# CHAPTER 3

- ❑ PART 1 - COUNTING METHOD
- ❑ PART 2 - PERMUTATION & COMBINATION
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# CHAPTER 3

# COUNTING METHODS

# [Part 1]

# Basic Counting Principles

- Counting principle is all about choices we might make given many possibilities.
- It is used to find the number of possible outcomes.
- It provides a basis of computing probabilities of discrete events.

# Basic Counting Principles

**Some sample of counting problems:**

- Problem 1 - How many ways are there to seat *n* couples at a round table, such that each couple sits together?
- Problem 2- How many ways are there to express a positive integer *n* as a sum of positive integers?
- Problem 3 - There are three boxes containing books. The first box contains 15 mathematics books by different authors, the second box contains 12 chemistry books by different authors, and the third box contains 10 computer science books by different authors. A student wants to take a book from one of the three boxes. In how many ways can the student do this?
- Problem 4 - The department will award a free computer to either a CS student or a CS professor. How many different choices are there, if there are 530 students and 15 professors?

# Basic Counting Principles

- There are a number of basic principles that we can use to solve such problems:
  1) **Addition Principle**
  2) **Multiplication Principle**

# Addition Principle

- Suppose that tasks $T_1, T_2, \ldots T_k$ can be done in $n_1, n_2, \ldots n_k$ ways, respectively.
- If all these tasks are **independent** of each other, then the number of ways to do one of these tasks is $n_1 + n_2 + \ldots + n_k$
- If a task can be done in $n_1$ ways and a second task in $n_2$ ways, and if these two tasks cannot be done at the same time, then there are $n_1 + n_2$ ways to do either task.

# Example 1

We want to find the number of integers between 5 and 50 that end with 1 or 7.

Solution:

- Let *T* denote this task. We divide *T* into the following tasks.
  - $T_1$: find all integers between 5 and 50 that end with 1.
  - $T_2$: find all integers between 5 and 50 that end with 7.
- $T_1$: find all integers between 5 and 50 that end with 1.
  ➔ 11, 21, 31, 41 => 4 ways
- $T_2$: find all integers between 5 and 50 that end with 7.
  ➔ 7, 17, 27, 37, 47 => 5 ways

.: $n_1 + n_2 = 4 + 5 = 9$ ways.

# Example 2

We want to find the number of integers between 4 and 100 that end with 3 or 5.

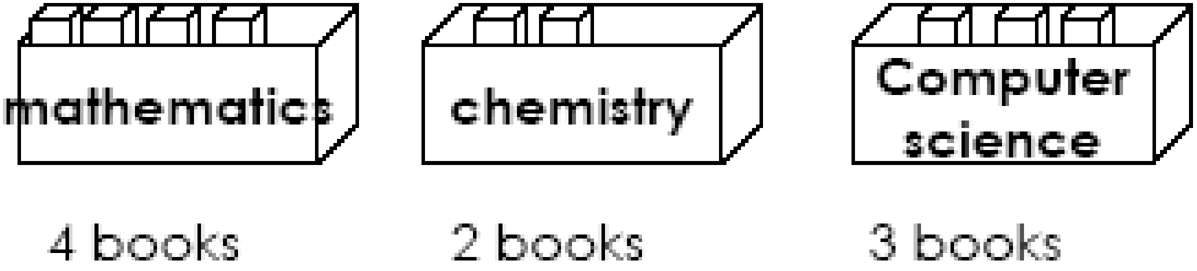
Solution:

Let $T$ denote this task. We divide $T$ into the following tasks.

- $T_1$: find all integers between 4 and 100 that end with 3.
- $T_2$: find all integers between 4 and 100 that end with 5.

- $T_1$: find all integers between 4 and 100 that end with 3.
  => 13, 23, 33, 43, 53, 63, 73, 83, 93 => 9 integers
- $T_2$: find all integers between 4 and 100 that end with 5.
  $\Rightarrow$5, 15, 25, 35, 45, 55, 65, 75, 85, 95 => 10 integers
- Task $T_1$ can be done in 9 ways, Task $T_2$ can be done in 10 ways.
- The number of ways to do one of these tasks is **9+10=19**
- Task $T$ can be completed in 19 ways.

# Example 3




- A student wants to take a book from one of the three boxes. In how many ways can the students do this?
- $T_1$: choose a mathematics book, 4 ways
- $T_2$: choose a chemistry book, 2 ways
- $T_3$: choose a comp. science book, 3 ways
- The number of ways to do one of these tasks is, 4+2+3= **9**

# Exercise

- There are 8 male students and 21 female students in Discrete Structure class. Among all of them, 7 students are Chinese and the rest are Malay.
  - In how many ways can we select 1 student - a boy or a girl?
  - In how many ways can we select 1 student - a Chinese or a Malay?

# Exercise - Solution

- In how many ways can we select 1 student - a boy or a girl?
  - To select a boy
    - 8 ways
  - To select a girl
    - 21 ways
  - The number of ways
    - **8+21=29 ways**

- In how many ways can we select 1 student - a Chinese or a Malay?
  - To select a Chinese
    - 7 ways
  - To select a Malay
    - 22 ways
  - The number of ways
    - **7+22=29 ways**

# Multiplication Principle

- A task *T* can be completed in *k* successive steps.

  **Step 1** can be completed in $n_1$ different ways.

  **Step 2** can be completed in $n_2$ different ways.

  **Step *k*** can be completed in $n_k$ different ways.
- Then the task *T* can be completed in $n_1.n_2.... n_k$ different ways.
- Suppose that a procedure can be broken down into two successive tasks. If there are $n_1$ ways to do the first task and $n_2$ ways to do the second task after the first task has been done, then there are $n_1n_2$ ways to do the procedure.

# Example 4

- Morgan is a lead actor in a new movie. She needs to shoot a scene in the morning in studio A and an afternoon scene in studio C. and no direct route from studio A to studio C. There are three roads, say $A_1$, $A_2$, and $A_3$, from studio A to studio B and four roads, say $B_1$, $B_2$, $B_3$, and $B_4$, from studio B to studio C. In how many ways can Morgan go from studio A to studio C?

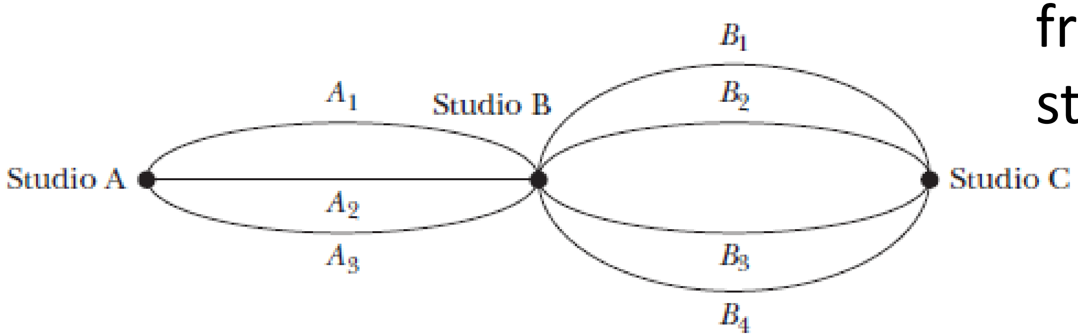



**FIGURE 7.1** Routes from studio A to studio C

**Solution**

There are 3 ways to go from studio *A* to studio *B* and 4 ways to go from studio *B* to studio *C*.

- $T1 = 3$, $T2 = 4$

The number of ways to go from studio *A* to studio *C* via studio *B* is $3 \times 4 = 12$.

# Example 5

- The letters *A, B, C, D*, and *E* are to be used to form strings of length 4. How many strings can be formed if we do not allow repetitions? For example, *BADE, ACBD, AEBC* ..

**Solution**

- $T_1$: choose the first letter ➔ 5 ways
- $T_2$: choose the second letter ➔ 4 ways
- $T_3$: choose the third letter ➔ 3 ways
- $T_4$: choose the fourth letter ➔ 2 ways
- There are 5.4.3.2= 120 strings.

# Example 6

- The letters *A, B, C, D*, and *E* are to be used to form strings of length 4. How many strings can be formed if we allow repetitions, For example, *BABB, AABB, ACEE* ..

**Solution**

- $T_1$: choose the first letter ➔ 5 ways
- $T_2$: choose the second letter ➔ 5 ways
- $T_3$: choose the third letter ➔ 5 ways
- $T_4$: choose the fourth letter ➔ 5 ways
- There are 5.5.5.5= 625 strings.

# Example 7

- The letters *A, B, C, D*, and *E* are to be used to form strings of length 4. How many strings begin with *A*, if repetitions are not allowed? For example, *ADEC, ACBD, AEBC* ..

**Solution**

- $T_1$: choose the first letter → *A* (1 way)
- $T_2$: choose the second letter → 4 ways
- $T_3$: choose the third letter → 3 ways
- $T_4$: choose the fourth letter → 2 ways
- There are 1.4.3.2= 24 strings.

# Exercise

- Danial, Kenny and Joseph are fighting over a turn to play a game that can only has 2 players at a time. In how many ways can we select 2 players at a time?

