CHAPTER 5

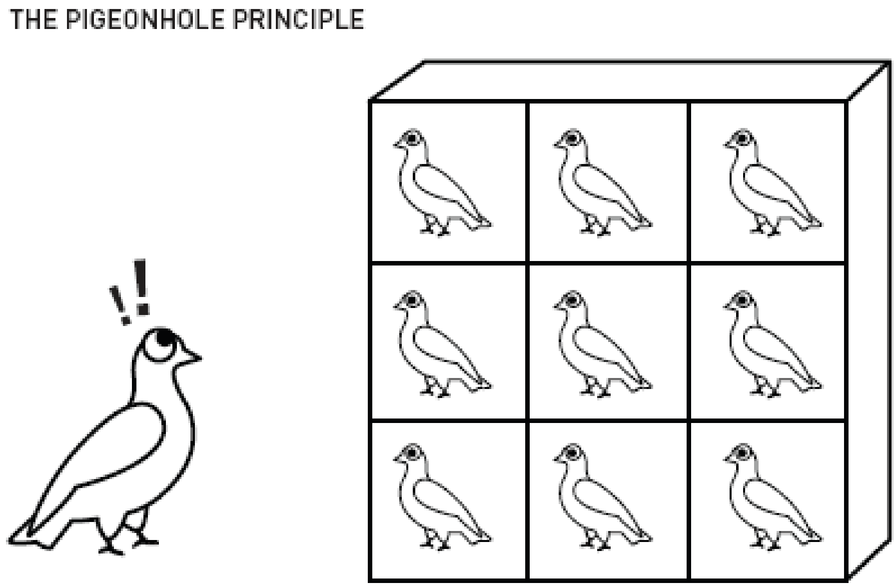
COUNTING METHODS (Part 3)

Pigeonhole Principle

Introduction

- The Pigeonhole Principle is a really simple concept
- Discovered in the 1800s
- Peter Gustav Lejeune Dirichlet was the youngest member of the Prussian Academy of Sciences, he worked at number theory and analysis
- He also came up with a simple little thing that he called The Dirichlet Drawer Principle (or Shoe Box Principle), but that we now call The Pigeonhole Principle.

Pigeonhole Principle



- Imagine 9 pigeonholes and 10 pigeons. A storm comes along, and all of the pigeons take shelter inside the pigeonholes.
- They could be arranged any number of ways. For instance, all 10 pigeons could be inside one hole, and the rest of the holes could be empty.
- What we know for sure, no matter what, is that there is at least one hole that contains more than one pigeon.

The principle works no matter what the particular number of pigeons and pigeonholes. As long as there are **(N - 1) number of pigeonholes, and (N) number of pigeons**, we know there will always be at least two pigeons in one hole.

Pigeonhole Principle (1st Form)

Pigeonhole Principle

(First Form)

If n pigeons fly into k pigeonholes and $k < n$, some pigeonhole contains at least two pigeons

Pigeonhole Principle (1^{st} Form)

- The principle tells nothing about how to locate the pigeonhole that contains 2 or more pigeons
- It only asserts the existence of a pigeonhole containing 2 or more pigeons
- To apply this principle, one must decide
  – which objects are the pigeons
  – Which objects are the pigeonholes

Example (PP - 1st Form)

1. Among 8 people there are at least two persons who have the same birthday.
   - Pigeonholes : Days (7) – Monday to Sunday
   - Pigeons: People (8)

2. Among 13 people there are at least two persons whose month of birth is same.
   - Pigeonholes : Months(12) – January to December
   - Pigeons: People (13)

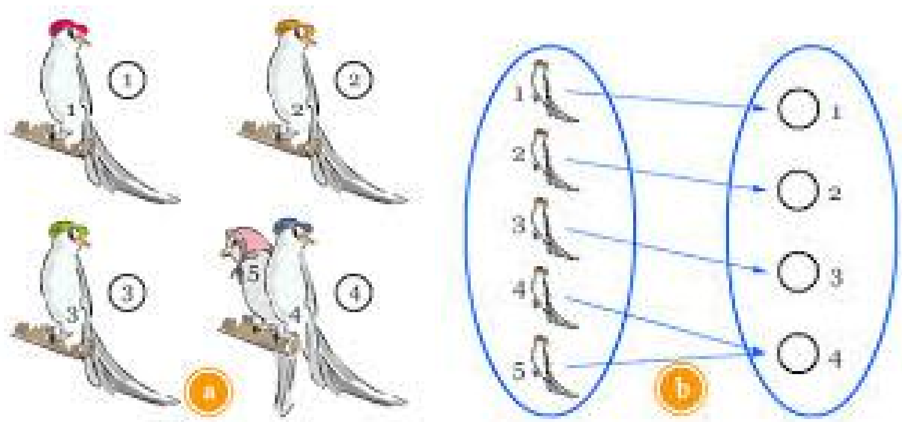
Example (PP - 1st Form)

3. In a party there are n people. Prove that there are at least two persons who know exactly the same number of people.
   - Pigeonholes : How many people a person can know which is at most n-1 (need to exclude him/herself)
   - Pigeons: People (n)
   - If a person knows i people then the person is put in the i-th box. There are n people. So there must be one box which contains 2 persons.

