

## CHAPTER 3

# COUNTING METHODS

## [Part 1]

# Basic Counting Principles

- Counting principle is all about choices we might make given many possibilities.
- It is used to find the number of possible outcomes.
- It provides a basis of computing probabilities of discrete events.

# Basic Counting Principles

## Some sample of counting problems:

- Problem 1 - How many ways are there to seat  $n$  couples at a round table, such that each couple sits together?
- Problem 2- How many ways are there to express a positive integer  $n$  as a sum of positive integers?

# Basic Counting Principles

- Problem 3 - There are three boxes containing books. The first box contains 15 mathematics books by different authors, the second box contains 12 chemistry books by different authors, and the third box contains 10 computer science books by different authors. A student wants to take a book from one of the three boxes. In how many ways can the student do this?

# Basic Counting Principles

Problem 4 - The department will award a free computer to either a CS student or a CS professor. How many different choices are there, if there are 530 students and 15 professors?

Problem 5 - A scholarship is available, and the student to receive this scholarship must be chosen from the Mathematics, Computer Science, or the Engineering Department. How many different choices are there for this student if there are 38 qualified students from the Mathematics Department, 45 qualified students from the Computer Science Department and 27 qualified students from the Engineering Department?

# Basic Counting Principles

- There are a number of basic principles that we can use to solve such problems:
  - 1) **Addition Principle**
  - 2) **Multiplication Principle**

# Addition Principle

- Suppose that tasks  $T_1, T_2, \dots, T_k$  can be done in  $n_1, n_2, \dots, n_k$  ways, respectively.
- If all these tasks are **independent** of each other, then the number of ways to do one of these tasks is  $n_1 + n_2 + \dots + n_k$



# Basic Counting Principles

- If a task can be done in  $n_1$  ways and a second task in  $n_2$  ways, and if these two tasks cannot be done at the same time, then there are  $n_1 + n_2$  ways to do either task.

# Example 1

We want to find the number of integers between 5 and 50 that end with 1 or 7.

**Solution:**

- Let  $T$  denote this task. We divide  $T$  into the following tasks.
  - $T_1$ : find all integers between 5 and 50 that end with 1.
  - $T_2$ : find all integers between 5 and 50 that end with 7.

- $T_1$ : find all integers between 5 and 50 that end with 1.  
     $\rightarrow 11, 21, 31, 41 \Rightarrow 4$  ways
  - $T_2$ : find all integers between 5 and 50 that end with 7.  
     $\rightarrow 7, 17, 27, 37, 47 \Rightarrow 5$  ways
- $\therefore n_1 + n_2 = 4 + 5 = 9$  ways.

## Example 2

We want to find the number of integers between 4 and 100 that end with 3 or 5.

**Solution:**

Let  $T$  denote this task. We divide  $T$  into the following tasks.

- $T_1$ : find all integers between 4 and 100 that end with 3.
- $T_2$ : find all integers between 4 and 100 that end with 5.

- $T_1$ : find all integers between 4 and 100 that end with 3.  
=> 13, 23, 33, 43, 53, 63, 73, 83, 93  
=> 9 integers
- $T_2$ : find all integers between 4 and 100 that end with 5.  
=> 5, 15, 25, 35, 45, 55, 65, 75, 85, 95  
=> 10 integers

- Task  $T_1$  can be done in 9 ways.
- Task  $T_2$  can be done in 10 ways.
- The number of ways to do one of these tasks is,

$$9+10=19$$

- Task  $T$  can be completed in 19 ways.

## Example 3



4 books



2 books



3 books

- A student wants to take a book from one of the three boxes.
- In how many ways can the students do this?

- $T_1$ : choose a mathematics book  
➤ 4 ways
- $T_2$ : choose a chemistry book  
➤ 2 ways
- $T_3$ : choose a computer science book.  
➤ 3 ways
- The number of ways to do one of these tasks is,  
➤  $4+2+3=9$



# Exercise

- There are 8 male students and 21 female students in Discrete Structure class. Among all of them, 7 students are Chinese and the rest are Malay.
  - In how many ways can we select 1 student - a boy OR a girl?
  - In how many ways can we select 1 student - a Chinese OR a Malay?