# Exercise - Solution

- In how many ways can we select 2 players at a time?
  - To select 1st player
    - 3 ways
  - To select 2nd player
    - 2 ways
  - The number of ways
    - 3•2=6 ways

- The possible order:

| Player 1 | Player 2 |
|---|---|
| Danial | Kenny |
| Danial | Joseph |
| Kenny | Joseph |
| Kenny | Danial |
| Joseph | Kenny |
| Joseph | Danial |

# Exercise

- Diana, Sherry, Devi and Mary are going to DSI using a  motorcycle. In how many ways can we select 2 peoples to ride the motorcycle?


- In how many ways can we select 2 peoples to ride the motorcycle?
  - To select 1st rider
  - 4 ways
- To select 2nd rider
  - 3 ways
- The number of ways
  - 4•3=12 ways

- The possible order

| Rider 1 | Rider 2 |
|---|---|
| Diana | Sherry |
| Diana | Devi |
| Diana | Mary |
| Sherry | Diana |
| Sherry | Devi |
| Sherry | Mary |
| Devi | Diana |
| Devi | Sherry |
| Devi | Mary |
| Mary | Diana |
| Mary | Sherry |
| Mary | Devi |

# Exercise

- There are 8 male students and 21 female students in Discrete Structure class. Among all of them, 7 students are Chinese and the rest are Malay.
  - In how many ways can we select 2 students - a boy and a girl?
  - In how many ways can we select 2 students - a Chinese and a Malay?

# Exercise - Solution

- In how many ways can we select 2 students - a boy and a girl?
  - To select a boy
    - 8 ways
  - To select a girl
    - 21 ways
  - The number of ways
    - **8•21=168 ways**

- In how many ways can we select 2 students - a Chinese and a Malay?
  - To select a Chinese
    - 7 ways
  - To select a Malay
    - 22 ways
  - The number of ways
    - **7•22=154 ways**

# Exercise

- Suppose that, in order to declare a variable name in a computer language, the name must have five characters. The first character must be a letter, and the remaining characters can be letters or digits. How many different variable names are possible?

# Exercise - Solution

- How many different variable names are possible?
  - ➢ To select 1st character
  - 26 ways ( 26 alphabets)
  - ➢ To select 2nd character
  - 36 ways (26 alphabets + 10 numbers)
  - ➢ To select 3rd character
  - 36 ways (26 alphabets + 10 numbers)
  - ➢ To select 4th character
  - 36 ways (26 alphabets + 10 numbers)
  - ➢ To select 5th character
  - 36 ways (26 alphabets + 10 numbers))

➢ The number of ways

$26 \bullet 36 \bullet 36 \bullet 36 \bullet 36 = 43{,}670{,}016$ ways

# Basic Counting Principles

- The counting problem that we have considered so far involved either the addition principle or the multiplication principle.
- Sometimes, however, we need to use both of these counting principles to solve a particular problem.

# Example 8

- How many 8-bit strings begin either 101 or 111?

**Solution**

- 1 0 1 _ _ _ _ _ → 2. 2. 2. 2. 2 = 32
- 1 1 1 _ _ _ _ _ → 2. 2. 2. 2. 2 = 32

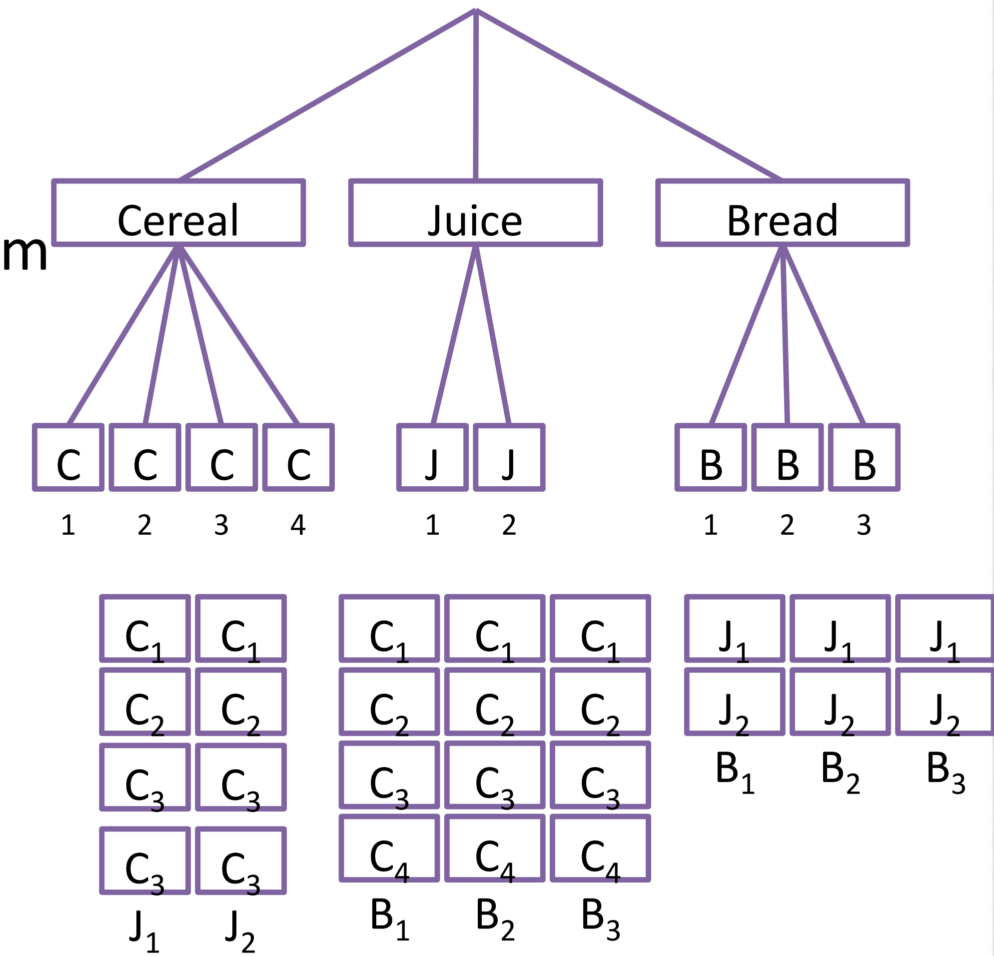
- 32 + 32 =64

# Example 9

- The following items are available for breakfast, 4 types of cereal, 2 types of juice and  3 types of bread. How many ways a breakfast can be prepared if exactly 2 items are selected from 2 different groups?

**Solution**

- A breakfast can be prepared in either of the following 3 ways:
  - ➢ A cereal & a juice, 4. 2 = 8
  - ➢ A cereal & a bread, 4. 3 =12
  - ➢ A juice & a bread, 2. 3 = 6
  - ➢ 8+12+6=26 ways



# Exercise

A six-person committee composed of Aina, Wan, Chan, Tan, Syed and Helmi is to be selected to hold as a chairperson, secretary, and treasurer.

a. In how many ways can this be done?
b. In how many ways can this be done if either Aina or Wan must be chairperson?
c. In how many ways can this be done if Syed must hold one of the position?
d. In how many ways can this be done if Tan and Helmi must hold any position?

# Solution

a. How many ways can this be done?

Select the chairperson (6 ways), secretary (5 ways), treasurer (4 ways)

- 6. 5. 4 =**120**

b. Aina or Wan must be chairperson.

Aina is chairperson 1. 5. 4 =20, Wan is chairperson 1. 5. 4 =20

- 20+20 =**40**

c. Syed must hold one of the position.

Syed is chairperson 1. 5. 4 =20, secretary 1. 5. 4 =20, treasurer 1. 5. 4 =20

- 20+20+20 =**60**

d. Tan and Helmi must hold any position.

- Tan - 3 ways, Helmi - 2 ways, remaining position - 4 ways
- 3. 2. 4 =**24**

# CHAPTER 3

# COUNTING METHODS (Part 2)

## Permutation & Combination

# Permutations & Combinations

- If the order **does** matter, it is a **Permutation**. Permutation means **arrangement** of things. The word **arrangement** is used, if the order of things *is considered*.

  ***"The combination to the safe was 472"***

  - Now we **do** care about the order. "724" would not work, nor would "247". It has to be exactly **4-7-2**.

- If the order **doesn't** matter, it is a **Combination**. Combination means **selection** of things. The word **selection** is used, when the order of things has *no importance*.

  ***"My fruit salad is a combination of apples, grapes and bananas"***

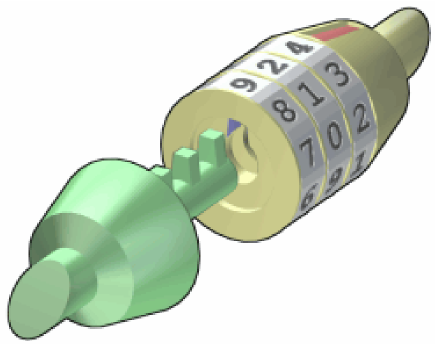
  - We don't care what order the fruits are in, they could also be "bananas, grapes and apples" or "grapes, apples and bananas", its the same fruit salad.

# Permutations

To help you to remember, think "**P**ermutation ... **P**osition"

There are basically two types of permutation:

- **Without repetition**: for example the first three people in a running race. You can't be first and second.
- **Repetition is Allowed**: such as the permutation lock (in picture). It could be "333".