Pigeonhole Principle (2nd Form)

Pigeonhole Principle

(Second Form)



If f is a function from a finite set X to a finite set Y and $|X| > |Y|$, then $f(x_1) = f(x_2)$ for some $x_1, x_2 \in X, x_1 \neq x_2$

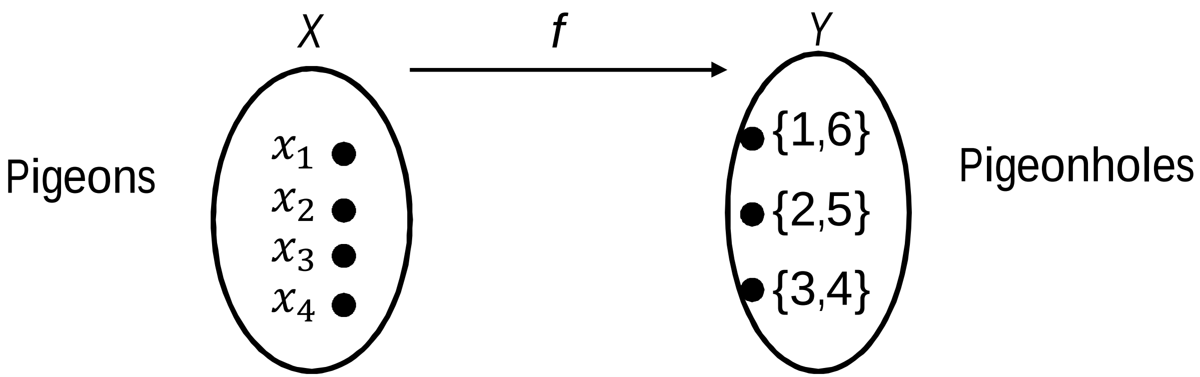
Pigeonhole Principle (2^{nd} Form)

- The 2^{nd} form can be reduced to the 1^{st} form by letting *X* be the set of pigeons and *Y* be the set of pigeonholes.
- Assign pigeon *x* to pigeonhole *f(x)*
- By the 1^{st} form principle, at least 2 pigeons, $x_1, x_2 \in X$, are assigned to the same pigeonhole; that is, $f(x_1) = f(x_2)$ for some $x_1, x_2 \in X, x_1 \neq x_2$

Example 1 (PP – 2nd Form)

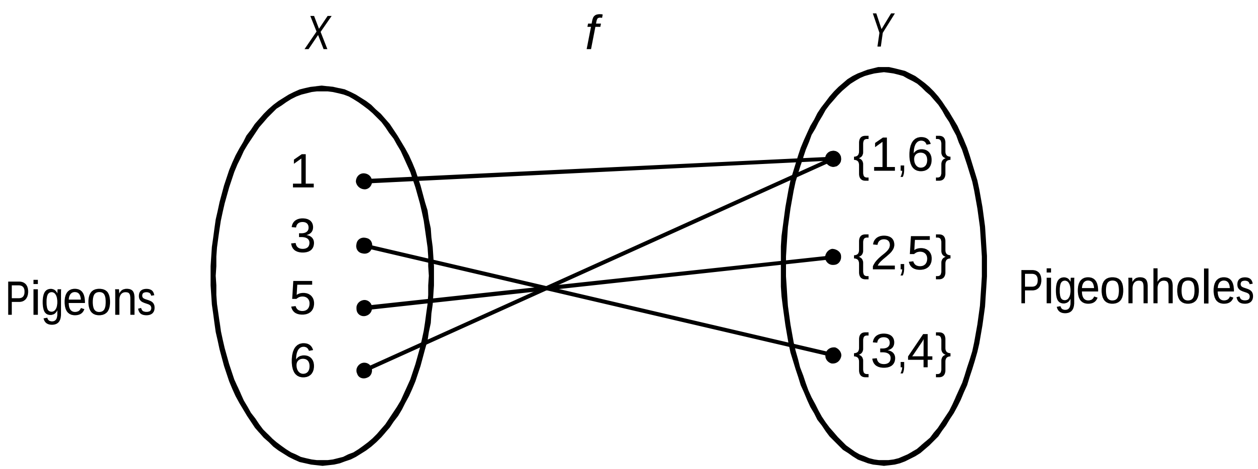
Let A = {1,2,3,4,5,6}. Show that if we choose any four distinct members of A, then for at least one pair of these four integers their sum is 7.

