Functions

Functions

- Let *X* and *Y* be nonempty sets
- A function *f* from *X* to *Y* is a relation from *X* to *Y* having the properties.
  - The domain of *f* is *X*
  - If $(x,y), (x,y') \in f$, then $y=y'$
    - (e.g. f(1)=b, f(2)=b is a function, but f(1)=a, f(1)=b is NOT a function)

Functions

- A function from X to Y is denoted, $f : X \to Y$
- The domain of f is the set X.
- The set Y is called the codomain or target of f.
- The set $\{ y \mid (x,y) \in f \}$ is called the range.

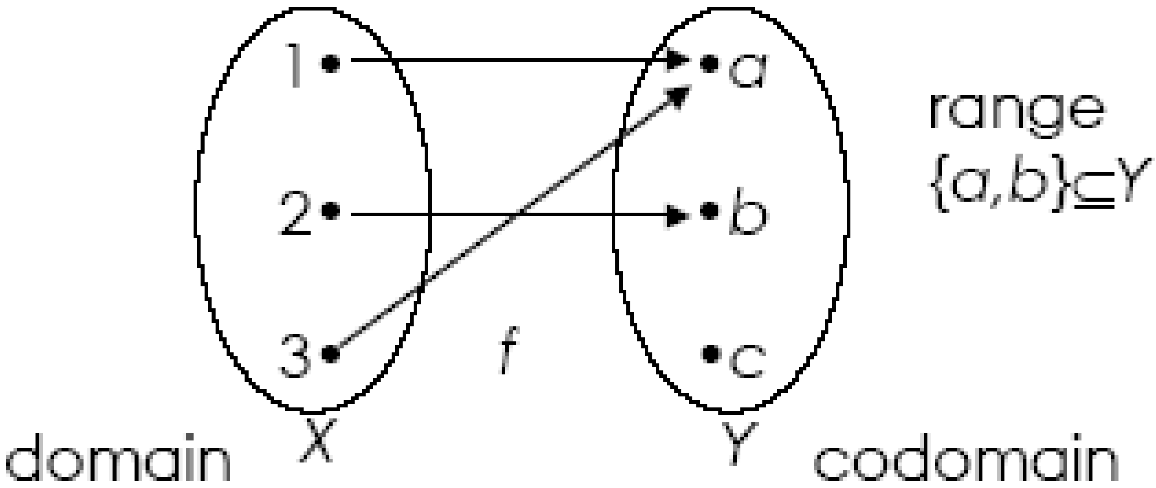
example

- The relation, f = { (1,a), (2,b), (3,a) } from X = { 1, 2, 3 } to Y= { a, b, c } is a function from X to Y.

- The domain of f is X
- The range of f is {a, b}

example

- $f = \{ (1,a), (2,b), (3,a) \}$



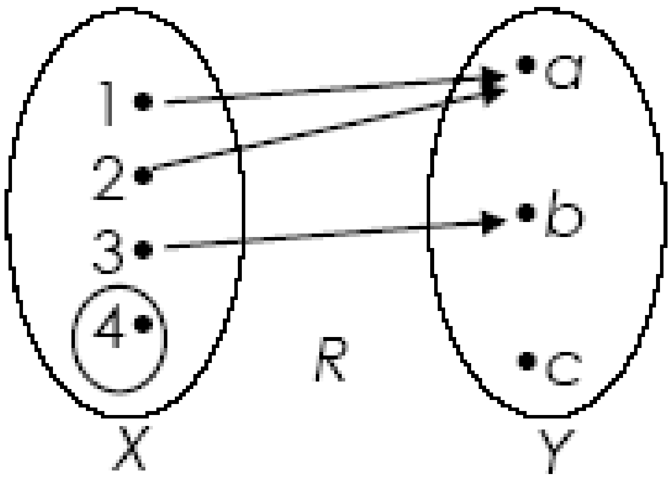
example

- The relation, $R = \{(1,a), (2,a), (3,b)\}$ from $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c\}$ is NOT a function from X to Y.