# Permutations
## (without Repetition)

**Example 1:**

- There are 6 permutations of three elements. If the elements are denoted A, B, C, the six permutations are: ABC, ACB, BAC, BCA, CAB, CBA

  = 3! = 3.2.1 = 6

**Example 2:**

- How many permutations of letters *ABCDEF* contain the substring *DEF*?

| | | | |
|---|---|---|---|
| ***DEF*** | ***A*** | ***B*** | ***C*** |

- 4! = 24

**Example 3:**

- How many permutations of letters *ABCDEF* contain the letters *DEF* together in any order?.

| | | | |
|---|---|---|---|
| ***DEF*** | ***A*** | ***B*** | ***C*** |

- *DEF* 3! =6
- *DEF, A, B, C* 4! = 24 6.24=144

# Linear and Circular Permutations

- We consider one of the objects as fixed and the remaining objects are arranged as in **linear permutation**. If we have '$n$' things, number of linear–arrangement is $n!$.
- If we have '$n$' things and circular arrangement is considered, then total number of **circular-permutations** is given by $(n-1)!$.

# Linear permutations n!

The single difference between circular and linear permutations is that circular permutations are the different orders in which a group of objects can be arranged in a **circle**, whereas linear permutations are the different orders in which a group of objects can be arranged in a **straight line**.

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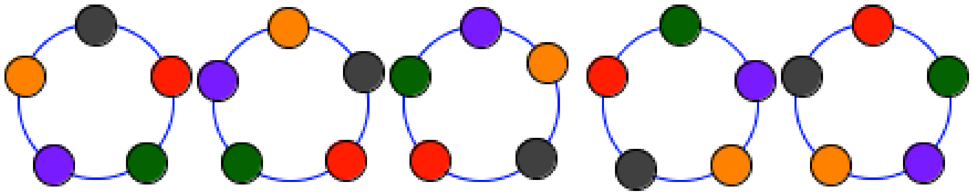
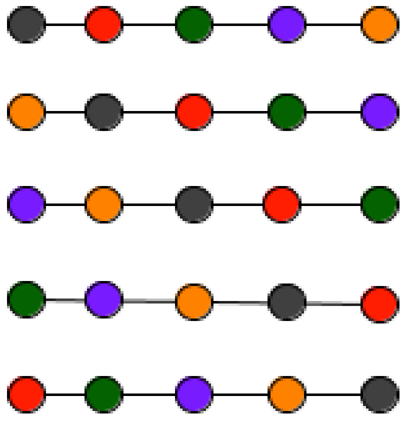
# Linear permutations n!

The following 5 arrangement of beads around a necklace is counted as 1 arrangement. This arrangement is black-red-green-purple-orange. You can put the black bead in the first position, second, third, fourth, and fifth and as long as it is followed by the other four beads in the same order, it will just be the same arrangement.



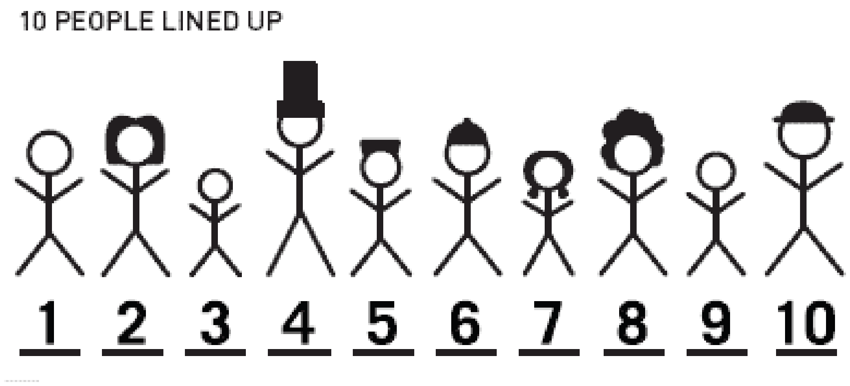
In a linear permutation, each of this arrangement is different as shown in the image below (imagine cutting the necklace).



The number of ways of arranging the 5 beads on a line is n! = 5! =120 ways

# Example

- Suppose we are expecting ten people seat in a round table for dinner. Let's think about how many ways we can line them up.
- There will be 10! ways (linear permutation) to line up ten guests. 10 for the first position, 9 for the second, 8 for the third, and so on.




$$P = 10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$$

# Circular permutations (n-1)!

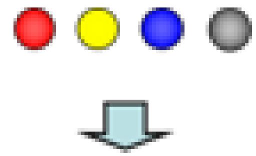
When objects are arranged in a circle, the counting technique used to find the number of permutations is called circular permutation.

To determine the number of circular permutations, we shall consider one object fixed and calculate the number of arrangements based on the remaining number of objects left.

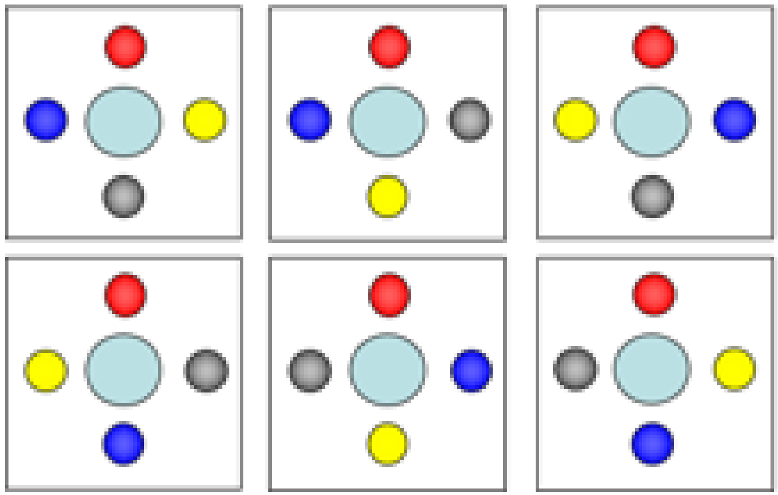
The number of circular permutations of n different objects is defined in symbols by:

$$P = (n-1)!$$

# Circular permutations (n-1)!



$$\frac{4!}{4} = 3! = 6$$



*Circular Permutation*

$$\frac{n!}{n} = (n-1)!$$

# Example

- How many circular permutation can you form with 10 objects

  **(n-1)! = (10-1)!=9!=362880**

- How many ways can four boys and two girls be seated at a round table?

  **(n-1)!=(6-1)!=5!=120**

# Example

If 6 persons are to be seated in a round table with 6 chairs, how many ways can they be seated?

$n = 6$

$P = (n - 1)!$

$= (6 - 1)!$

$= 5!$

$= 120$ ways

# $r$-Permutations

(Permutations without repetition)

- An $r$-permutations of $n$ (distinct) elements $x_1,\ldots, x_n$ is an ordering of an $r$-element subset of $\{x_1,\ldots, x_n\}$.

- The number of $r$-permutations of a set of $n$ distinct elements is,

$$P(n,r) = {}^{n}P_{r} = {}_{n}P_{r} = \frac{n!}{(n-r)!}$$

# Example

- 2- permutations of a, b, c are,
  - *ab, ac, ba, bc, ca, cb*

$$P(3,2) = \frac{3!}{(3-2)!} = \frac{3!}{1!} = 3.2 = 6$$

n = 3 letters a, b, c
r = 2 letters formed

# Example

- In how many ways can we select a chairperson, vice-chairperson, secretary, and treasurer from a group of 10 persons?

$$P(10,4) = \frac{10!}{(10-4)!} = \frac{10!}{6!} = 10.9.8.7 = 5040$$

n = 10 candidates
r = 4 positions

# Example

- How many dance pairs, (dance pairs means a pair (W,M), where W stands for a women and M for man), can be formed from a group of 6 women and 10 men?

$$P(10,6) = \frac{10!}{(10-6)!} = \frac{10!}{4!} = 10.9.8.7.6.5 = 151200$$

n = 10 men

r = 6 women coupled to men

# *r*-Permutations

(Permutations with repetitions allowed)

- *r*-permutations of a set of *n* distinct elements if repetitions are allowed.

$$P(n,r) = n^r$$

- In other words:
  - There are ***n*** possibilities for the first choice,
  - THEN there are ***n*** possibilities for the second choice,
  - and so on, multiplying each time.
- Which is easier to write down using an exponent of ***r***:

$$n \times n \times \ldots (r \text{ times}) = n^r$$

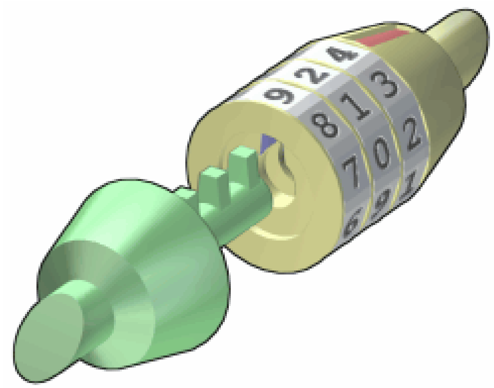
# Example

- In the lock, there are 10 numbers to choose from (0,1,..9) and you choose 3 of them:

$$\mathbf{10 \times 10 \times \ldots\ (3\ times)}$$

$$\mathbf{= 10^3}$$

$$\mathbf{= 1{,}000\ permutations}$$



# Example

- How many five-letters word can be formed from the letters *A-Z*?
  - $P(26,5) = 26^5$

n = 26 letters A-Z
r = 5 letters formed

# Permutations
(***n*** objects, ***k*** different types)

- A collection of n objects of k different types.
- The total number of different arrangements of these n objects is,

$$P(n) = \frac{n!}{(n_1!n_2!...n_k!)}$$

# Example

- Find the number of different ways the letters of the word ASSESSMENT can be arranged?

