SCSI1013: Discrete Structures

CHAPTER 1

SET THEORY

[Part 1: Set & Subset]

Introduction

- Why are we studying sets
  - The concept of set is basic to all of mathematics and mathematical applications.
  - Serves as a basis of description of higher concept and mathematical reasoning
  - Set is fundamental in many areas of Computer Science.
    - For us, sets are useful to understand the principles of *counting and probability theory*

Sets

- The concept of set is basic to all of mathematics and mathematical applications.
- A set is a well-defined collection of distinct objects.
- These objects are called members or elements of the set.

Sets

- Well-defined means that we can tell for certain whether an object is a member of the collection or not.
- If a set is finite and not too large, we can describe it by listing the elements in it.

Example

- *A* is a set of all positive integers less than 10, *A*={1, 2, 3, 4, 5, 6, 7, 8, 9}
- *B* is a set of first 5 positive odd integers, *B*={1, 3, 5, 7, 9}
- *C* is a set of vowels, *C*={*a*, *e*, *i*, *o*, *u*}

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Defining Sets

- This can be done by:
  - Listing ALL elements of the set within braces.
  - Listing enough elements to show the pattern then an ellipsis.
  - Use set builder notation to define "rules" for determining membership in the set.

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Example

1. Listing ALL elements. $A = \{1, 2, 3, 4\}$ explicitly
2. Demonstrating a pattern. $\mathbb{N} = \{1, 2, 3, \ldots\}$ implicitly
3. Using set builder notation. $P = \{x \mid x \in \mathbb{R} \text{ and } x \notin \mathbb{C}\}$ implicitly

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Sets

- A set is determined by its elements and not by any particular order in which the element might be listed.
- **Example**,
  A={1, 2, 3, 4},
  A might just as well be specified as
  {2, 3, 4, 1} or {4, 1, 3, 2}

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Sets

- The elements making up a set are assumed to be distinct, we may have duplicates in our list, only one occurrence of each element is in the set.
- **Example**

  $\{a, b, c, a, c\} \longrightarrow \{a, b, c\}$

  $\{1, 3, 3, 5, 1\} \longrightarrow \{1, 3, 5\}$

Sets

- Use uppercase letters *A, B, C …* to denote sets, lowercase denote the elements of set.
- The symbol $\in$ stands for 'belongs to'
- The symbol $\notin$ stands for 'does not belong to'

Example

- $X=\{ a, b, c, d, e \}$, $b \in X$ and $m \notin X$
- $A=\{\{1\}, \{2\}, 3, 4\}$, $\{2\} \in A$ and $1 \notin A$

Sets

- If a set is a large finite set or an infinite set, we can describe it by listing a property necessary for memberships
- Let S be a set, the notation,

  $A= \{x \mid x \in S, P(x)\}$ or $A= \{x \in S \mid P(x)\}$

  means that A is the set of all elements x of S such that x satisfies the property P.

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Example

- Let A={1, 2, 3, 4, 5, 6}, we can also write A as,
  $A=\{x \mid x \in Z,\ 0 < x < 7\}$
  if Z denotes the set of integers.

- Let $B=\{x \mid x \in Z, x>0\}$, $B=\{1, 2, 3, 4, \ldots\}$

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Example

The set of natural numbers: $\mathbb{N} = \{1, 2, 3, \ldots\}$
The set of integers: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
The set of positive integers: $\mathbb{Z}^{+} = \{1, 2, 3, \ldots\}$

The set of Rational Numbers (fractions): $\frac{1}{2}, \frac{2}{3}, \frac{5}{7}, etc \in \mathbb{Q}$

More formally: $\mathbb{Q} = \left\{ \frac{a}{b} \middle| a, b \in \mathbb{R},\ b \neq 0 \right\}$

The set of Irrational Numbers: $\sqrt{2}, \pi$, or e are irrational

The Real numbers $= \mathbb{R} =$ the union of the rational numbers with the irrational numbers

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Some Symbols Used With Set Builder Notation

The standard form of notation for this is called "set builder notation".

