CHAPTER 1

SET THEORY

[Part 2: Operation on Set]

Union

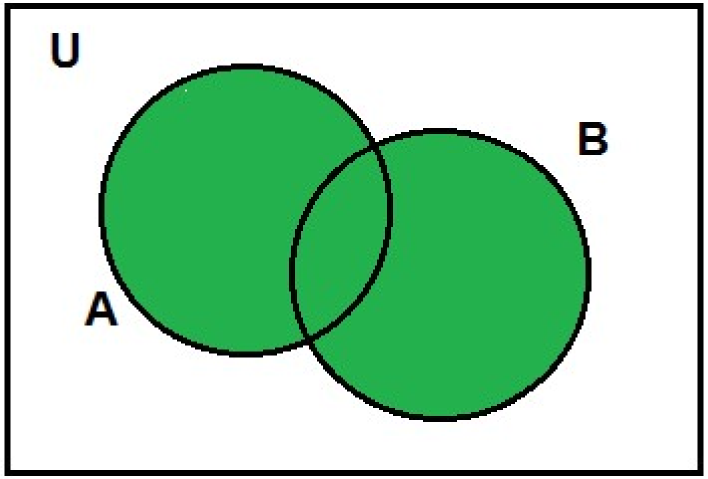
- The union of two sets A and B, denoted by $A \cup B$, is defined to be the set

$$A \cup B = \{ x \mid x \in A \text{ or } x \in B \}$$

- The union consists of all elements belonging to either A or B (or both)

Union

- Venn diagram of $A \cup B$



Example

A={1, 2, 3, 4, 5}, B={2, 4, 6} and C={8, 9}

$A \cup B = \{1, 2, 3, 4, 5, 6\}$

$A \cup C = \{1, 2, 3, 4, 5, 8, 9\}$

$B \cup C = \{2, 4, 6, 8, 9\}$

$A \cup B \cup C = \{1, 2, 3, 4, 5, 6, 8, 9\}$

Union

- If A and B are finite sets, the cardinality of $A \cup B$,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Intersection

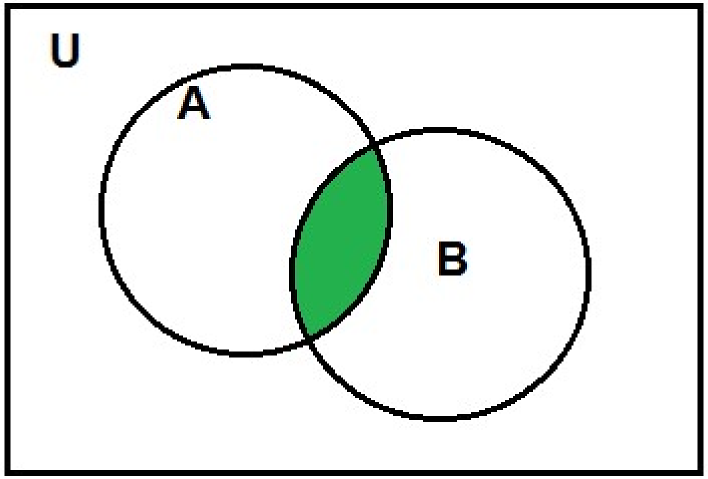
- The intersection of two sets A and B, denoted by $A \cap B$, is defined to be the set

$$A \cap B = \{ x \mid x \in A \text{ and } x \in B \}$$

- The intersection consists of all elements belonging to both A and B.

Intersection

- Venn diagram of $A \cap B$



Example

A={1, 2, 3, 4, 5, 6}, B={2, 4, 6, 8, 10} and
C={ 1, 2, 8, 10 }

$$A \cap B = \{2, 4, 6\}$$
$$A \cap C = \{1, 2\}$$
$$C \cap B = \{2, 8, 10\}$$
$$A \cap B \cap C = \{2\}$$

Exercise

Problem 1

There are two types of ice cream cones, chocolate and vanilla. You and 24 of your friends (25 total people) are going to buy ice cream cones. If 15 people buy vanilla cones, and 20 people buy chocolate cones, how many people bought both chocolate and vanilla ice cream cones?