- 10 letters, A (1), S(4), E(2), M(1), N(1), T(1)

$$P(10) = \frac{10!}{(1!\ 4!\ 2!\ 1!\ 1!\ 1!)} = 75600$$

n = 10 letters - ASSESSMENT
$n_1$ =letter A, $n_2$ =letter S…., $n_5$ =letter T

# Exercise

In how many ways can 10 distinct books be divided among 3 students if Khairin gets 4 books and Nurina and Sarah each get 3 books.

# Solution

- Put the books in some fixed order. Now consider orderings of 4 K's, 3 N's and 3 S's,
- Example: K K K K N N N S S S

$$P(10) = \frac{10!}{4!\ 3!\ 3!} = 4200$$

# Combinations

- There are also two types of combinations (remember the order does **not** matter now):
- **Without repetition**: such as combination of subjects to enroll (Maths., English, Music)
- **Repetition is Allowed**: such as coins in your pocket (5,5,5,10,10)

# Combinations
## (without Repetition)

- Given a set $X=\{x_1,\dots, x_n\}$ containing n (distinct) elements.
- An *r*-combination of *X* is an unordered selection of *r*-elements of *X*.
- The number of *r*-combinations of a set of *n* distinct elements is,

$$C(n,r)={}^{n}C_{r}={}_{n}C_{r}=\binom{n}{r}=\frac{n!}{r!(n-r)!}$$

# Example

- In how many ways can we select a committee of three from a group of 10 distinct persons?

$$C(10,3) = \frac{10!}{3!(10-3)!} = \frac{10!}{3!7!} = 120$$

n = 10 candidates
r = 3 positions

# Example

- In how many ways can we select a committee of 2 women and 3 men from a group of 5 distinct women and 6 distinct men?

$$C(5,2) = \frac{5!}{2!3!} = 10$$

$$C(6,3) = \frac{6!}{3!3!} = 20$$

n = 5 candidates (women)
r = 2 positions

n = 6 candidates (men)
r = 3 positions

$$10.20 = 200$$

# Example

- How many 8-bit strings contain exactly four 1's ?

$$C(8,4) = \frac{8!}{4!4!} = 70$$

n = 8-bit strings
r = 4 binary one

# Exercise

- A committee of six is to be made from 4 students and 8 lecturers. In how many ways can this be done:
  - If the committee contains exactly 3 students?
  - If the committee contains at least 3 students?

# Solution (a)

- If the committee contains exactly 3 students?
- Select 3 students $C(4,3) = \dfrac{4!}{3!\ 1!} = 4$
- Select 3 lecturers $C(8,3) = \dfrac{8!}{3!\ 5!} = 56$
- 4.56 = 224

Committee members = 6

n = 4 students
r = 3 positions

n = 8 lecturers
r = 6-3 = 3 positions

# Solution (b)

- If the committee contains at least 3 students?
- We have to consider 2 cases
  - Case 1: 3 students and 3 lecturers
  - Case 2: 4 students and 2 lecturers

Maximum no. of students = 4

- Case 1: 3 students and 3 lecturers
  - 224 ways    Solution (a)
- Case 2: 4 students and 2 lecturers

n = 4 students
r = 4 positions

$$C(4,4) = 1 \qquad C(8,2) = \frac{8!}{2!6!} = 28$$

n = 8 lecturers
r = 2 positions

  - 1.28 = 28
- Case 1+ case 2 = 224 + 28 = 252

# Exercise

- A student is required to answer 7 out of 12 questions, which are divided into two groups, each containing 6 questions.
- The student is not permitted to answer more than 5 questions from either group.
- In how many different ways can the student choose the 7 questions?

# Solution

| Number of question from group A | | Number of question from group B |
|---|---|---|
| 5 | C(6,5).C(6,2)=90 | 2 |
| 4 | C(6,4).C(6,3)=300 | 3 |
| 3 | C(6,3).C(6,4)=300 | 4 |
| 2 | C(6,2).C(6,5)=90 | 5 |

**90+300+300+90 = 780**

# Exercise

There is a shipment of 50 microprocessors of which four are defective.

- In how many ways can we select a set of 4 microprocessors?
- In how many ways can we select a set of 4 microprocessor containing at least 1 defective microprocessor?

# Solution (a)

In how many ways can we select a set of 4 microprocessors?

⇒Order doesn't matter, therefore use **combination.**
⇒Repetition not allowed.

$$C(n,r)=\frac{n!}{r!(n-r)!}$$

$$C(50,4)=\frac{50!}{4!(50-4)!}=230,300$$

# Solution (b)

To select a set of 4 microprocessors containing at least I defective microprocessor?

⇒Order doesn't matter, therefore use **combination.**
⇒Repetition not allowed.
⇒Different tasks.

| Not - Defective | | Defective |
|---|---|---|
| 3 | C(46,3).C(4,1) =15180*4 = 60720 | 1 |
| 2 | C(46,2).C(4,2) =1035*6 = 6210 | 2 |
| 1 | C(46,1).C(4,3) =46*4 = 184 | 3 |
| 0 | C(46,0).C(4,4) =1*1 = 1 | 4 |
| Solution : 60720+6210+184+1 = 67115 ways | | |

# Combinations
## (Repetition Allowed)

The number of r-combinations of n objects with repetitions allowed is,

$$C(n+r-1,r)=\frac{(n+r-1)!}{r!(n-1)!}$$

where ***n*** is the number of things/objects to choose from, and you choose ***r*** of them (repetition allowed, order doesn't matter)

# Example

- There are five flavors of ice-cream: **banana, chocolate, lemon, strawberry and vanilla**.
- You can have three scoops.
- How many variations will there be?

- Let's use letters for the flavors:

  {b, c, l, s, v}

- Example selections would be
  - {c, c, c} (3 scoops of chocolate)
  - {b, l, v} (one each of banana, lemon and vanilla)
  - {b, v, v} (one of banana, two of vanilla)
- (And just to be clear: There are ***n*=5** things to choose from, and you choose ***r* = 3** of them. Order does not matter, and you **can** repeat!)

$$C(n+r-1,r)=\binom{n+r-1}{r}=\frac{(n+r-1)!}{r!(n-1)!}$$

$$C(5+3-1,3)=\binom{5+3-1}{3}=\frac{(5+3-1)!}{3!(5-1)!}=\frac{7!}{3!\times 4!}=\frac{5040}{6\times 24}=35$$

# Example

A bakery makes 4 different varieties of donuts. Khairin wants to buy 6 donuts. How many different ways can he do it?

$$C(4+6-1,6)=\frac{(4+6-1)!}{6!(4-1)!}=\frac{9!}{6!3!}=84$$

# Exercise

There is a box containing identical blue, green, pink, yellow, red and dark blue balls. In how many ways we can select 4 balls?

# Solution

$$C(6+4-1,4)=\frac{(6+4-1)!}{4!(6-1)!}=\frac{9!}{4!5!}=126$$

# Summary

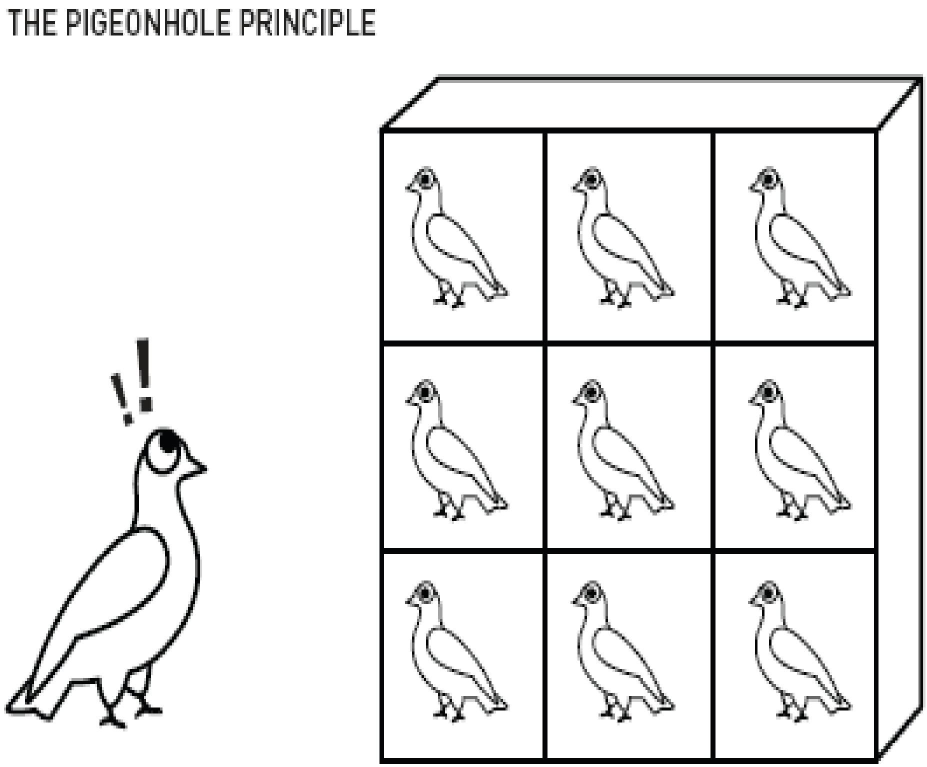
Which formula to use?