# Exercise - Solution

- In how many ways can we select 2 students - a boy and a girl?

- To select a boy
  - 8 ways
- To select a girl
  - 21 ways
- The number of ways
  - **$8+21=29$  ways**

- In how many ways can we select 2 students - a Chinese and a Malay?

- To select a Chinese
  - 7 ways
- To select a Malay
  - 22 ways
- The number of ways
  - **$7+22=29$  ways**

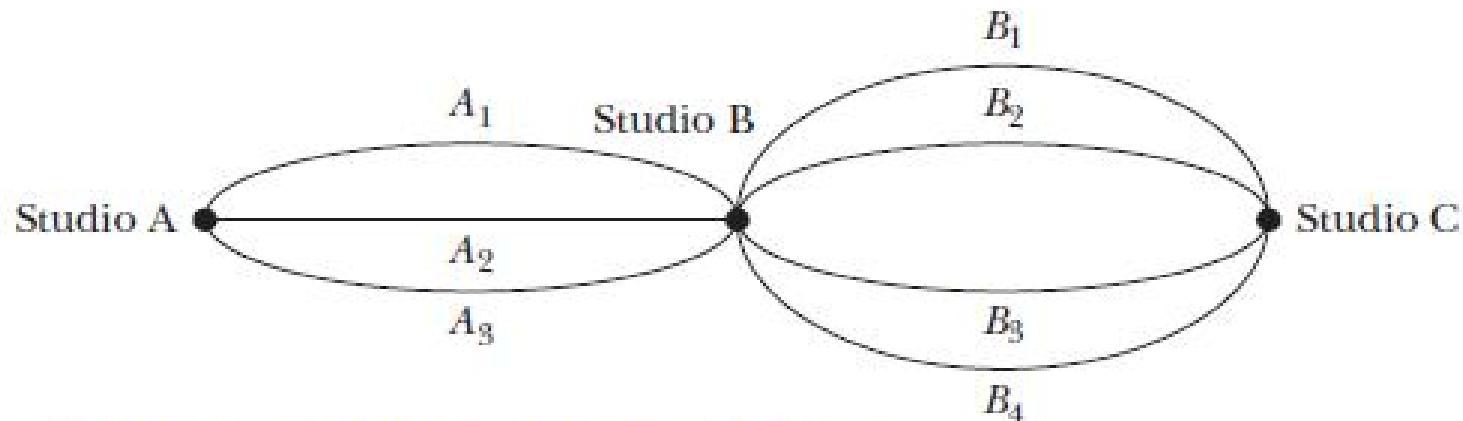
# Multiplication Principle

- A task  $T$  can be completed in  $k$  successive steps.  
**Step 1** can be completed in  $n_1$  different ways.  
**Step 2** can be completed in  $n_2$  different ways.  
**Step  $k$**  can be completed in  $n_k$  different ways.
- Then the task  $T$  can be completed in  $n_1 \cdot n_2 \cdot \dots \cdot n_k$  different ways.

Suppose that a procedure can be broken down into two successive tasks. If there are  $n_1$  ways to do the first task and  $n_2$  ways to do the second task after the first task has been done, then there are  $n_1 n_2$  ways to do the procedure.

# Example 4

- Morgan is a lead actor in a new movie. She needs to shoot a scene in the morning in studio A and an afternoon scene in studio C. She looks at the map and finds that there is no direct route from studio A to studio C. Studio B is located between studios A and C. Morgan's friends Brad and Jennifer are shooting a movie in studio B. There are three roads, say  $A_1$ ,  $A_2$ , and  $A_3$ , from studio A to studio B and four roads, say  $B_1$ ,  $B_2$ ,  $B_3$ , and  $B_4$ , from studio B to studio C. In how many ways can Morgan go from studio A to studio C and have lunch with Brad and Jennifer at Studio B?