Problem 2

A group of 50 people go to the candy store to buy candy bars. Each person buys at least one bar. The store sells two types of candy bars, Sweet and Tasty. If 45 people buy both types of Candy Bars, and 47 people buy at least one Sweet bar each, how many people bought only Tasty candy bars?

Problem 3

There are 49 people that own pets. 15 people own only dogs, 10 people own only cats, five people own only cats and dogs, and 3 people own cats, dogs, and snakes. How many total snakes are there?

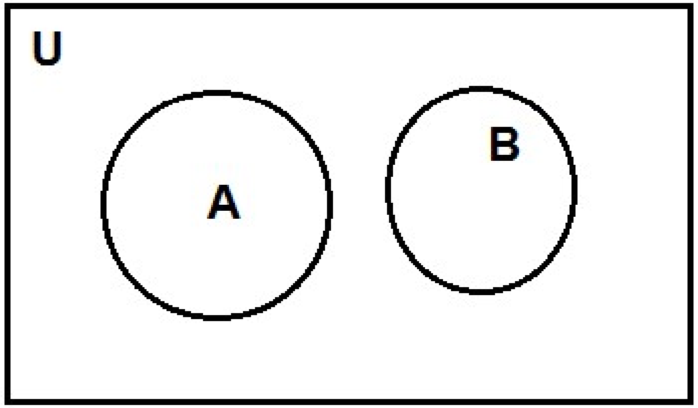
Disjoint

- Two sets *A* and *B* are said to be disjoint if,

  $$A \cap B = \varnothing$$

Disjoint

- Venn diagram, $A \cap B = \varnothing$



Example

A = {1, 3, 5, 7, 9, 11}

B = {2, 4, 6, 8, 10}

$A \cap B = \varnothing$

Difference

- The set

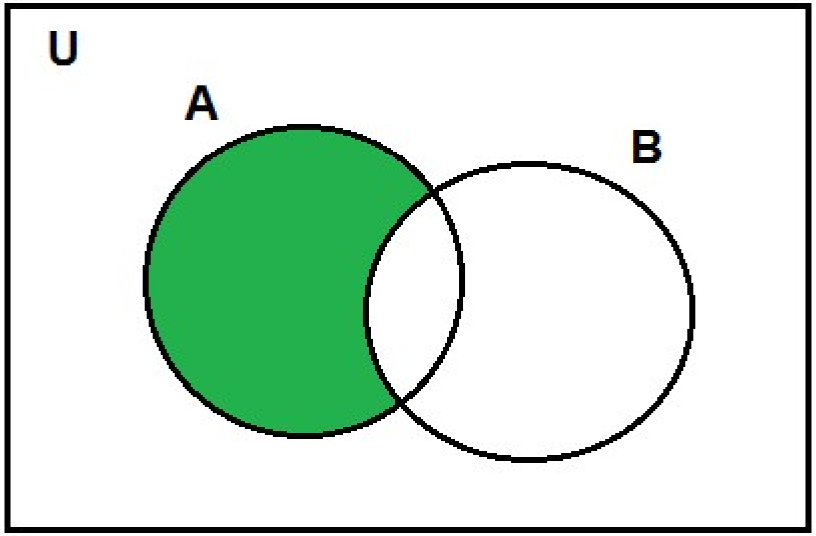
  $A - B = \{x \mid x \in A \text{ and } x \notin B\}$

  is called the difference.

- The difference $A - B$ consists of all elements in A that are not in B.