| | **Permutations** - order matters | **Combinations** - order does not matter |
|---|---|---|
| Repetition is allowed | $P_n = n^r$ | $C(n+r-1,r) = \frac{(n+r-1)!}{r!(n-1)!}$ |
| Repetition is not allowed | $P(n,r) = \frac{n!}{(n-r)!}$ | $C(n,r) = \frac{n!}{r!(n-r)!}$ |

# Part 3 - Pigeonhole Principle

# Introduction

- The Pigeonhole Principle is a really simple concept
- Discovered in the 1800s




- Imagine 9 pigeonholes and 10 pigeons. A storm comes along, and all of the pigeons take shelter inside the pigeonholes.
- They could be arranged any number of ways. For instance, all 10 pigeons could be inside one hole, and the rest of the holes could be empty.
- What we know for sure, no matter what, is that there is at least one hole that contains more than one pigeon.

# Pigeonhole Principle (PP)

The principle works no matter what the particular number of pigeons and pigeonholes. As long as there are **(N - 1) number of pigeonholes, and (N) number of pigeons**, we know there will always be at least two pigeons in one hole.

# PP ($1^{st}$ Form)

Pigeonhole Principle

*(First Form)*

If $n$ pigeons fly into $k$ pigeonholes and $k < n$, some pigeonhole contains at least two pigeons

# PP (1st Form)

- The principle tells nothing about how to locate the pigeonhole that contains 2 or more pigeons
- It only asserts the existence of a pigeonhole containing 2 or more pigeons
- To apply this principle, one must decide
  - Which objects are the pigeons
  - Which objects are the pigeonholes

# Example (PP - 1st Form)

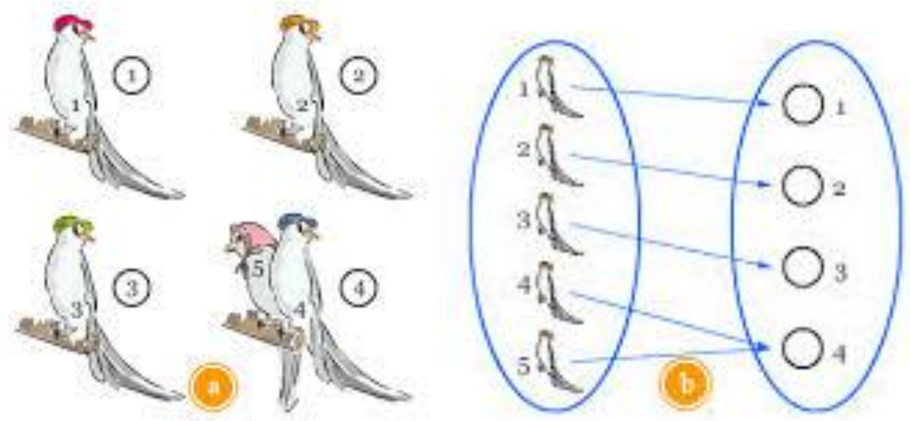
1. Among 8 people there are at least two persons who have the same birthday.
   - Pigeonholes : Days (7) – Monday to Sunday
   - Pigeons: People (8)

2. Among 13 people there are at least two persons whose month of birth is same.
   - Pigeonholes : Months(12) – January to December
   - Pigeons: People (13)

# PP (2nd Form)

Pigeonhole Principle

(Second Form)




If $f$ is a function from a finite set $X$ to a finite set $Y$ and $|X| > |Y|$, then $f(x_1) = f(x_2)$ for some $x_1, x_2 \in X$, $x_1 \neq x_2$

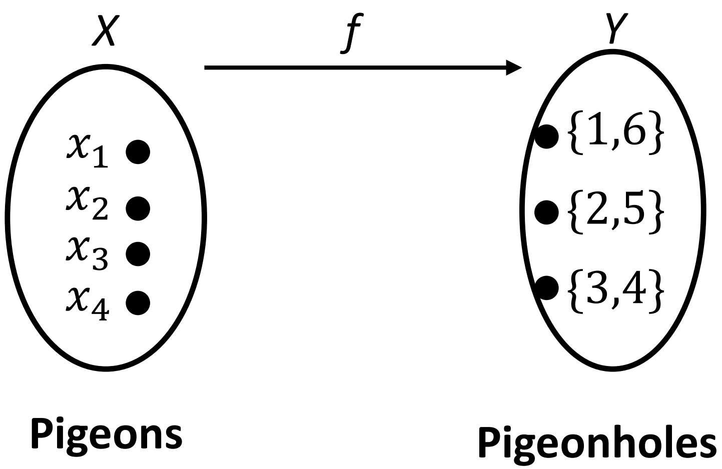
# PP (2nd Form)

- The 2nd form can be reduced to the 1st form by letting *X* be the set of pigeons and *Y* be the set of pigeonholes.
- Assign pigeon *x* to pigeonhole *f(x)*
- By the 1st form principle, at least 2 pigeons, $x_1, x_2 \in X$ , are assigned to the same pigeonhole; that is, $f(x_1) = f(x_2)$ for some $x_1, x_2 \in X, x_1 \neq x_2$

# Example 1 (PP – 2nd Form)

Let $A = \{1,2,3,4,5,6\}$. Show that if we choose any four distinct members of A, then for at least one pair of these four integers their sum is 7.

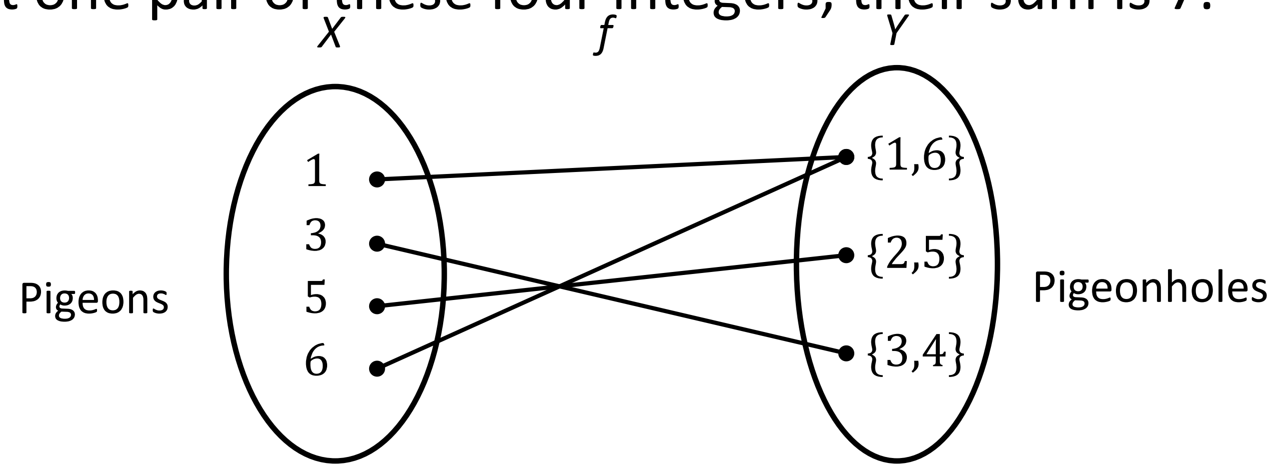
- Notice that {1,6}, {2,5} and {3,4} are the only pairs of distinct integers such that their sum is 7. **--> $x_1 \neq x_2$**
- Let $X = \{x_1, x_2, x_3, x_4\}$ be any subset of four distinct elements of *A*.
- Let $Y = \{\{1,6\}, \{2,5\}, \{3,4\}\}$, a set of 3 distinct elements and a part of *A*. $y_1 = \{1,6\}, y_2 = \{2,5\}, y_3 = \{3,4\}$



# Example 1 (PP – 2nd Form)

continue

- Define $f: X \to Y$ by $f(a) = y_i$ if $a \in y_i$. For example, if $a = 1 \in X$, then $f(1) = \{1,6\}$.
- For $X = \{1,3,5,6\}$, see in the figure below.
- Now $|X|$=4 and $|Y|$=3. Then by 2nd form principle, at least two distinct elements of $X$ **must be** mapped to the same element of $Y$.
- Hence, if we choose any four distinct members of $A$, then for at least one pair of these four integers, their sum is 7.



