th entry of $M$.

# Symmetric Relations

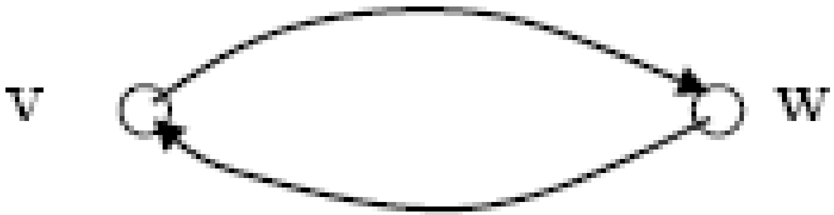
The matrix of relation $M_R$ is symmetric if $M_R = M_R^T$

example

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} = M_R^T$$

# Symmetric Relations

The digraph of a symmetric relation has the property that whenever there is a directed edge from v to w, there is also a directed edge from w to v.



# Example 13

- The relation $R = \{ (a,a), (b,c), (c,b), (d,d) \}$ on $X = \{ a, b, c, d \}$

$$\boxed{\begin{array}{l} (b,c) \in R \\ (c,b) \in R \end{array}}$$

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

symmetric

# Antisymmetric

A relation $R$ on set $A$ is antisymmetric if $a \neq b$, whenever $aRb$, then $b \not R a$. In other word if whenever $aRb$, then $bRa$ then it implies that $a=b$

$$\forall a,b \in A,\ (a,b) \in R \land a \neq b \rightarrow (b, a) \notin R$$

Or

$$\forall a,b \in A,\ (a,b) \in R \land (b, a) \in R \rightarrow a = b$$

# Antisymmetric Relations

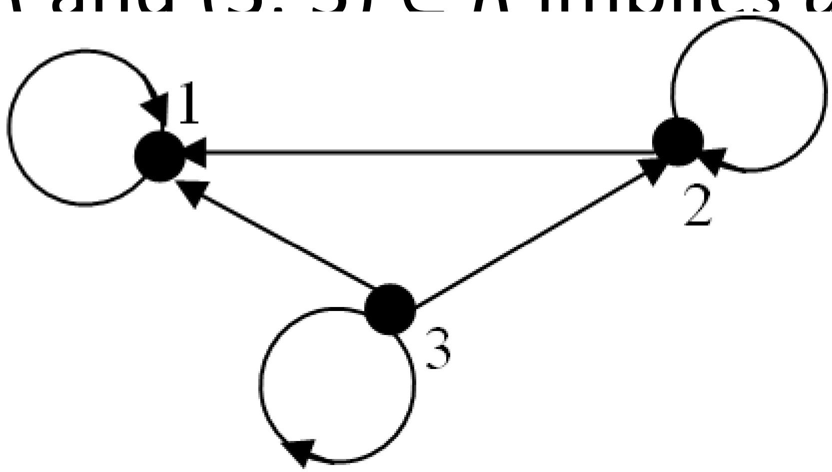
- Matrix $M_R = [M_{ij}]$ of an antisymmetric relation $R$ satisfies the property that if $i \neq j$, then $m_{ij}=0$ or $m_{ji}=0$
- If $R$ is antisymmetric relation, then for different vertices $i$ and $j$ there cannot be an edge from vertex $i$ to vertex $j$ and an edge from vertex $j$ to vertex $i$
- At least one directed relation and one way