**FIGURE 7.1** Routes from studio A to studio C

- There are 3 ways to go from studio *A* to studio *B* and 4 ways to go from studio *B* to studio *C*.
  - $T1 = 3$
  - $T2 = 4$
- The number of ways to go from studio *A* to studio *C* via studio *B* is  $3 \times 4 = 12$ .

## Example 5

- The letters *A*, *B*, *C*, *D*, and *E* are to be used to form strings of length 4.
- How many strings can be formed if we do not allow repetitions?
- For example:
  - *BADE*, *ACBD*, *AEBC* ..

- $T_1$ : choose the first letter  $\rightarrow$  5 ways
- $T_2$ : choose the second letter  $\rightarrow$  4 ways
- $T_3$ : choose the third letter  $\rightarrow$  3 ways
- $T_4$ : choose the fourth letter  $\rightarrow$  2 ways
- There are  $5.4.3.2 = 120$  strings.



## Example 6

- The letters *A*, *B*, *C*, *D*, and *E* are to be used to form strings of length 4.
- How many strings can be formed if we allow repetitions?
- For example:
  - *BABB*, *AABB*, *ACEE* ..

- $T_1$ : choose the first letter  $\rightarrow$  5 ways
- $T_2$ : choose the second letter  $\rightarrow$  5 ways
- $T_3$ : choose the third letter  $\rightarrow$  5 ways
- $T_4$ : choose the fourth letter  $\rightarrow$  5 ways
- There are  $5.5.5.5 = 625$  strings.

## Example 7

- The letters *A*, *B*, *C*, *D*, and *E* are to be used to form strings of length 4.
- How many strings begin with *A*, if repetitions are not allowed?
- For example:
  - *ADEC*, *ACBD*, *AEBC* ..

- $T_1$ : choose the first letter  $\rightarrow$  A (1 way)
- $T_2$ : choose the second letter  $\rightarrow$  4 ways
- $T_3$ : choose the third letter  $\rightarrow$  3 ways
- $T_4$ : choose the fourth letter  $\rightarrow$  2 ways
- There are  $1.4.3.2 = 24$  strings.

# Exercise

- Danial, Kenny and Joseph are fighting over a turn to play a game that can only has 2 players at a time. In how many ways can we select 2 players at a time?
- Diana, Sherry, Devi and Mary are going to DSI using a motorcycle. In how many ways can we select 2 peoples to ride the motorcycle?

# Exercise

- Suppose that, in order to declare a variable name in a computer language, the name must have five characters. The first character must be a letter, and the remaining characters can be letters or digits. How many different variable names are possible?

# Exercise - Solution



- In how many ways can we select 2 players at a time?
  - To select 1<sup>st</sup> player
    - 3 ways
  - To select 2<sup>nd</sup> player
    - 2 ways
  - The number of ways
    - $3 \cdot 2 = 6$  ways

- The possible order:

Player 1	Player 2
Danial	Kenny
Danial	Joseph
Kenny	Joseph
Kenny	Danial
Joseph	Kenny
Joseph	Danial

■ In how many ways can we select 2 peoples to ride the motorcycle?

- To select 1<sup>st</sup> rider
  - 4 ways
- To select 2<sup>nd</sup> rider
  - 3 ways
- The number of ways
  - $4 \cdot 3 = 12$  ways



● The possible order

Rider 1	Rider 2
Diana	Sherry
Diana	Devi
Diana	Mary
Sherry	Diana
Sherry	Devi
Sherry	Mary
Devi	Diana
Devi	Sherry
Devi	Mary
Mary	Diana
Mary	Sherry
Mary	Devi



# Exercise - Solution

- How many different variable names are possible?

