SCSI1013: Discrete Structures

CHAPTER 2 – Part 3

Recurrence Relation

Recurrence vs Recursive?

- A recurrence is a recursive description of a function
- Like all recursive structures, a recurrence consists of one or more base cases and one or more recursive cases
- Performance of recursive algorithms typically specified with **recurrence equations**

Definition

- A **recurrence relation** is an equation that defines a sequence {a_n, a_{n+1} , . . .} based on a rule that gives the next term as a function of one or more of the previous term in the sequence for all integers *n* with $n \geq n_0$,( ) $a_0,\ a_1, \ldots$
- A sequence is called a **solution** of a recurrence relation if it terms satisfy the recurrence relation.

Simple Recurrence Relation

The simplest form of a **recurrence relation** is the case where the next term depends only on the immediately previous term.

3, 8, 13, 18, 23, . . .

The above sequence shows a pattern, given initial condition, a_1 =3:

3, 3+5, 8+5, 13+5, 18+5, . . .
a_1 , a_2, a_3, a_4, a_5, ...

generated from an equation:

$$a_n = a_{n-1} + 5, \qquad n \geq 2$$

n_{th} term of a sequence

Recurrence relation can be used to compute any n-*th* term of the sequence, given the initial condition, a_0:

$a_n = a_{n-1} + 5$, $n \geq 2$

3, 8, 13, 18, 23, **28,** $a_{n-1} + 5, \ldots$

$a_2 = a_1 + 5$, $3 + 5 = 8$
$a_3 = a_2 + 5$, $8 + 5 = 13$
$a_4 = a_3 + 5$, $13 + 5 = 18$
⋮
$a_6 = a_5 + 5$, $23 + 5 = 28$

Example 1

Consider the following sequence:

3, 9, 27, 81, 243, . . .

The above sequence shows a pattern:

3^1, 3^2, 3^3, 3^4, 3^5, . . .
$a_1,\ a_2,\ a_3,\ a_4,\ a_5, \ldots$

Recurrence relation is defined by:

$$a_n = 3^n, n \geq 1$$

Example 2

Given initial condition, $a_0 = 1$ and recurrence relation:

$\mathbf{a_n = 1 + 2a_{n-1}}$ **,** $\mathbf{n \geq 1}$

First few sequence are:

$a_1 = 1 + 2\ (1) = 3$
$a_2 = 1 + 2(3) = 7$
$a_3 = 1 + 2(7) = 15$

1, 3, 7, 15, 31, 63, ...

Example 3

Given initial condition, $a_0 = 1$, $a_1 = 2$ and recurrence relation:

$\mathbf{a_n = 3(\ a_{n-1} + a_{n-2}\ ),\ n \geq 2}$

First few sequence are:

$a_2 = 3(\ 2 + 1) = 9$
$a_3 = 3(\ 9 + 2) = 33$
$a_4 = 3(\ 33 + 9) = 126$

1, 2, 9, 33, 126, 477, 1809, 6858, 26001,...

Exercise

Find few sequence for recurrence relation given:

(Let's do it up to a_6)

1) $a_n = 6a_{n-1} - 9a_{n-2}$, where $a_0 = 2$, $a_1 = 3$

2) $a_n = 2a_{n-1} - a_{n-2}$, where $a_0 = 5$, $a_1 = 3$

Example 4

For a photo shoot, the staff at a company have been arranged such that there are 10 people in the front row and each row has 7 more people than in the row in front of it. Find the recurrence relation and compute number of staff in the first 5 rows.

Example 4 -Solution

Notice that the difference between the number of people in successive rows is a constant amount.

This means that the n_{th} term of this sequence can be found using:

$$a_n = a_{n-1} + 7,\ n \geq 2 \text{ with } a_1 = 10$$

Example 4 -Solution

Number of staff in the first 5 rows:

$a_1 = 10$,
$a_2 = a_1 + 7$, $10 + 7 = 17$
$a_3 = a_2 + 7$, $17 + 7 = 24$
$a_4 = a_3 + 7$, $24 + 7 = 31$
$a_5 = a_4 + 7$, $31 + 7 = 38$

10, 17, 24, 31, 38

Example 5

Find a recurrence relation and initial condition for

1, 5, 17, 53, 161, 485, …

Solution:
Look at the differences between terms:

4, 12, 36, 108, …

the difference in the sequence is growing by a factor of 3.

Example 5 -Solution

However the original sequence is not.

1(3)=**3**, 5(3)=**15**,17(3)=**51**, . . .

1, 5, 17, 53, 161, 485, ...

It appears that we always end up with 2 less than the next term.

So, the recurrence relation is defined by:

$a_n = 3(a_{n-1}) + 2$, $n \geq 1$, with initial condition, $a_0 = 1$

Example 6

A depositor deposits RM 10,000 in a savings account at a bank yielding 5% per year with interest compounded annually. How much money will be in the account after 30 years? Let P_n denote the amount in the account after n years.

Example 6 - Solution

Derive the following **recurrence relation**:

$$P_n = P_{n-1} + 0.05P_{n-1} = 1.05P_{n-1}$$

Where, P_n = Current balance and P_{n-1} = Previous year balance and 0.05 is the compounding interest.

Example 6 - Solution

Initial condition, P_0 = 10,000. Then,

$P_1 = 1.05P_0$
$P_2 = 1.05P_1 = (1.05)^2P_0$
$P_3 = 1.05P_2 = (1.05)^3P_0$
...
$P_n = 1.05P_{n-1} = (1.05)^nP_0$,

now we can use this formula to calculate n_{th} term without iteration

Example 6 - Solution

Let us use this formula to find P_{30} under the initial condition P_0 = 10,000:

$P_{30} = (1.05)^{30}(10,000) = 43,219.42$

After 30 years, the account contains RM 43,219.42.

Exercise

Consider the following sequence:

1, 5, 9, 13, 17

Find the recurrence relation that defines the above sequence.

Exercise

A basketball is dropped onto the ground from a height of 15 feet. On each bounce, the ball reaches a maximum height 55% of its previous maximum height.

a)Write a recursive formula, a_n , that completely defines the height reached on the n_{th} bounce, where the first term in the sequence is the height reached on the ball's first bounce.

b)How high does the basketball reach after the 4_{th} bounce? Give your answer to two decimal places.