# Example 14

- Let $R$ be a relation on $A = \{1, 2, 3\}$ defined as $(a, b) \in R$ if $a \geq b$, $a, b \in A$ is an antisymmetric relation because for all $a, b \in A$, $(a, b) \in R$ and $a \neq b$, then $(b, a) \notin R$, for example

$$(3, 2) \in R \text{ but } (2, 3) \notin R$$

$$(3, 3) \in R \text{ and } (3, 3) \in R \text{ implies } a = b$$



# Example 15

- The relation $R$ on $X = \{ 1, 2, 3, 4 \}$ defined by,

$$(x,y) \in R \quad \text{if } x \leq y,\ x,y \in X$$

$$(1,2) \in R$$
$$(2,1) \notin R$$

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

antisymmetric

30

# Example 16

The relation

$R = \{ (a,a), (b,b), (c,c) \}$

on $X = \{ a, b, c \}$

$R$ has no members of the form $(x,y)$ *with* $x \neq y$, then R is antisymmetric

$$\begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{matrix} a & b & c \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

# Asymmetric

Arelation is asymmetric if and only if it is both antisymmetric and irreflexive.

The matrix $M_{R} = [m_{ij}]$ of an asymmetric relation $R$ satisfies the property that

If $m_{ij} = 1$ then $m_{ji} = 0$

$m_{ii} = 0$ for all $i$ (the main diagonal of matrix $M_R$ consists entirely of 0's or otherwise)

# Digraph

- If *R* is asymmetric relation, then the digraph of *R* cannot simultaneously have an edge from vertex *i* to vertex *j* and an edge from vertex *j* to vertex *i*
- All edges are "one way street" and no loop at every vertex

# Example 17

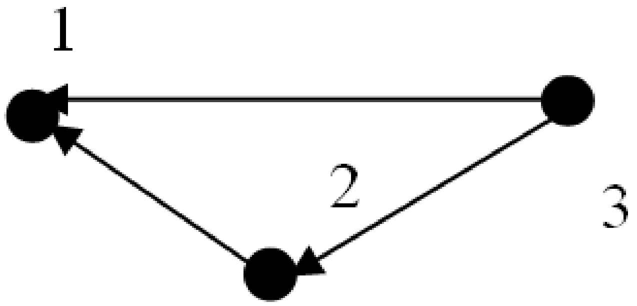
- Let $R$ be the relation on A = {1, 2, 3} defined by $(a, b) \in R$ if $a > b$, $a,b \in A$ is an asymmetric relation because,

$$(2, 1) \in R \text{ but } (1, 2) \notin R$$

$$(3, 1) \in R \text{ but } (1, 3) \notin R$$

$$(3, 2) \in R \text{ but } (2, 3) \notin R$$

$$(1, 1) \notin R,\ (2, 2) \notin R,\ (3, 3) \notin R$$



# Not Symmetric

- Let $R$ be a relation on a set $A$. Then $R$ is called ***not symmetric***, if for all $a$, $b \in A$, if $(a, b) \in R$, there exist $(b, a) \notin R$.

$$\exists a,b \in A, (a, b) \in R \rightarrow (b, a) \notin R$$

# Not Symmetric AND not antisymmetric

- Let $R$ be a relation on a set $A$. Then $R$ is called ***not symmetric*** and ***not antisymmetric***, if for all $a$, $b \in A$, if $(a, b) \in R$, there exist $(b, a) \notin R$ and if $(a, b) \in R$, there exist $(b, a) \in R$.

$$\exists a,b \in A, (a, b) \in R \rightarrow (b, a) \in R$$

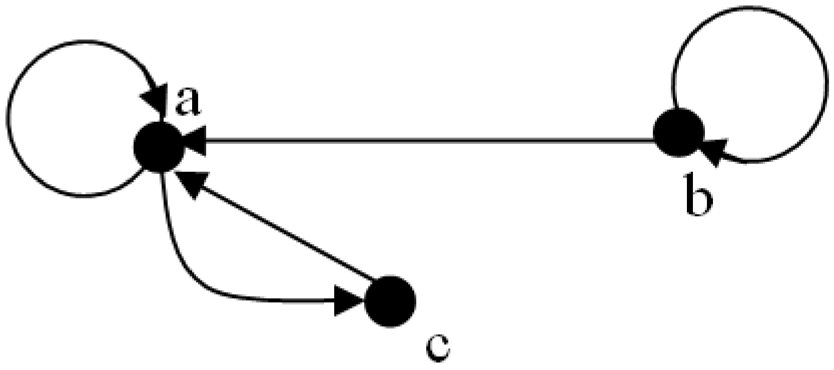
AND

$$\exists a,b \in A, (a, b) \in R \rightarrow (b, a) \notin R$$

# Example 18

- Relation $R = \{(a, c), (b, b), (c, a), (b, a), (a, a)\}$ on $A = \{a, b, c\}$ is not symmetric and not antisymmetric relation because there is,