# PP (3rd Form)

Pigeonhole Principle

*(Third Form)*

*The Generalized Pigeonhole Principle*

Ceiling function that takes as input a **real number** *x* and gives as output the least **integer** ceiling (*x*) = ⌈*x*⌉ that is greater than or equal to *x*

Let *f* be a function from a finite set *X* to a finite set *Y*. Suppose that $|X| = n$ and $|Y| = m$. Let $k = \left\lceil \frac{n}{m} \right\rceil$. Then there are at least $k$ values $a_1, \ldots, a_k \in X$ such that $f(a_1) = f(a_2) = \ldots = f(a_k)$

# PP (3rd Form)

- Let $Y = \{y_1, \dots, y_m\}$.
- There are at most $k-1$ values $x \in X$ with $f(x) = y_1$; there are at most $k-1$ values $x \in X$ with $f(x) = y_2$; … ; there are at most $k-1$ values $x \in X$ with $f(x) = y_m$.
- Thus there are at most $m(k-1)$ members in the domain of $f$.
- $m(k-1) < m\frac{n}{m} = n$. Therefore, there are at least $k$ values, $a_1, \dots, a_k \in X$, such that $f(a_1) = f(a_2) = \cdots = f(a_k)$

# Example 1 (PP – 3rd Form)

Suppose that there are 50 people in a room. Then at least 5 of these people must have their birthday in the same month.

- Pigeons – people ($n = 50$).
- Pigeonholes – months ($m = 12$).
- Thus

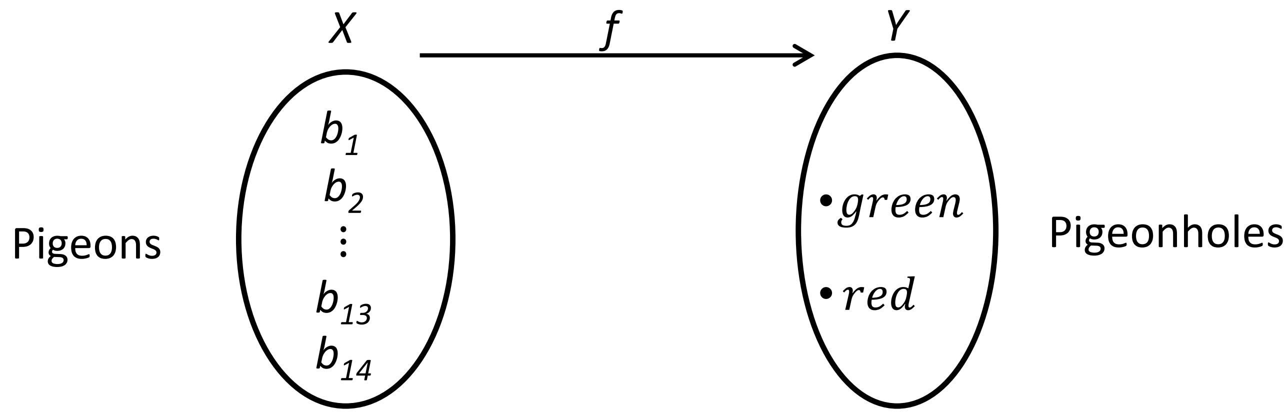
$$k = \left\lceil \frac{n}{m} \right\rceil = \left\lceil \frac{50}{12} \right\rceil = \mathbf{4.16} = 5$$

# Exercise

A box that contains 8 green balls and 6 red balls is kept in a completely dark room. What is the least number of balls one must take out from the box so that at least 2 balls will be the same colour?

# Solution

- Let X be the set of all balls in the box and Y={green, red}.
- Define a function $f: X \to Y$ by $f(b) = green$, if the colour of the ball is green and $f(b) = red$, if the colour of the ball is red.
- If we take subset A of 3 balls of X, then $|A| > |Y|$
- By the pigeonhole principle, at least two elements of A must be assigned the same value in Y
- Therefore, at least 2 of the balls of A must have the same colour



# CHAPTER 3

# [Part 4: Probability Theory]

# Probability Theory

- Let E be an event.
- The probability of *E*, *P(E) is*

$$P(E) = \sum_{x \in E} P(x)$$

- We read this as *"P(E)* equals the sum, over all *x* such that *x* is in *E*, of *P(x)*"

# Probability Axioms

Let $S$ be a sample space. A probability function $P$ from the set of all events in $S$ to the set of real numbers satisfies the following three axioms:

For all events $A$ and $B$ in $S$,

1. $0 \text{ £ } P(A) \text{ £ } 1$
2. $P(Ø) = 0$ and $P(S) = 1$
3. If $A$ and $B$ are disjoint, the $P(A \text{ È } B) = P(A) + P(B)$

A function $P$ that satisfies these axioms is called a probability distribution or a probability measure.

# Complementary Probabilities

- The complement of an event $A$ in a sample space $S$ is the set of all outcomes in $S$ except those in $A$.

$$P(A') = 1 - P(A)$$

- **Example:**

  Let the probability for getting a prize in a lucky draw is 0.075. Thus, the probability of *not* getting a prize in a lucky draw is 1 - 0.075 = 0.925.

# The Uniform Probability Distribution

- The probability of an event occurring is:

$$P(E) = \frac{|E|}{|S|}$$

where:

- **E** is the set of desired events.
- **S** is the set of all possible events.
- Note that $0 \leq |E| \leq |S|$:
  - Thus, the probability will always between 0 and 1.
  - An event that will never happen has probability 0.
  - An event that will always happen has probability 1.

# Example

A coin is flipped four times and the outcome for each flip is recorded.

i) List all the possible outcomes in the sample space.

ii) Find the event (E) that contain only the outcomes in which 1 tails appears.

i. All possible outcomes: [heads(H), tails (T)]

| | |
|---|---|
| HHHH | HHHT |
| HHTH | HTHH |
| THHH | HHTT |
| HTTH | HTHT |
| THHT | TTHH |
| THTH | HTTT |
| TTHT | TTTH |
| THTT | TTTT |

ii. The event E that contains 1 tails

E={ HHHT, HHTH, HTHH, THHH }

# Example

Two fair dice are rolled. Find the event (E) that the sum of the numbers on the dice is 7.

- The number on top face of each die is 1,2,3,4,5,6.
- Let A={1,2,3,4,5,6}
- The sample space is,

  $S = \{ (a, b) \in A \times A \mid a, b \in A \}$
- The event E that the sum of the numbers on the dice is 7 is,

  $E = \{(1,6), (6,1), (2,5), (5,2), (3,4), (4,3)\}$

# Example

A coin is flipped and a dice is rolled.

i) List the members of the sample space.

ii) List the members of the event "the coin shows a head and the dice shows a number less than 4".

- The sample space,

  (H,1) (H,2) (H,3) (H,4) (H,5) (H,6)
  (T,1) (T,2) (T,3) (T,4) (T,5) (T,6)

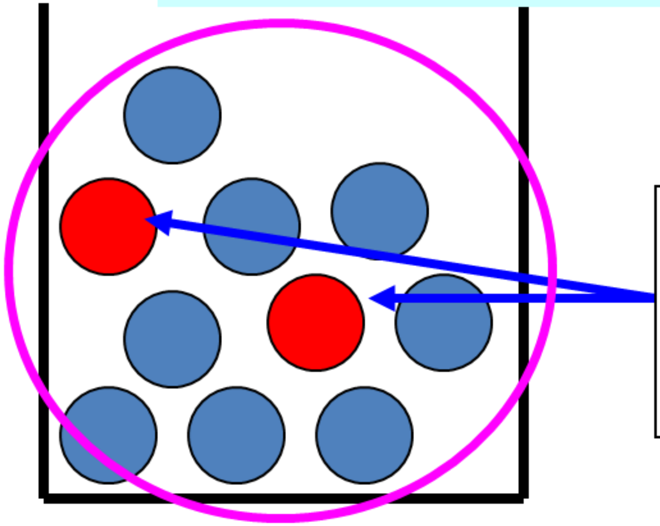
- The members of the event "the coin shows a head and the die shows a number less than 4,

  (H,1) (H,2) (H,3)

# Example

- Question: What is the probability of picking a red marble out of a bowl with 2 red and 8 blue?

$$p(\text{red}) = \frac{\text{outcomes in which red occurs}}{\text{total number of outcomes}}$$



There are 2 outcomes that are red

There are 10 total possible outcomes

$p(\text{red}) = 2 / 10 = 0.2$

# Example

- What is the probability that if a fair coin is tossed 6 times you will get
  - Less than 2 heads
  - At least 2 heads

The number of possible outcome is: 2.2.2.2.2.2 = 64

- Let $A$ be the event that less than 2 H's are observed,

  $A$= { TTTTTT, HTTTTT, THTTTT, TTHTTT, TTTHTT, TTTTHT, TTTTTH }

  Then, $P(A)$ = 7/64

- Let $B$ be the event that at least 2 H's are observed,

  $P(B)$ = 1-7/64 = 57/64

## Probability of a General Union of Two Events

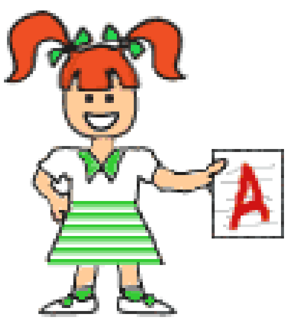
If $S$ is any sample space and $A$ and $B$ are any events in $S$, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

# Example

In a math class of 30 students, 17 are boys and 13 are girls. On a unit test, 4 boys and 5 girls made an A grade. If a student is chosen at random from the class, what is the probability of choosing a girl or an A student?