- The domain of R, $\{ 1,2,3 \}$ is not equal to X.

example

- $R = \{(1,a), (2,a), (3,b)\}$



There is no arrow from 4

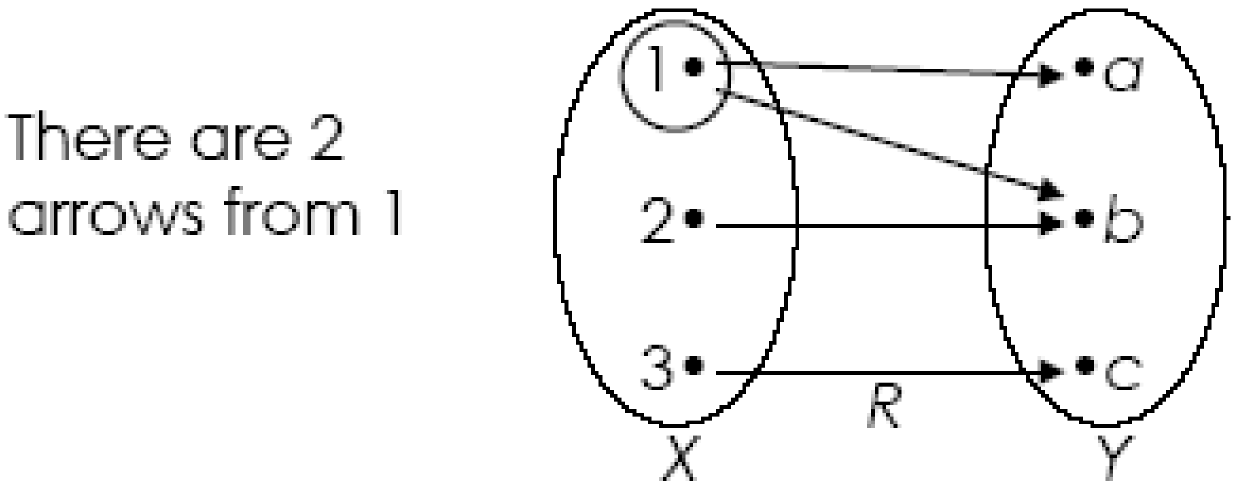
example

- The relation, $R = \{(1,a), (2,b), (3,c), (1,b)\}$ from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$ is NOT a function from X to Y

- (1,a) and (1,b) in R but a ≠ b.

example

- R= {(1,a), (2,b), (3,c), (1,b)}



example

- For the function, $f = \{(1,a), (2,b), (3,a)\}$
- We may write
  - $f(1)=a$, $f(2)=b$, $f(3)=a$
- Notation $f(x)$ is used to define a function

example

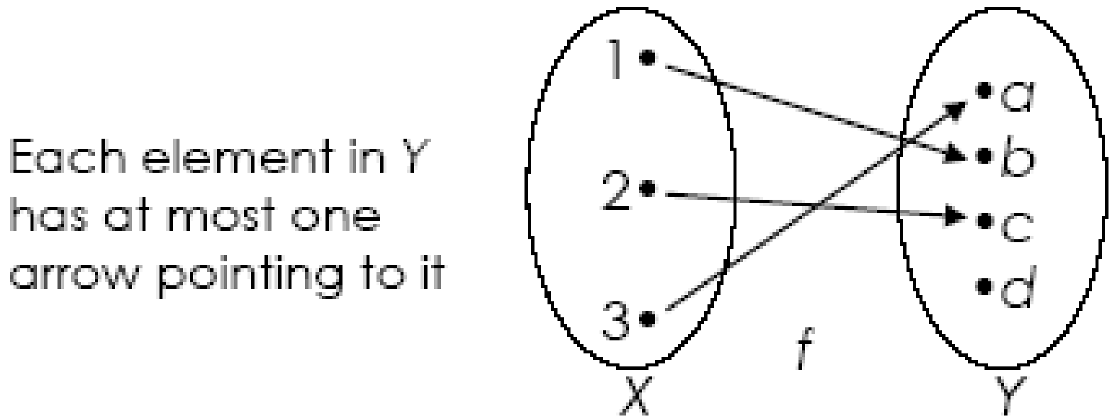
- $f(x) = x2$
- $f(2)=4$, $f(-3.5)=12.25$, $f(0)=0$
- $f = \{(x, x2)\mid x$ is a real number$\}$

One-to-one

- A function f from X to Y, is said one-to one (or injective) if for each $y \in Y$, there is at most one $x \in X$, with $f(x)=y$.
- For all $x1$, $x2$, if $f(x1) = f(x2)$, then $x1=x2$.
- $\forall x1 \ \forall x2 \ ((f(x1) = f(x2)) \rightarrow (x1=x2))$

example

- The function, $f = \{ (1,b), (3,a), (2,c) \}$ from $X = \{ 1, 2, 3 \}$ to $Y = \{ a, b, c, d \}$ is one-to-one.