For instance, $\{x \mid x$ is an odd positive integer$\}$ represents the set $\{1, 3, 5, 7, 9, \ldots\}$

$\{x \mid x$ is an odd positive integer$\}$ is read as

"the set consisting of all x such that x is an odd positive integer".

The vertical bar, " | ", stands for "such that"

Other "short-hand" notation used in working with sets

"$\forall$" stands for "for every"
"$\exists$" stands for "there exists"
"$\cup$" stands for "union"
"$\cap$" stands for "intersection"
"$\subseteq$" stands for "is a subset of"
"$\subset$" stands for "is a (proper) subset of"
"$\not\subset$" stands for "is a not a (proper) subset of"
"$\varnothing$" stands for the "empty set"
"$\in$" stands for "is an element of"
"$\notin$" stands for "is not an element of"
"$\times$" stands for "cartesian cross product"
"$=$" stands for "is equal to"

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Subset

- If every element of A is an element of B, we say that A is a subset of B and write $A \subseteq B$.

  $A=B$, if $A \subseteq B$ and $B \subseteq A$

- The empty set ($\varnothing$) is a subset of every set.

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Example

A={1, 2, 3}

Subset of A,

∅, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3} {1, 2, 3}

Note:
A is a subset of *A*

Exercise

Answer true or false

a) $\{x\} \subseteq \{x\}$
b) $\{x\} \in \{x\}$
c) $\{x\} \in \{x,\{x\}\}$
d) $\{x\} \subseteq \{x,\{x\}\}$

Proper Subsets

If $A \subseteq B$ and B contains an element that is not in A, then we say "A is a **proper subset** of B": $A \subset B$ or $B \supset A$.

Formally: $A \subseteq B$ means $\forall x\ [x \in A \rightarrow x \in B]$.

For all sets: $A \subseteq A$.

Note: If *A* is a subset of *B* and *A* does not equal B, we say that *A* is a proper subset of *B* ($A \subseteq B$ and $A \neq B$ ($B \not\subseteq A$))

Example

- A={1, 2, 3}

- Proper subset of A,

  ∅, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}

Note:
A proper subset of a set *A* is a subset of *A* that is not equal to *A* ($\{1,2,3\} \not\subset A$)

Example

Let a set, B={1, 2, 3, 4, 5, 6} and the subset, A = {1, 2, 3}.

$\Rightarrow$ A is proper subset of B.

Example

A={a,b,c,d,e,f,g,h}

B={b,d,e}

C={a,b,c,d,e}

D={r,s,d,e}

- Proper subset of A ?? B and C

Empty Sets

The **empty set** $\varnothing$ or {} but not {$\varnothing$} is the set without elements.

Note:

- Empty set has no elements
- Empty set is a subset of any set
- There is exactly one empty set
- Properties of empty set:
  $A \cup \varnothing = A$, $A \cap \varnothing = \varnothing$
  $A \cap A' = \varnothing$, $A \cup A' = U$
  $U' = \varnothing$, $\varnothing' = U$

Example

$\varnothing$ = $\{x \mid x$ is a real number and $x^2 = -3\}$

$\varnothing$ = $\{x \mid x$ is positive integer and $x^3 < 0\}$

Equal Sets

The sets A and B are **equal** (A=B) if and only if each element of A is an element of B and vice versa.

Formally: A=B means $\forall x\ [x \in A \leftrightarrow x \in B]$.

Example

$A=\{a, b, c\}$, $B=\{b, c, a\}$, $A=B$

$C=\{1, 2, 3, 4\}$

$D=\{x \mid x$ is a positive integer and $2x < 10\}$,

$C=D$

Exercise

Determine whether each pair of sets is equal

a) {1, 2, 2, 3}, {1, 3, 2}
b) $\{x \mid x^2 + x = 2\}$, {1, -2}
c) $\{x \mid x$ is a real number and $0 < x \leq 2\}$, {1,2}

Equivalent Sets

Two sets, A and B, are equivalent if there exists a one-to-one correspondence between them.

When we say sets "have the same size", we mean that they are equivalent.