Difference

- Venn diagram of $A - B$



Example

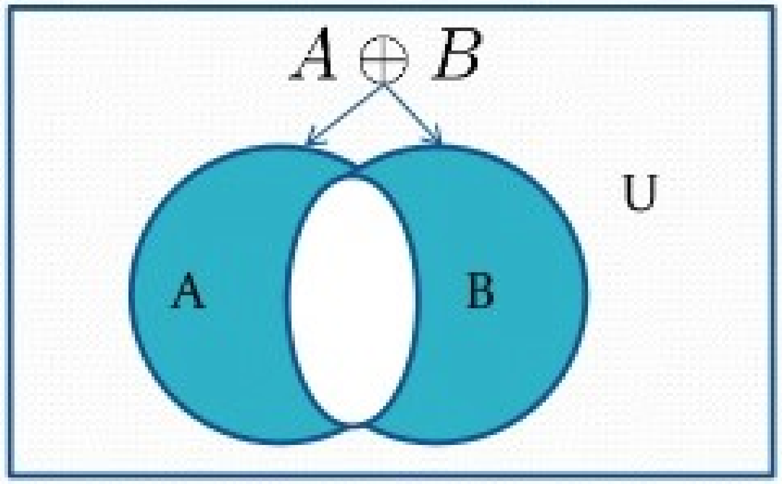
$A = \{ 1, 2, 3, 4, 5, 6, 7, 8 \}$

$B = \{ 2, 4, 6, 8 \}$

$A - B = \{ 1, 3, 5, 7 \}$

Symmetric Difference

The symmetric difference of set *A* and set *B*, denoted by $A \oplus B$ is the set **(A - B) U (B - A)**



Venn Diagram

Symmetric Difference

- Formal definition for the symmetric difference of two sets:

  $A \oplus B = \{ x \mid (x \in A \text{ or } x \in B) \text{ and } x \notin A \cap B\}$

  $A \oplus B = (A \cup B) - (A \cap B)$ ← Important!

- Further examples
  - $\{1, 2, 3\} \oplus \{3, 4, 5\} = \{1, 2, 4, 5\}$
  - {New York, Washington} ⊕ {3, 4} = {New York, Washington, 3, 4}
  - $\{1, 2\} \oplus \varnothing = \{1, 2\}$
    - The symmetric difference of any set S with the empty set will be the set S

Example

$$U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

$$A = \{1, 2, 3, 4, 5\};\ B = \{4, 5, 6, 7, 8\}$$

$$A \oplus B = (A - B) \cup (B - A) = \{1, 2, 3, 6, 7, 8\}$$

B-A $= \{6, 7, 8\}$

$$A - B = \{1, 2, 3\}$$

Complement

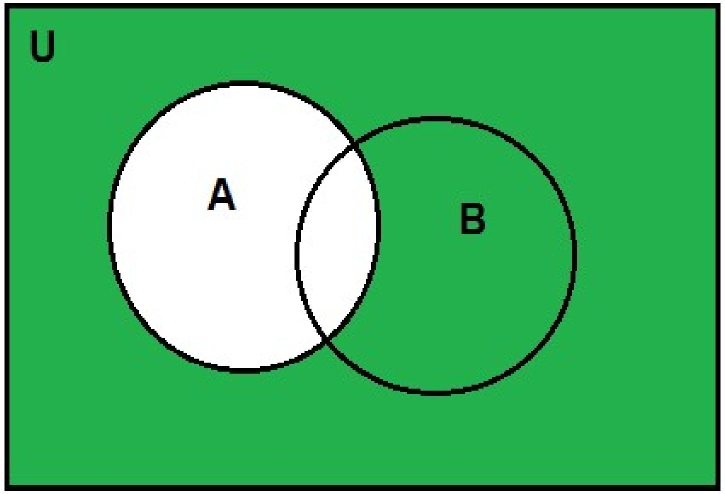
- The complement of a set A with respect to a universal set U, denoted by A' is defined to be