$(a,c), (c,a) \in R$ and also $(b,a) \in R$ but $(a,b) \notin R$



# Example 19

- The relation $R = \{ (a,a), (b,c), (c,b), (d,d) \}$ on $X = \{ a, b, c, d \}$

$$(b,c) \in R$$
$$(c,b) \in R$$

|   | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | 1 | 0 | 0 | 0 |
| $b$ | 0 | 0 | 1 | 0 |
| $c$ | 0 | 1 | 0 | 0 |
| $d$ | 0 | 0 | 0 | 1 |

Symmetric and

not antisymmetric

# Example 20

1. Let $A=Z$, the set of integers and let $R=\{(a,b)\in A\times A \mid a<b\}$. So that $R$ is the relation "less than".

Is $R$ symmetric, asymmetric or antisymmetric?

2. Let $A=\{1,2,3,4\}$ and let $R = \{(1,2), (2,2), (3,4), (4,1)\}$

Determine whether $R$ symmetric, asymmetric or antisymmetric.

# Solution

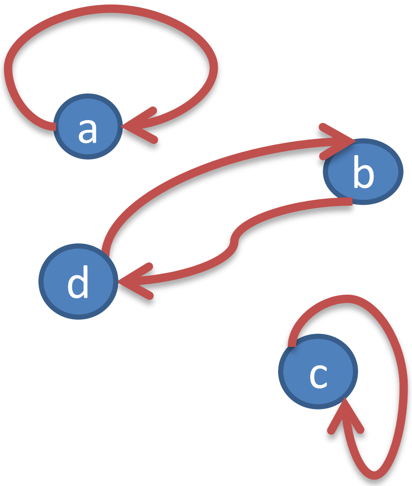
Question 1

- Symmetric : If $a<b$, then it is not true that $b<a$, so *R* is not symmetric
- Assymetric : If $a<b$ then $b>a$ (*b* is greater than *a*), so *R* is assymetric
- Antisymmetric : If $a \neq b$, then either $a>b$ or $b>a$, so *R* is antisymmetric

Question 2

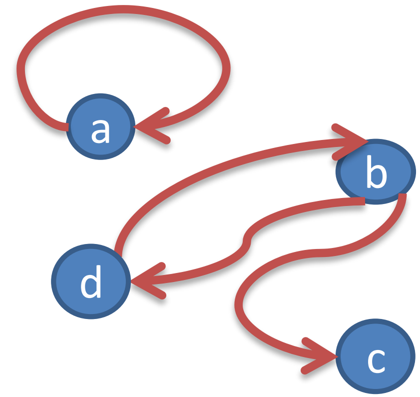
- *R* is not symmetric since $(1,2) \in R$, but $(2,1) \notin R$
- *R* is not asymmetric , since $(2,2) \in R$
- *R* is antisymmetric, since $a \neq b$, either $(a,b) \notin R$ or $(b,a) \notin R$