Solution:




Probabilities: P(girl or A)= P(girl) + P(A) - P(girl and A)

$$= \frac{13}{30} + \frac{9}{30} - \frac{5}{30}$$

$$= \frac{17}{30}$$

# Example 7

- Suppose that a student is selected at random among 90 students, where 35 are over 100 pounds, 20 are boys, and 15 are over 100 pounds and boys.

- What is the probability that the student selected is over 100 pounds or a boy?

- $P(A)$ = students over 100 pounds = 35/90
- $P(B)$ = students are boys = 20/90
- $P(A \cap B)$ = students over 100 pounds and boys = 15/90
- $P(A \cup B)$ = students over 100 pounds or a boy

  $= P(A) + P(B) - P(A \cap B) = 35/90 + 20/90 - 15/90 =$ **40/90**

# Exercise

The probability of a person having a car accident is 0.09. The probability of a person driving while drunk is 0.32 and probability of a person having a car accident while drunk is 0.15. What is the probability of a person driving while drunk or having a car accident?

# Solution

**Solution:**

- Let A = drunk ; B = accident
- Then,

$$P\ (A \text{ or } B) = P(A) + P(B) - P\ (A \text{ and } B)$$
$$= 0.32 + 0.09 - 0.15$$
$$\mathbf{= 0.26}$$

# Mutually Exclusive Events

- Events A and B are mutually exclusive if

$$A \cap B = \emptyset$$

- If A and B are mutually exclusive events,

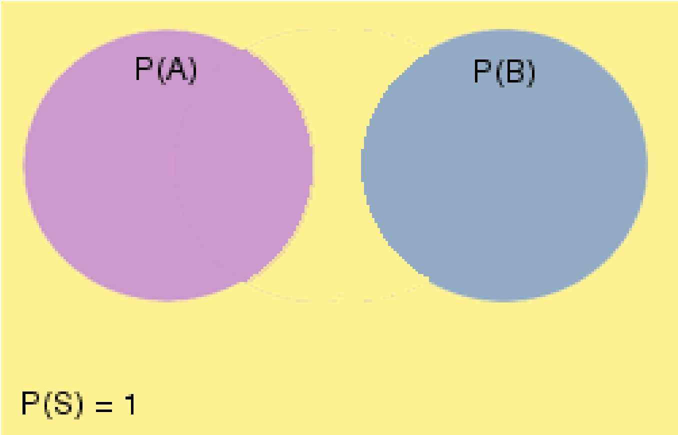
$$P(A \cup B) = P(A) + P(B)$$

- Two events are mutually exclusive if they cannot occur at the same time.

# Mutually Exclusive Events (cont.)

**Mutually Exclusive Events**

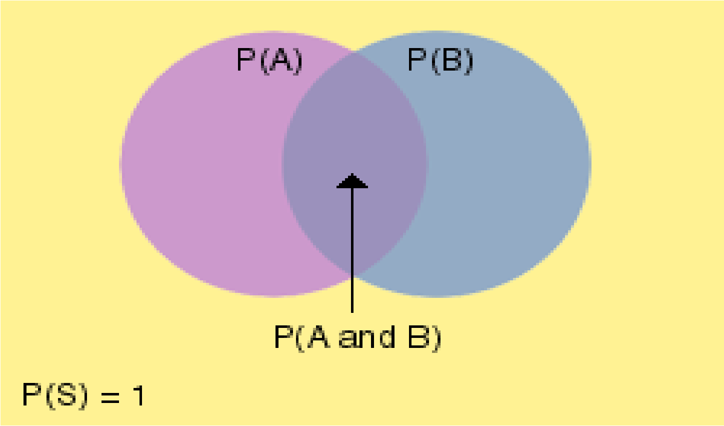
Two events are mutually exclusive if they cannot occur at the same time (i.e., they have no outcomes in common).




In the Venn Diagram above, the probabilities of events A and B are represented by two disjoint sets (i.e., they have no elements in common).

**Non-Mutually Exclusive Events**

Two events are non-mutually exclusive if they have one or more outcomes in common.




In the Venn Diagram above, the probabilities of events A and B are represented by two intersecting sets (i.e., they have some elements in common).

Note: In each Venn diagram above, the sample space of the experiment is represented by S, with P(S) = 1.

# Example

**Question:** Two fair dice are rolled. Find the probability of getting doubles or the sum of 5?

**Solution:**

- Let A denote the event "get doubles"
- Let B denote the event "get the sum of 5"
- A and B are mutually exclusive (you cannot get doubles and the sum of 5 simultaneously)

- A ={(1,1) (2,2) (3,3) (4,4) (5,5) (6,6)}
  $P(A) = 6/36 = 1/6$

- B ={(1,4) (2,3) (3,2) (4,1)}
  $P(B) = 4/36 = 1/9$

- The probability of getting doubles or the sum of 5 is,
  $P(A \cup B) = 1/6 + 1/9 = 5/18$

# Example

A glass jar contains 1 **RED**, 3 **GREEN**, 2 **BLUE** and 4 **YELLOW** marbles. If a single marble is chosen at random from the jar, what is the probability that it is **YELLOW** or **GREEN** ?

**Solution:**

Probabilities:

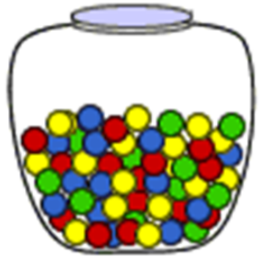
$$P(\text{yellow}) = \frac{4}{10}$$

$$P(\text{green}) = \frac{3}{10}$$

$$P(\text{yellow or green}) = P(\text{yellow}) + P(\text{green})$$

$$= \frac{4}{10} + \frac{3}{10}$$

$$= \frac{7}{10}$$



# Conditional Probability

- Let A and B be events, and assume that P(B)>0.
- The conditional probability of A given B is,

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

**Definition:** The **conditional probability** of an event **A** in relationship to an event **B** is the probability that event **A** occurs given that event **B** has already occurred. The notation for conditional probability is **P(A|B)** [pronounced as ***the probability of event A given B***].

# Example

Suppose that we roll 2 fair dice. Find the probability of **getting a sum of 7, given that the digit in the first die is greater than in the second.**

# Solution

- The sample space $S$ consists of $6 \times 6 = 36$ outcomes.
- Let A be the event "the sum of digits of the 2 dice is 7"

$$A = \{ (6,1), (5,2), (4,3), (3,4), (2,5), (1,6) \}$$

# Solution (cont.)

- Let B be the event "the digit in the first die is greater than the second"

$B = \{ (6,1), (6,2), (6,3), (6,4), (6,5), (5,1), (5,2), (5,3), (5,4), (4,1), (4,2), (4,3), (3,1), (3,2), (2,1) \}$

$P(B) = 15/36$

# Solution (cont.)

- Let C be the event "the sum of digits of the 2 dice is 7 but the digit in the first die is greater than the second"

$C = \{ (6,1), (5,2), (4,3) \} = A \cap B$

$P(A \cap B) = 3/36$

# Solution (cont.)

- The probability of getting a sum of 7, given that the digit in the first die is greater than in the second is,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{(3/36)}{(15/36)} = \frac{1}{5}$$

# Exercise

Question: Weather records show that the probability of high barometric pressure is 0.85 and the probability of rain and high barometric pressure is 0.15. What is the probability of rain given high barometric pressure?

# Solution

- Let ***H*** denote the event "rain".
- Let ***T*** denote the event "high barometric pressure".
- The probability of rain given high barometric pressure is,

$$P\left(H \middle| T\right) = \frac{P(H \cap T)}{P(T)} = \frac{0.15}{0.85} = 0.1765$$

# Bayes' Theorem

Suppose that a sample space ***S*** is a union of mutually disjoint events $B_1, B_2, B_3, \ldots, B_n$, suppose ***A*** is an event in ***S***, and suppose ***A*** and all the $\boldsymbol{B_k}$ have nonzero probabilities, where $k$ is an integer with $1 \le k \le n$. Then

$$P\left(B_k \middle| A\right) = \frac{P\left(A \middle| B_k\right) P(B_k)}{\sum_{i=1}^{n} P\left(A \middle| B_i\right) P(B_i)}$$

# Example

The ELISA test is used to detect antibodies in blood and can indicate the presence of the HIV virus. Approximately 15% of the patients at one clinic have the HIV virus. Among those that have the HIV virus, approximately 95% test positive on the ELISA test. Among those that do not have the HIV virus, approximately 2% test positive on the ELISA test. **Find the probability that a patient has the HIV virus if the ELISA test is positive.**

# Solution

- Let $H$ denote the event "patient has the HIV virus".
- Let $T$ denote the event "patient does not have the HIV virus".
- Thus,

$$P(H) = 0.15$$
$$P(T) = 0.85$$

- Let *Pos* denote the feature "tests positive"

$$P(Pos|H) = 0.95$$
$$P(Pos|T) = 0.02$$

- The probability that a patient has the HIV virus if the ELISA test is positive is

$$\begin{aligned} P(H \mid Pos) &= \frac{P(Pos \mid H)P(H)}{P(Pos \mid H)P(H) + P(Pos \mid T)P(T)} \\ &= \frac{(0.95)(0.15)}{(0.95)(0.15) + (0.02)(0.85)} \\ &= 0.893 \end{aligned}$$