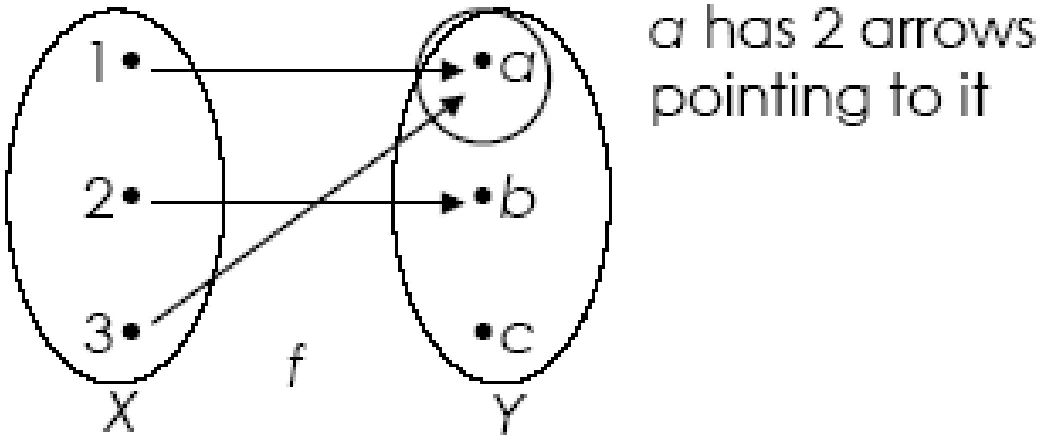


example

- The function, $f = \{ (1,a), (2,b), (3,a) \}$ from $X = \{ 1, 2, 3 \}$ to $Y = \{ a, b, c\}$ is NOT one-to-one.
- $f(1) = a = f(3)$

Example

- $f = \{ (1,a), (2,b), (3,a) \}$



example

- Show that the function,

  $f(n) = 2n+1$

  on the set of positive integers is one-to one.

example

- For all positive integer, n_1 and n_2 if $f(n_1) = f(n_2)$, then $n_1 = n_2$.
- Let, $f(n_1) = f(n_2)$, $f(n) = 2n+1$

  then $2n_1 + 1 = 2n_2 + 1$ (-1)

  $2n_1 = 2n_2$ (÷2)

  $n_1 = n_2$
- This shows that f is one-to-one.

example

- Show that the function,

  $f(n) = 2^n - n^2$

  from the set of positive integers to the set of integers is NOT one-to-one.

example

- Need to find 2 positive integers, n_1 and n_2

  $n_1 \neq n_2$ with $f(n_1) = f(n_2)$.

  trial and error,

  $f(2) = f(4)$

  f is not one-to-one.

Onto

- If f is a function from X to Y and the range of f is Y, f is said to be onto Y (or an onto function or a surjective function)
- For every $y \in Y$, there exists at least one $x \in X$ such that $f(x)=y$

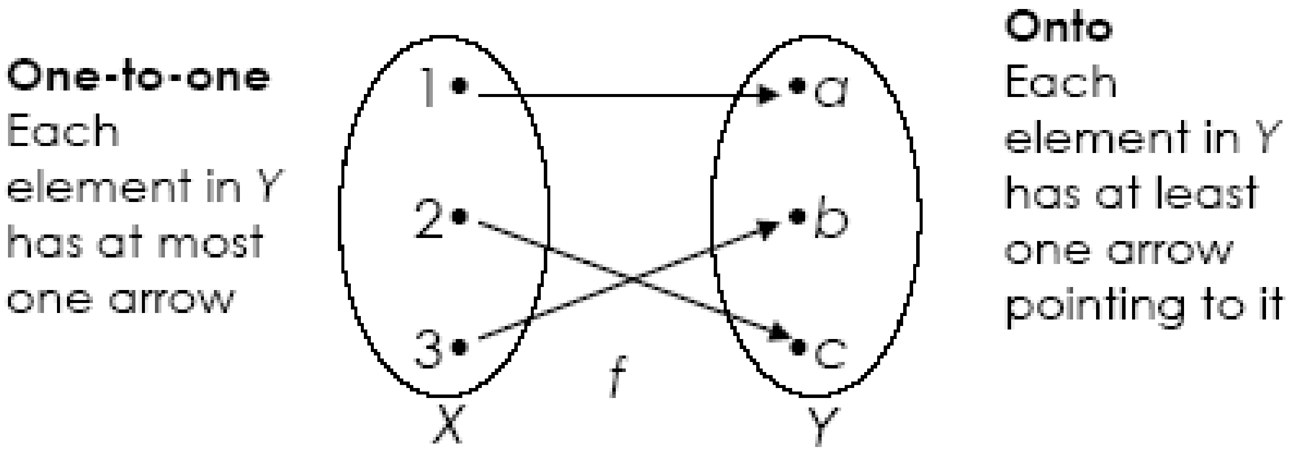
$$\forall y \in Y \, \exists x \in X \, (f(x)=y)$$