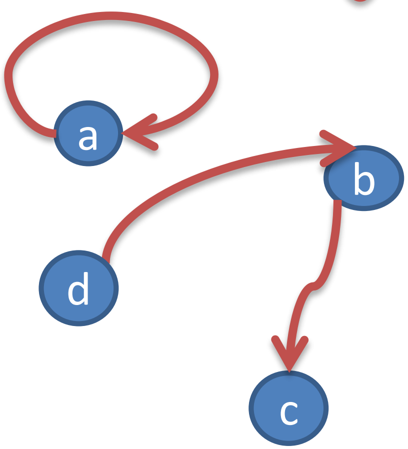
# Summary on Symmetric




Symmetric

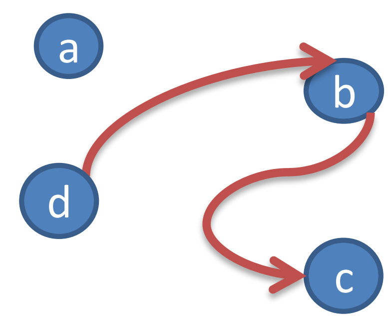
Not Symmetric and not Antisymmetric




Antisymmetric

Asymmetric



# Exercise 14

# Exercise 15

Let A = { 1,2,3,4). Construct the matrix of relation of *R*. Then, determine whether the relation is symmetric, assymetric, antisymmetric, not symmetric or not antisymmetric.

(i) *R* = { (1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,4))

(ii) *R* = { (1,3), (1,1), (3,1), (1,2), (3,3), (4,4)}

(iii) *R* ={ (1,2), (1,3), (1,1), (3,3), (3,2), (1,4), (4,2), (3,4)}

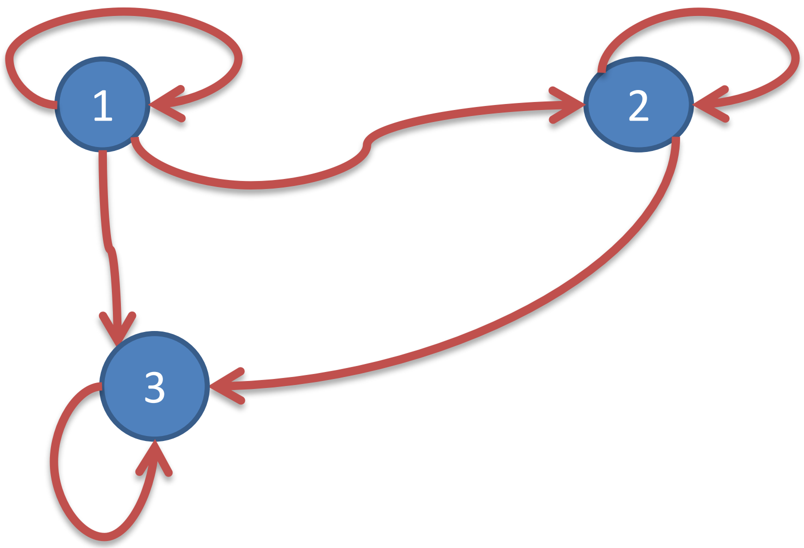
# Transitive Relations

- A relation $R$ on set $A$ is transitive if for all $a,b \in A$, $(a,b) \in R$ and $(b,c) \in R$ implies that $(a,c) \in R$
- In the diagraph of $R$, $R$ is a transitive relation if and only if there is a directed edge from one vertex $a$ to another vertex $b$, an if there exits a directed edge from vertex $b$ to vertex $c$, then there must exists a directed edge from $a$ to $c$

# Example 21

*R*={(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)}

The diagraph:



# Exercise 16

# Transitive Relations

The relation R on a set A is transitive if an only if $R^2 \subseteq R$ .