- To select 1<sup>st</sup> character
  - 26 ways ( 26 alphabets)
- To select 2<sup>nd</sup> rider
  - 36 ways (26 alphabets + 10 numbers)
- To select 3<sup>rd</sup> character
  - 36 ways (26 alphabets + 10 numbers)
- To select 4<sup>th</sup> rider
  - 36 ways (26 alphabets + 10 numbers)
- To select 5<sup>th</sup> character
  - 36 ways (26 alphabets + 10 numbers))

- The number of ways

$26 \cdot 36 \cdot 36 \cdot 36 \cdot 36 = 43,670,016$  ways

# Basic Counting Principles

- The counting problem that we have considered so far involved either the addition principle or the multiplication principle.
- Sometimes, however, we need to use both of these counting principles to solve a particular problem.

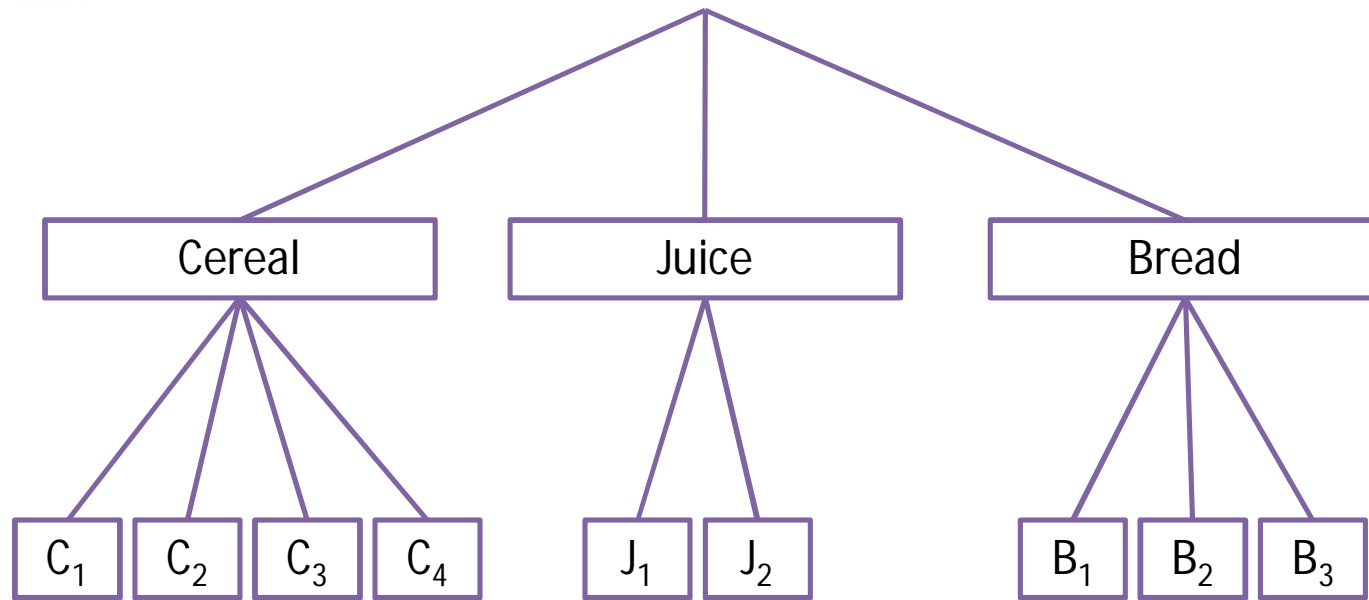
## Example 8

- How many 8-bit strings begin either 101 or 111?
  - $\underline{1} \underline{0} \underline{1} \underline{\quad} \underline{\quad} \underline{\quad} \underline{\quad} \underline{\quad} \rightarrow 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$
  - $\underline{1} \underline{1} \underline{1} \underline{\quad} \underline{\quad} \underline{\quad} \underline{\quad} \underline{\quad} \rightarrow 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$
  - $32 + 32 = 64$

## Example 9

- The following items are available for breakfast.
  - 4 types of cereal
  - 2 types of juice
  - 3 types of bread
- How many ways a breakfast can be prepared if exactly 2 items are selected from 2 different groups?

