CHAPTER 2 – Part 3

Recurrence Relation

Definition

- A **recurrence relation** is an equation that defines a sequence $\{a_n, a_{n+1}, \ldots\}$ based on a rule that gives the next term as a function of one or more of the previous term in the sequence for all integers n with $n \geq n_0$,( ) $a_0,\ a_1, \ldots$.
- A sequence is called a **solution** of a recurrence relation if it terms satisfy the recurrence relation.

Simple Recurrence Relation

The simplest form of a **recurrence relation** is the case where the next term depends only on the immediately previous term.

$$3, 8, 13, 18, 23, \ldots$$

The above sequence shows a pattern, given initial condition, $a_1 = 3$:

$$\begin{array}{cccccc} 3, & 3+5, & 8+5, & 13+5, & 18+5, & \ldots \\ a_1, & a_2, & a_3, & a_4, & a_5, & \ldots \end{array}$$

generated from an equation:

$$a_n = a_{n-1} + 5, \qquad n \geq 2$$

n_{th} term of a sequence

Recurrence relation can be used to compute any n-*th* term of the sequence, given the initial condition, a_0:

$$a_n = a_{n-1} + 5\ ,\ \ n \geq 2$$

3, 8, 13, 18, 23, **28,** $a_{n-1} + 5$, . . .

$$a_2 = a_1 + 5\ ,\ \ 3 + 5 = 8$$
$$a_3 = a_2 + 5\ ,\ \ 8 + 5 = 13$$
$$a_4 = a_3 + 5\ ,\ \ 13 + 5 = 18$$
$$\vdots$$
$$a_6 = a_5 + 5\ ,\ \ 23 + 5 = 28$$

Example 1

Consider the following sequence:

$$3, 9, 27, 81, 243, \ldots$$

The above sequence shows a pattern:

$$3^1,\ 3^2,\ 3^3,\ 3^4,\ 3^5, \ldots$$

$$a_1,\ a_2,\ a_3,\ a_4,\ a_5, \ldots$$

Recurrence relation is defined by:

$$a_n = 3^n, n \geq 1$$

Example 2

Given initial condition, $a_0 = 1$ and recurrence relation:

$$\boldsymbol{a_n = 1 + 2a_{n-1}\ ,\ n \geq 1}$$

First few sequence are:

$a_1 = 1 + 2\,(1) = 3$
$a_2 = 1 + 2(3) = 7$
$a_3 = 1 + 2(7) = 15$

1, 3, 7, 15, 31, 63, ...

Example 3

Given initial condition, a_0 = 1, a_1 = 2 and recurrence relation:

$$\boldsymbol{a_n = 3(a_{n-1} + a_{n-2}),\ n \geq 2}$$

First few sequence are:

$a_2 = 3(2 + 1) = 9$
$a_3 = 3(9 + 2) = 33$
$a_4 = 3(33 + 9) = 126$

1, 2, 9, 33, 126, 477, 1809, 6858, 26001,...

Exercise

1) $a_n = 6a_{n-1} - 9a_{n-2}$, where $a_0 = 2$, $a_1 = 3$

2) $a_n = 2a_{n-1} - a_{n-2}$, where $a_0 = 5$, $a_1 = 3$

Example 4

For a photo shoot, the staff at a company have been arranged such that there are 10 people in the front row and each row has 7 more people than in the row in front of it. Find the recurrence relation and compute number of staff in the first 5 rows.

Example 4 -Solution

Notice that the difference between the number of people in successive rows is a constant amount.

This means that the n_{th} term of this sequence can be found using:

$$\boldsymbol{a_n} = \boldsymbol{a_{n-1}} + 7, \quad n \geq 2 \text{ with } \boldsymbol{a_1 = 10}$$

Example 4 -Solution

Number of staff in the first 5 rows*:*

$\boldsymbol{a_1 = 10}$,
$\boldsymbol{a_2 = a_1}$ + 7, 10 + 7 = 17
$\boldsymbol{a_3 = a_2}$ + 7, 17 + 7 = 24
$\boldsymbol{a_4 = a_3}$ + 7, 24 + 7 = 31
$\boldsymbol{a_5 = a_4}$ + 7, 31 + 7 = 38

10, 17, 24, 31, 38

Example 5

Find a recurrence relation and initial condition for

1, 5, 17, 53, 161, 485, …

Solution:

Look at the differences between terms:

4, 12, 36, 108, …

the difference in the sequence is growing by a factor of 3.

Example 5 -Solution

However the original sequence is not.

1(3)=**3**, 5(3)=**15**,17(3)=**51**, . . .

1, 5, 17, 53, 161, 485, ...

It appears that we always end up with 2 less than the next term.

So, the recurrence relation is defined by:

$\boldsymbol{a_n = 3(a_{n-1}) + 2,\ n \geq 1}$, **with initial condition,** $\boldsymbol{a_0 = 1}$

Example 6

A depositor deposits RM 10,000 in a savings account at a bank yielding 5% per year with interest compounded annually. How much money will be in the account after 30 years? Let P_n denote the amount in the account after n years.

Example 6 - Solution

Derive the following **recurrence relation**:

$$P_n = P_{n-1} + 0.05P_{n-1} = 1.05P_{n-1}$$

Where, P_n = Current balance and P_{n-1} = Previous year balance and 0.05 is the compounding interest.

Example 6 - Solution

Initial condition, $P_0 = 10{,}000$. Then,

$P_1 = 1.05P_0$
$P_2 = 1.05P_1 = (1.05)^2P_0$
$P_3 = 1.05P_2 = (1.05)^3P_0$
...
$P_n = 1.05P_{n-1} = (1.05)^nP_0$,

now we can use this formula to calculate n_{th} term without iteration

Example 6 - Solution

Let us use this formula to find P_{30} under the initial condition $P_0 = 10{,}000$:

$$P_{30} = (1.05)^{30}(10{,}000) = 43{,}219.42$$

After 30 years, the account contains RM 43,219.42.

Exercise

Consider the following sequence:

1, 5, 9, 13, 17

Find the recurrence relation that defines the above sequence.

Exercise

A basketball is dropped onto the ground from a height of 15 feet. On each bounce, the ball reaches a maximum height 55% of its previous maximum height.

a)Write a recursive formula, a_n , that completely defines the height reached on the n_{th} bounce, where the first term in the sequence is the height reached on the ball's first bounce.

b)How high does the basketball reach after the 4_{th} bounce? Give your answer to two decimal places.