Every element of $R^2$ is an element of $R$

If $(a,c) \in R^2$ then $(a,c) \in R$

The matrix of the relation $M_R$ of $R^2$

$$M_{R^2} = M_R \otimes M_R$$

$\otimes$ is the product of boolean

# Boolean Algebra

| + | 1 | 0 |
|---|---|---|
| 1 | 1 | 1 |
| 0 | 1 | 0 |

| . | 1 | 0 |
|---|---|---|
| 1 | 1 | 0 |
| 0 | 0 | 0 |

- The matrix $M_R = [m_{ij}]$ and $M_{R^2} = [n_{ij}]$
- The relation R on a set A is transitive

If $n_{ij} = 1$ then $m_{ij} = 1$

# Example 22

The relation *R* on *A*={1,2,3} defined by *(a,b)*$\in$*R* if *a*$\leq$*b, a,b*$\in$*A*, is a transitive. The matrix of relation $M_{R,}$

$$M_R = \begin{matrix} & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 3 & 0 & 0 & 1 \end{matrix}$$

The product of boolean,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that,(1,2) and (2,3)$\in$*R*, (1,3)$\in$$R^2$ *and (1,3)* $\in$*R.Thus* $R^2 \subseteq R$

# Example 23

The relation $R$ on $A$={a,b,c,d} IS r={(a,a), (b,b), (c,c), (d,d), (a,c), (c,b)} is not transitive. The matrix of relation $M_{R,}$

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The product of boolean,

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \otimes \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Note that,(a,c) and (c,b)$\in R$, (a,b)$\in R^2$ but (a,b)$\notin R$.Thus $R^2$ is not a subset of R and the relation is not tansitive

# Example 24

Let $R$ be a relation on $A=\{1,2,3\}$ is defined by $(a,b)\in R$ if $a\leq b$, $a,b\in A$. Find $R$. Is $R$ a transitive relation?

Solution:

$R=\{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$

$R$ is a transitive relation because

(1,2) and $(2,2)\in R$, $(1,2)\in R$

**(1,2) and $(2,3)\in R$, $(1,3)\in R$**

**(1,3) and $(3,3)\in R$, $(1,3)\in R$**

(2,2) and $(2,3)\in R$, $(2,3)\in R$

(2,3) and $(3,3)\in R$, $(2,3)\in R$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad R^2 \subseteq R$$

# Equivalence Relations

Relation $R$ on set $A$ is called an equivalence relation if it is a **reflexive, symmetric and transitive**.

**Example 25**

Let $R$={(1,1), (1,3), (2,2), (3,1), (3,3)} on {1,2,3}, the matirx of the relation $M_R$,

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

All the main diagonal matrix elements are 1 and the matris is reflexive.

# Equivalence Relations

**Example 25(Cont.)**

The transpose matrix $M_R$,$M_R^T$ is equal to $M_R$, so $R$ is symmetric

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array} \qquad M_R^T = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

The product of boolean show that the matrix is transitive.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

# Partial Order Relations

Relation $R$ on set $A$ is called a partial order relation if it is a reflexive, antisymmetric and transitive.

**Example 26**

Let $R$ be a relation on a set $A$={1,2,3} defined by $(a,b) \in R$ if $a \leq b$, $a,b \in R$.

$R$={(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)}

$R$ is reflexive, antisymmetric and transitive.

So $R$ is a partial order relation.

# Exercise 17

The relation *R* on the set {1,2,3,4,5} defined by the rule $(x,y) \in R$ if $x+y \leq 6$

(i) List the elements of R

(ii) Find the domain of R

(iii) Find the range of R

(iv) Is the relation of R refelxive, symmetric, assymmetric, antisymmetric, transitive, and/or equivalence relation or partial order?

# Exercise 18

The relation *R* on the set {1,2,3,4,5} defined by the rule $(x,y) \in R$ if 3 divides $x$-$y$

(i) List the elements of R

(ii) Find the domain of R

(iii) Find the range of R

(iv) Is the relation of R refelxive, symmetric, assymmetric, antisymmetric, transitive, and/or equivalence relation or partial order?

# Exercise 19

The relation *R* on the set {1,2,3,4,5} defined by the rule $(x,y) \in R$ if $x=y-1$

(i) List the elements of R

(ii) Find the domain of R

(iii) Find the range of R

(iv) Is the relation of R refelxive, symmetric, assymmetric, antisymmetric, transitive, and/or equivalence relation or partial order?