Example

Set A: {A, B, C, D, E} and **Set** B: {1, 2, 3, 4, 5}

Note:
- An **equivalent set** is simply a **set** with an **equal** number of elements.
- The **sets** do not have to have the same exact elements, just the same number of elements.

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Finite Sets

A set A is finite

if it is empty
or
if there is a natural number *n* such that set A is equivalent to {1, 2, 3, . . . *n*}.

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Example

$A = \{1, 2, 3, 4\}$

$B = \{x \mid x \text{ is an integer}, 1 \leq x \leq 4\}$

Note:
There exists a nonnegative integer n such that *A* has *n* elements (*A* is called a finite set with *n* elements)

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Infinite Sets

A set *A* is infinite

if there is NOT a natural number *n* such that set *A* is equivalent to {1, 2, 3, . . . *n*}.

Infinite sets are uncountable.
Are all infinite sets equivalent?

A set is infinite if it is equivalent to a proper subset of itself!

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Example

$Z = \{x \mid x$ is an integer$\}$
or $Z = \{..., -3, -2, -1, 0, 1, 2, 3,...\}$

$S=\{x \mid x$ is a real number and $1 \leq x \leq 4\}$

Example

$C = \{5, 6, 7, 8, 9, 10\}$ *(finite set)*

$B = \{x \mid x$ is an integer, $10 < x < 20\}$ *(finite set)*

$D = \{x \mid x$ is an integer, $x > 0\}$ *(infinite set)*

Universal Set

- Sometimes we are dealing with sets all of which are subsets of a set *U*.
- This set *U* is called a universal set or a universe.
- The set *U* must be explicitly given or inferred from the context.

Universal Set

Typically we consider a set A
a part of a **universal set** $\mathcal{U}$,
which consists of all possible elements.
To be entirely correct we should say
$\forall x \in \mathcal{U}\ [x \in A \leftrightarrow x \in B]$
instead of
$\forall x\ [x \in A \leftrightarrow x \in B]$ for A=B.

Note that $\{ x \mid 0<x<5\}$ is can be ambiguous.
Compare $\{ x \mid 0<x<5, x \in N \}$ with $\{ x \mid 0<x<5, x \in Q \}$

Example

- The sets A={1,2,3}, B={2,4,6,8} and C={5,7}
- One may choose U={1,2,3,4,5,6,7,8} as a universal set.
- Any superset of U can also be considered a universal set for these sets A, B, and C.
  For example, $U=\{x \mid x$ is a positive integer$\}$

Cardinality of Set

- Let S be a finite set with n distinct elements, where $n \geq 0$.
- Then we write $|S|=n$ and say that the cardinality (or the number of elements) of S is n.
- **Example**
  A= {1, 2, 3}, $|A|=3$
  B= {a,b,c,d,e,f,g}, $|B|=7$

Exercise

- If M is finite, determine the $|M|$
  - If M= {1, 2, 3, 4}
  - If M = {4, 4, 4}
  - If M= {}
  - If M= {ø,{ø}, {ø, {ø}}}

Power Set

- The set of all subsets of a set A, denoted $P(A)$, is called the power set of A.

$$P(A)= \{X \mid X \subseteq A\}$$

- If $|A|=n$, then $|P(A)| = 2^n$

Example

- A={1,2,3}
- The power set of A,
  $P(A)$= {∅, {1}, {2}, {3}, {1,2}, {1,3}, {2,3}, {1,2,3} }
- Notice that $|A| = 3$, and $|P(A)| = 2^3 = 8$

Exercise

- Let X= {1, 2, 2, {1}, a}
- Find:
  - $|X|$
  - Proper subset of X
  - Power set of X

Exercise

- List the member of P({a, b, c, d}). Which are proper subset of {a, b, c, d}?

Summary

How to Think of Sets

The elements of a set do not have an ordering, hence {a,b,c} = {b,c,a}
The elements of a set do not have multitudes, hence {a,a,a} = {a,a} = {a}
All that matters is: "Is x an element of A or not?"
The size of A is thus the number of *different* elements