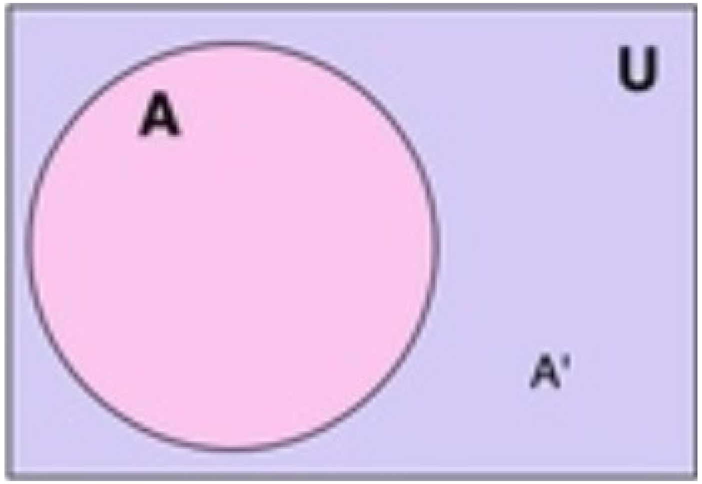
$$A' = \{x \in U \mid x \notin A\}$$

$$A' = U - A$$

Complement

Venn diagram of *A'*





Example

Let U be a universal set,

U= { 1, 2, 3, 4, 5, 6, 7 }

A= { 2, 4, 6 }

$A' = U - A = \{ 1, 3, 5, 7 \}$

Exercise

- Let,

  $U = \{ a, b, c, d, e, f , g, h, i, j, k, l, m \}$

  $A = \{ a, c, f, m\}$

  $B = \{ b, c, g, h, m \}$

- Find:

  $A \cup B$, $A \cap B$, $| A \cup B|$, $A - B$ dan A'.

Exercise

Let the universe be the set U={1, 2, 3, 4,.....,10}.

Let A={1, 4, 7, 10}, B={1, 2, 3, 4, 5} and

C={2, 4, 6, 8}.

List the elements of each set:

a) U'

b) $B' \cap (C - A)$

c) $B - A$

d) $(A \cup B) \cap (C - B)$

Properties of Sets

- Commutative laws

$$A \cap B = B \cap A$$

$$A \cup B = B \cup A$$

Properties of Sets

- Associative laws

$$A \cap (B \cap C) = (A \cap B) \cap C$$

$$A \cup (B \cup C) = (A \cup B) \cup C$$

Properties of Sets

- Distributive laws

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Properties of Sets

- Absorption laws

$$A \cup (A \cap B) = A$$

$$A \cap (A \cup B) = A$$

Properties of Sets

- Idempotent laws

$$A \cap A = A$$

$$A \cup A = A$$

Properties of Sets

- Complement laws

$$(A')' = A$$

$$A \cap A' = \varnothing$$

$$A \cup A' = \mathrm{U}$$

$$\varnothing' = U$$

$$U' = \varnothing$$

Properties of Sets

- De Morgan's laws

$$(A \cap B)' = A' \cup B'$$

$$(A \cup B)' = A' \cap B'$$

Properties of Sets

- Properties of universal set

$$A \cup U = U$$

$$A \cap U = A$$

Properties of Sets

- Properties of empty set

$$A \cup \varnothing = A$$

$$A \cap \varnothing = \varnothing$$

Example

- Let A, B and C denote the subsets of a set S and let C' denote a complement of C in S.

- If $A \cap C = B \cap C$ and $A \cap C' = B \cap C'$, then prove that $A = B$

Example

$$
\begin{aligned}
A &= A \cap S \\
&= A \cap (C \cup C') \\
&= (A \cap C) \cup (A \cap C') && \text{by distributivity} \\
&= (B \cap C) \cup (B \cap C') && \text{by the given conditions} \\
&= B \cap (C \cup C') && \text{by distributivity} \\
&= B \cap S \\
&= B
\end{aligned}
$$

Simplification using the Set Law

Simplify the set

$(((A \cup B) \cap C)' \cup B')' =$

$= ((A \cup B) \cap C)'' \cap B''$	[DeMorgan]
$= ((A \cup B) \cap C) \cap B$	[Double Complement]
$= (A \cup B) \cap (C \cap B)$	[Associativity of $\cap$]
$= (A \cup B) \cap (B \cap C)$	[Commutativity of $\cap$]
$= ((A \cup B) \cap B) \cap C$	[Associativity of $\cap$]
$= B \cap C$	[Absorption]