- Notice that {1,6}, {2,5} and {3,4} are the only pairs of distinct integers such that their sum is 7.
- Let $X = \{x_1, x_2, x_3, x_4\}$ be any subset of four distinct elements of A.
- Let Y = {{1,6}, {2,5}, {3,4}}, a set of 3 distinct elements and a part of A. y_1 = {1,6}, y_2 = {2,5}, y_3 = {3,4}



Example 1 (PP – 2^{nd} Form) continue

- Define $f: X \rightarrow Y$ by $f(a) = y_i$ if $a \in y_i$. For example, if $a = 1 \in X$, then $f(1) = \{1,6\}$.
- For $X = \{1,3,5,6\}$, see in the figure below.
- Now $|X|$=4 and $|Y|$=3. Then by 2^{nd} form principle, at least two distinct elements of X must be mapped to the same element of Y.
- Hence, if we choose any four distinct members of A, then for at least one pair of these four integers, their sum is 7.



Example 2 (PP – 2nd Form)

Using instant messaging, every Sunday evening 10 friends communicate with each other. Instant messaging allows a person to open separate window for each person he or she is communicating with. Then at any time at least 2 of these 10 friends must be communicating with the same number of friends.

- Let $X = \{x_1, x_2, \dots, x_{10}\}$ be the set of 10 friends.
- For each x_i, let n_i be the number of friends they are communicating with with $i = 1,2, \dots, 10$.
- A person may not be communicating with any person or may be communicating with as many as 9 people.
- Thus, $0 \leq n_i \leq 9$, $i = 1,2, \dots, 10$.

Example 2 (PP – 2nd Form) continue

- If we take $Y = \{0,1,2, \dots, 9\}$, then we cannot apply the pigeonhole principle because the number of elements in *X* and the number of elements in the *Y* are the same.
- Suppose that one of the friends, say x_i, is not communicating with any other friend. Then $n_i = 0$.
- The remaining people can communicate with at most 8 other people.
- Thus, $0 \leq n_i \leq 8, i = 1,2, \dots, 10$. Then $Y = \{0,1,2, \dots, 8\}$.
- Set *X* is the pigeons and set *Y* is the pigeonholes. Then $|X| = 10$ and $|Y| = 9$. Then by 2nd form principle, at least two distinct elements of *X* must be mapped to the same element of *Y*.

Pigeonhole Principle (3rd Form)

Pigeonhole Principle

(Third Form)

The Generalized Pigeonhole Principle

Ceiling function that takes as input a **real number** *x* and gives as output the least **integer** ceiling (*x*) = $\lceil x \rceil$ that is greater than or equal to *x*

Let *f* be a function from a finite set *X* to a finite set *Y*. Suppose that $|X| = n$ and $|Y| = m$. Let $k = \left\lceil \frac{n}{m} \right\rceil$. Then there are at least k values $a_1, \ldots, a_k \in X$ such that $f(a_1) = f(a_2) = \ldots = f(a_k)$

Pigeonhole Principle (3rd Form)

- To prove – argue by contradiction.
- Let $Y = \{y_1, \ldots, y_m\}$.
- Suppose that the conclusion is false. Then there are at most $k-1$ values $x \in X$ with $f(x) = y_1$; there are at most $k-1$ values $x \in X$ with $f(x) = y_2$; … ; there are at most $k-1$ values $x \in X$ with $f(x) = y_m$.
- Thus there are at most $m(k-1)$ members in the domain of f.

Pigeonhole Principle (3rd Form)

- However

$$m(k-1) < m\frac{n}{m} = n$$

  is a contradiction

- Therefore, there are at least k values, $a_1, \ldots, a_k \in X$, such that

$$f(a_1) = f(a_2) = \cdots = f(a_k)$$

Example 1 (PP – 3rd Form)

Suppose that there are 50 people in a room. Then at least 5 of these people must have their birthday in the same month.