- A breakfast can be prepared in either of the following 3 ways:
  - A cereal & a juice
    - $4 \cdot 2 = 8$
  - A cereal & a bread
    - $4 \cdot 3 = 12$
  - A juice & a bread
    - $2 \cdot 3 = 6$
  - $8 + 12 + 6 = 26$  ways



C <sub>1</sub> J <sub>1</sub>	C <sub>1</sub> J <sub>2</sub>
C <sub>2</sub> J <sub>1</sub>	C <sub>2</sub> J <sub>2</sub>
C <sub>3</sub> J <sub>1</sub>	C <sub>3</sub> J <sub>2</sub>
C <sub>3</sub> J <sub>1</sub>	C <sub>3</sub> J <sub>2</sub>

C <sub>1</sub> B <sub>1</sub>	C <sub>1</sub> B <sub>2</sub>	C <sub>1</sub> B <sub>3</sub>
C <sub>2</sub> B <sub>1</sub>	C <sub>2</sub> B <sub>2</sub>	C <sub>2</sub> B <sub>3</sub>
C <sub>3</sub> B <sub>1</sub>	C <sub>3</sub> B <sub>2</sub>	C <sub>3</sub> B <sub>3</sub>
C <sub>4</sub> B <sub>1</sub>	C <sub>4</sub> B <sub>2</sub>	C <sub>4</sub> B <sub>3</sub>

J <sub>1</sub> B <sub>1</sub>	J <sub>1</sub> B <sub>2</sub>	J <sub>1</sub> B <sub>3</sub>
J <sub>2</sub> B <sub>1</sub>	J <sub>2</sub> B <sub>2</sub>	J <sub>2</sub> B <sub>3</sub>

## Example 10

A six-person committee composed of Aina, Wan, Chan, Tan, Syed and Helmi is to be selected to hold as a chairperson, secretary, and treasurer.

- In how many ways can this be done?
- In how many ways can this be done if either Aina or Wan must be chairperson?
- In how many ways can this be done if Syed must hold one of the position?
- In how many ways can this be done if Tan and Helmi must hold any position?

## Example 10 - Solution

- In how many ways can this be done?
  - Select the chairperson (6 ways)
  - Select the secretary (5 ways)
  - Select the treasurer (4 ways)
  - $6 \cdot 5 \cdot 4 = 120$



- In how many ways can this be done if either Aina or Wan must be chairperson?
  - If Aina is chairperson  $5 \cdot 4 = 20$
  - If Wan is chairperson  $5 \cdot 4 = 20$
  - $20 + 20 = 40$

- OR
  - ✓ Select the chairperson (2 ways)
  - ✓ Select the secretary (5 ways)
  - ✓ Select the treasurer (4 ways)
  - ✓  $2 \cdot 5 \cdot 4 = 40$

- In how many ways can this be done if Syed must hold one of the position?
  - If Syed is chairperson  $5 \cdot 4 = 20$
  - If Syed is secretary  $5 \cdot 4 = 20$
  - If Syed is treasurer  $5 \cdot 4 = 20$
  - $20 + 20 + 20 = 60$

- OR

- Assign Syed for any position is 3 ways
- Fill the highest remaining position is 5 ways
- Fill the last position is 4 ways
- $3 \cdot 5 \cdot 4 = 60$

- In how many ways can this be done if Tan and Helmi must hold any position?
  - Assign Tan - 3 ways
  - Assign Helmi - 2 ways
  - Fill the remaining position - 4 ways
  - $3 \cdot 2 \cdot 4 = 24$

# Exercise

- How many eight-bit strings have either the second or the fourth bit 1 (or both)?
- How many license plates of 2 letters from A to Z, followed by 3 digits from 0 to 9 can be made, if repetition of letters is not allowed?

# Exercise

- How many 5-digit telephone numbers can be constructed using digit 0-9 if each number starts with 67 and no digit appears more than once?