# Example

- Hana, Amir and Dani write a program that schedule tasks for manufacturing toys.
- The table shows the percentage of code written by each person and the percentage of buggy code for each person.

| | Coder | | |
|---|---|---|---|
| | Hana | Amir | Dani |
| % of code | 30 | 45 | 25 |
| % of bugs | 3 | 2 | 5 |

- Given that a bug was found, **find the probability that it was in the program code written by Dani.**

# Solution

- Let
  - H denotes the event of "code written by Hana"
  - A denotes the event of "code written by Amir"
  - D denotes the event of "code written by Dani"
  - B denotes the event of "a bug found in code"
- Since Hana wrote 30% of the code

  P(H) = 0.3
- Similarly,

  P(A) = 0.45

  P(D) = 0.25
- If Hana wrote the code, the table shows that 3% of bugs was found. Thus,

  P(B|H) = 0.03
- Similarly,

  P(B|A) = 0.02

  P(B|D) = 0.05
- The probability that a bug was found in the code written is

$$P(B) = P(B|H)P(H) + P(B|A)P(A) + P(B|D)P(D)$$
$$= (0.03)(0.3) + (0.02)(0.45) + (0.05)(0.25)$$
$$= 0.0305$$

If a bug was found, the probability that it was in the code written by Dani is

$$P(D|B) = \frac{P(B|D)P(D)}{P(B)}$$
$$= \frac{(0.05)(0.25)}{0.0305}$$
$$= 0.4098$$

# Exercise

A department store has three branches that sells clothes. The customers can return the clothes if they bought the clothes in wrong sizes, the clothes have defects or if they simply change their mind. Suppose that out of all of the returned clothes from last month, **half are from branch A**, **3/10 from branch B** and **1/5 from branch C** (the details shown in Table 1).

**Table 1**: Data on returned cloths by branch.

| | Branch A | Branch B | Branch C |
|---|---|---|---|
| Wrong size | 3/5 | 1/3 | 3/8 |
| Defects | 1/10 | 1/2 | 1/4 |
| Change mind | 3/10 | 1/6 | 3/8 |

i. What are the probabilities that the customers from branch A return the cloth because of they changed their mind?

ii. If it was discovered that the customer return because of wrong size, what is the probability that he or she return it at branch C?

iii. If it was discovered that the customer return because of defects, what is the probability that he or she return it at branch B?

# Solution

- Let
  - A denotes the event of "returned clothes from branch A"
  - B denotes the event of "returned clothes from branch B"
  - C denotes the event of "returned clothes from branch C"
  - S denotes the event of "wrong size"
  - D denotes the event of "defect"
  - M denotes the event of "change mind"
- Since Branch A received half of all returned clothes

  P(A) = 0.5
- Similarly,

  P(B) = 0.3

  P(C) = 0.2

| | Branch A | Branch B | Branch C |
|---|---|---|---|
| Wrong size | 3/5 | 1/3 | 3/8 |
| Defects | 1/10 | 1/2 | 1/4 |
| Change mind | 3/10 | 1/6 | 3/8 |

**** half are from branch A**, **3/10 from branch B** and **1/5 from branch C**

# Solution (cont.)

If the customers from Branch A return due to change mind:

$P(M|A) = 0.3$

Similarly,

$P(M|B) = 0.167$

$P(M|C) = 0.375$

| | Branch A | Branch B | Branch C |
|---|---|---|---|
| Wrong size | 3/5 | 1/3 | 3/8 |
| Defects | 1/10 | 1/2 | 1/4 |
| Change mind | 3/10 | 1/6 | 3/8 |

The probability that return due to change mind is

$$P(M) = P(M \mid A)P(A) + P(M \mid B)P(B) + P(M \mid C)P(C)$$
$$= (0.3)(0.5) + (0.167)(0.3) + (0.375)(0.2)$$
$$= 0.275$$

i) What are the probabilities that the customers from branch A return because of they **changed their mind**?

- Note that 'given condition' is "the customers from branch A return"
- Hence $P(M|A) = 0.3$

# Solution (cont.)

- If the customers from Branch A return due to wrong size:

  $P(S|A) = 0.6$

  – Similarly,

  $P(S|B) = 0.333$

  $P(S|C) = 0.375$

| | Branch A | Branch B | Branch C |
|---|---|---|---|
| Wrong size | 3/5 | 1/3 | 3/8 |
| Defects | 1/10 | 1/2 | 1/4 |
| Change mind | 3/10 | 1/6 | 3/8 |

The probability that return due to wrong size is

$$P(S) = P(S|A)P(A) + P(S|B)P(B) + P(S|C)P(C)$$
$$= (0.6)(0.5) + (0.333)(0.3) + (0.375)(0.2)$$
$$= 0.475$$

ii) If it was discovered that the customer return because of **wrong size**, what is the probability that he or she return it at branch C?

$$P(C|S) = \frac{P(S|C)P(C)}{P(S)}$$
$$= \frac{(0.375)(0.2)}{0.475}$$
$$= 0.158$$

# Solution (cont.)

- If the customers from Branch A return due to defects:

  $P(D|A) = 0.1$

  – Similarly,

  $P(D|B) = 0.5$

  $P(D|C) = 0.25$

| | Branch A | Branch B | Branch C |
|---|---|---|---|
| Wrong size | 3/5 | 1/3 | 3/8 |
| Defects | 1/10 | 1/2 | 1/4 |
| Change mind | 3/10 | 1/6 | 3/8 |

The probability that return due to defects is

$$P(D) = P(D|A)P(A) + P(D|B)P(B) + P(D|C)P(C)$$
$$= (0.1)(0.5) + (0.5)(0.3) + (0.25)(0.2)$$
$$= 0.25$$

iii) If it was discovered that the customer return because of **defects**, what is the probability that he or she return it at branch B?

$$P(B|D) = \frac{P(D|B)P(B)}{P(D)}$$
$$= \frac{(0.5)(0.3)}{0.25}$$
$$= 0.6$$

# Independent Events

- If the probability of event A does not depend on event B in the sense that P(A | B)=P(A), we say that A and B are independent events.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = P(A)$$

$$P(A \cap B) = P(A).P(B)$$

# Example

A dresser drawer contains one pair of socks with each of the following colors: **blue, brown, red, white and black.** Each pair is folded together in a matching set. You reach into the sock drawer and choose a pair of socks without looking. You replace this pair and then choose another pair of socks. What is the probability that you will choose the red pair of socks both times?

**Solution:**

Probabilities:

$$P(\text{red}) = \frac{1}{5}$$

$$P(\text{red and red}) = P(\text{red}) \cdot P(\text{red})$$

$$= \frac{1}{5} \cdot \frac{1}{5}$$

$$= \frac{1}{25}$$

# Example

A card is chosen at random from a deck of **52 cards**. It is then replaced and a second card is chosen. What is the probability of choosing a jack and an eight?

Probabilities:

$$P(\text{jack}) = \frac{4}{52}$$

$$P(8) = \frac{4}{52}$$

$$P(\text{jack and } 8) = P(\text{jack}) \cdot P(8)$$

$$= \frac{4}{52} \cdot \frac{4}{52}$$

$$= \frac{16}{2704}$$

$$= \frac{1}{169}$$

# Exercise

Farid and Azie take a final examination in discrete structure. The probability that Farid passes is 0.70 and the probability that Azie passes is 0.95. Assume that the events “Farid passes” and “Azie passes” are independent. Find the probability that Farid or Azie, or both, passes the final exam.

# Solution

- Let *F denotes the event "Farid pass the final exam"*
- Let *A denotes the event " Azie pass the final exam"*

➔We are asked to compute $P(F \cup A)$

$$P(F \cup A) = P(F) + P(A) - P(F \cap A)$$

$$P(F \cap A) = P(F) \bullet P(A) = (0.70)(0.95) = 0.665$$

- $P(F \cup A) = P(F) + P(A) - P(F \cap A)$

  $= 0.70 + 0.95 - \mathbf{0.665}$

  $= 0.985$

# Exercise

- Halim and Aina take a final examination in Fortran.
- The probability that Halim passes is 0.85, and the probability that Aina passes is 0.70.
- Assume that the events "Halim passes the final exam" and "Aina passes the final exam" are independent.

- Find the probability that Halim does not pass.
- Find the probability that both pass.
- Find the probability that both fail.
- Find the probability that at least one passes.

# Solution

- Let *H denotes the event "Halim pass the exam"*
- Let *A denotes the event " Aina pass the exam"*
- Given, $P(H) = 0.85$ ; $P(A) = 0.70$

i) Probability Halim does not pass the exam:

$$P(H') = 1 - P(H) = 1 - 0.85 = 0.15$$

ii) Probability that both pass:

$$P(H \cap A) = P(H) \bullet P(A) = (0.85)(0.70) = 0.595$$

iii) Probability that both fail:

$$P(H' \cap A') = P(H') \bullet P(A') = (1-0.85)(1-0.70) = (0.15)(0.3) = 0.045$$

iv) Probability that at least one passes:

$$P(\text{at least one pass}) = 1 - P(H' \cap A') = 1 - (0.045) = 0.955$$