- Pigeons – people ($n = 50$).
- Pigeonholes – months ($m = 12$).
- Thus

$$k = \left\lceil \frac{n}{m} \right\rceil = \left\lceil \frac{50}{12} \right\rceil = 5$$

More on 3rd form Pigeon hole

- A common type of problem asks for the minimum number of *pigeon (n)* such that at least *k* of these pigeon must be in one of pigeonhole (m) when these objects are distributes,
- From

$$\frac{n}{m} \geq k \;;$$

The smallest integer *n* with $\frac{n}{m} > k - 1$; $n = m(k-1) + 1$

Example

- What is the minimum number of students required in a course to be sure that at least six will receive the same grade, if there five possible grades, A, B,C,D and F?

pigeon (n)– the number of student ?

pigeon hole (m) – the grades = 5

k= 6

$$\therefore n = m(k-1)+1 = 5(6-1)+1 = 26$$

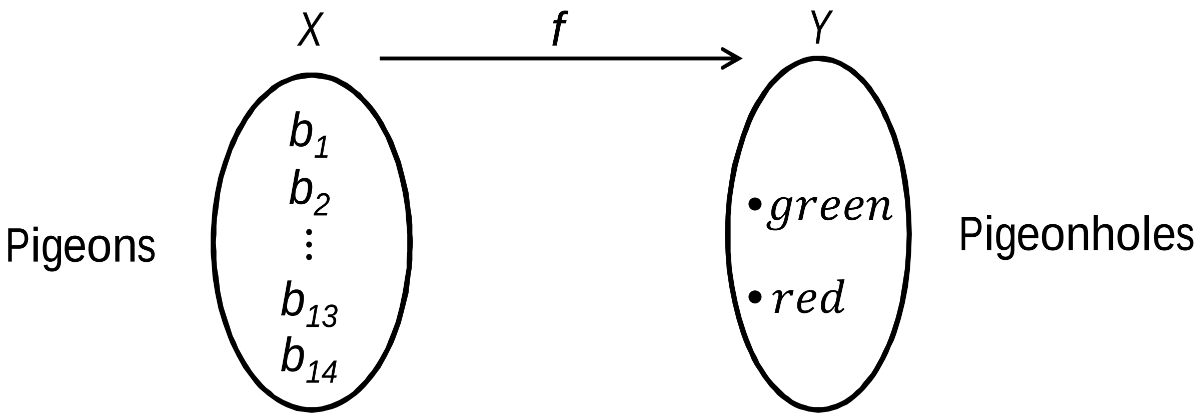
More Examples on Pigeonhole

A box that contains 8 green balls and 6 red balls is kept in a completely dark room. What is the least number of balls one must take out from the box so that at least 2 balls will be the same colour?

Solution:

Let X be the set of all balls in the box and Y={green, red}.

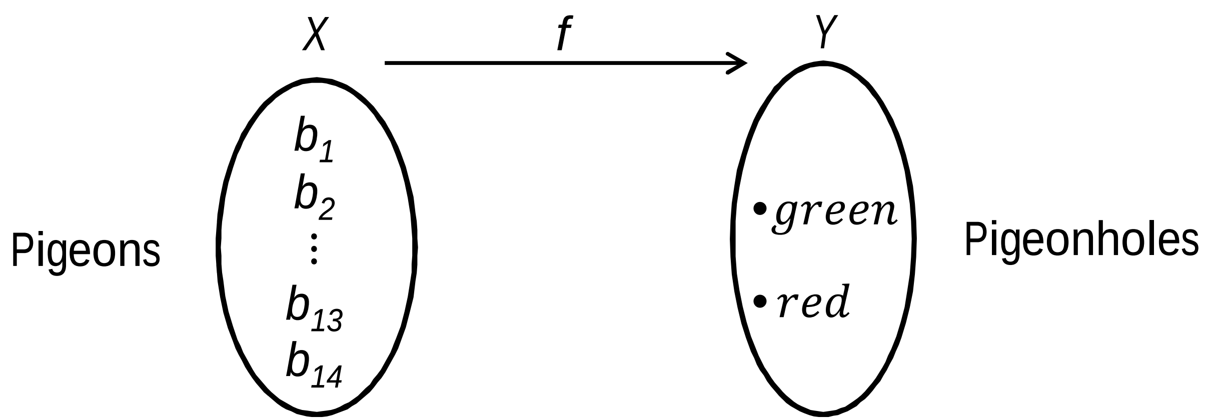
Define a function $f: X \to Y$ by $f(b) = green$, if the colour of the ball is green and $f(b) = red$, if the colour of the ball is red.



More Examples on Pigeonhole continue

Solution:

- If we take subset A of 3 balls of X, then $|A| > |Y|$
- By the pigeonhole principle, at least two elements of A must be assigned the same value in Y
- Therefore, at least 2 of the balls of A must have the same colour



Exercise

1. There are 400 students in a programming class. Show that at least 2 of them were born on the same day of a month.
2. Let $A = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$ be a set of seven integers. Show that if these numbers are divided by 6, then at least two of them must have the same remainder.
3. Let $A = \{1,2,3,4,5,6,7,8\}$. Show that if you choose any five distinct members of A, then there will be two integers such that their sum is 9.
4. From the integers in the set $\{1,2,3, \cdots, 19,20\}$, what is the least number of integers that must be chosen so that at least one of them is divisible by 4?