Exercise

- Let A, B and C be sets.
- Show that

$$(A \cup (B \cap C))' = A' \cap (B' \cup C')$$

Exercise

- Let A, B and C be sets such that $A \cap B = A \cap C$ and $A \cup B = A \cup C$

- Prove that $B = C$

Generalized Union/Intersection

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

Generalized Union/Intersection

Ex. Suppose that:

$$A_i = \{1,2,3,\ldots,i\} \quad i = 1,2,3,\ldots$$

$$\bigcup_{i=1}^{\infty} A_i = ? \qquad \bigcup_{i=1}^{\infty} A_i = Z^{+}$$

$$\bigcap_{i=1}^{\infty} A_i = ? \qquad \bigcap_{i=1}^{\infty} A_i = \{1\}$$

Cartesian Product

- Let A and B be sets.

- An ordered pair of elements $a \in A$ dan $b \in B$ written (a, b) is a listing of the elements a and b in a specific order.

- The ordered pair (a, b) specifies that a is the first element and b is the second element.

Cartesian Product

- An ordered pair (a, b) is considered distinct from ordered pair (b, a), unless $a=b$.

- Example $(1, 2) \neq (2, 1)$

Cartesian Product

- The Cartesian product of two sets A and B, written $A \times B$ is the set,

$$A \times B = \{(a,b) \mid a \in A, b \in B\}$$

- For any set A,

$$A \times \emptyset = \emptyset \times A = \emptyset$$

Example

$A = \{a, b\}$, $B = \{1, 2\}$.

$A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$

$B \times A = \{(1, a), (1, b), (2, a), (2, b)\}$

Cartesian Product

- if $A \neq B$, then $A \times B \neq B \times A$.
- if $|A| = m$ and $|B| = n$, then $|A \times B| = mn$.

Example

- $A = \{1, 3\}$, $B = \{2, 4, 6\}$.

$A \times B = \{(1, 2), (1, 4), (1, 6), (3, 2), (3, 4), (3, 6)\}$
$B \times A = \{(2, 1), (2, 3), (4, 1), (4, 3), (6, 1), (6, 3)\}$

$A \neq B$, $A \times B \neq B \times A$
$|A| = 2$, $|B| = 3$, $|A \times B| = 2.3 = 6$.

Cartesian Product

- The Cartesian product of sets $A_1, A_2, \ldots., A_n$ is defined to be the set of all n-tuples $(a_1, a_2, \ldots a_n)$ where $a_i \in A_i$ for $i=1,\ldots,n$;

- It is denoted $A_1 \times A_2 \times \ldots. \times A_n$

  $|A_1 \times A_2 \times \ldots. \times A_n| = |A_1|.|A_2| \ldots. |A_n|$

Example

- $A = \{a, b\}$, $B=\{1, 2\}$, $C=\{x, y\}$

  $$A \times B \times C = \{(a,1,x),(a,1,y), (a,2,x), (a,2,y), (b,1,x), (b,1,y), (b,2,x), (b,2,y)\}$$

- $|A \times B \times C| = 2.\ 2.\ 2 = 8$

Exercise

- Let $A= \{w, x\}$, B={1, 2} and C={KB, SD, PS}.
- Find $|A\times B|$, $|B\times C|$, $|A\times C|$, $|A\times B\times C|$, $|B\times C\times A|$, $|A\times B\times A\times C|$
- Determine the following set,

  a) $A\times B$, $B\times C$, $A\times C$

  b) $A\times B\times C$

  c) $B\times C\times A$

  d) $A\times B\times A\times C$

Exercise

- Let $X = \{1,2\}$, $Y=\{a\}$ and $Z=\{b,d\}$.
- List the elements of each set.

  a) $X \times Y$

  b) $Y \times X$

  c) $X \times Y \times Z$

  d) $X \times Y \times Y$

  e) $X \times X \times X$

  f) $Y \times X \times Y \times Z$