example

- The function, $f = \{ (1,a), (2,c), (3,b) \}$ from $X = \{ 1, 2, 3 \}$ to $Y = \{ a, b, c \}$ is one-to-one and onto Y.

example

- $f = \{ (1,a), (2,c), (3,b) \}$

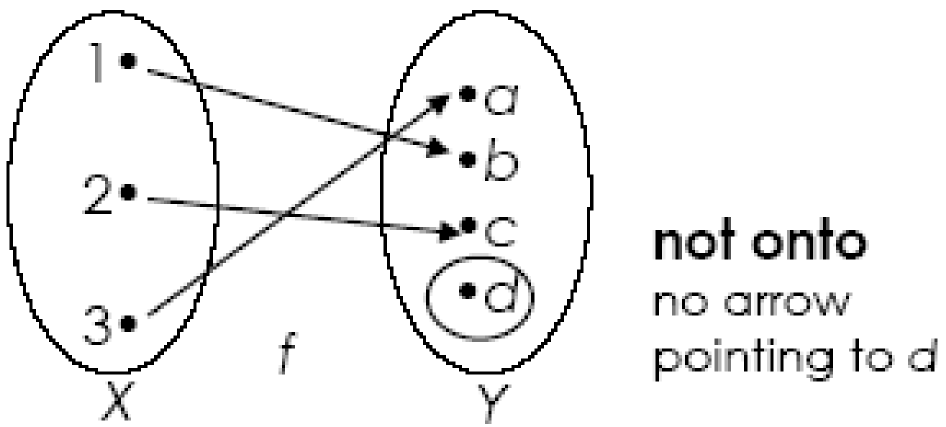


example

- The function, $f = \{ (1,b), (3,a), (2,c) \}$ is not onto $Y = \{a, b, c, d\}$
- It is onto $\{a, b, c\}$

example

- $f = \{ (1,b), (3,a), (2,c) \}$



Bijection

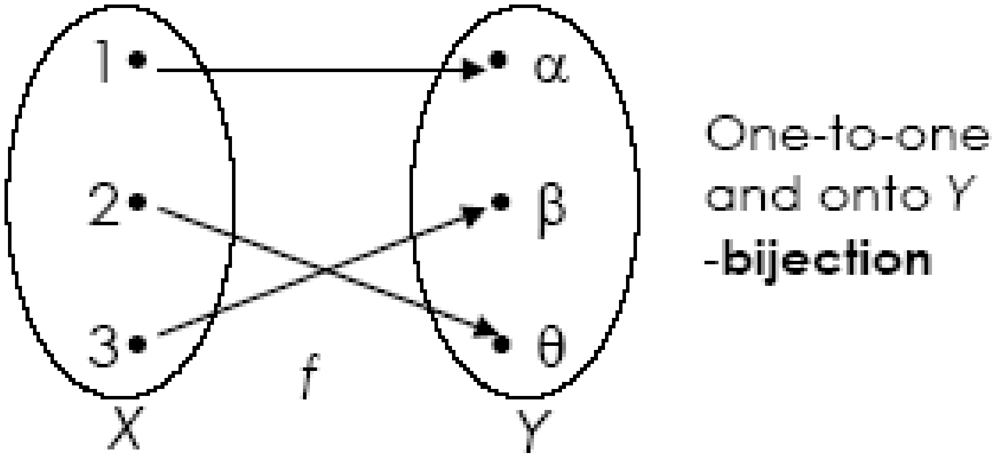
- *f* is called one-to-one correspondence (or bijective or bijection) if *f* is both one-to-one and onto.

example

- The function, $f = \{ (1,\alpha), (2,\theta), (3,\beta) \}$ from $X = \{ 1, 2, 3 \}$ to $Y = \{\alpha, \beta, \theta \}$ is one-to-one and onto Y.
- The function f is a bijection

example

- f = { (1,α), (2,θ), (3,β) }



exercise

Determine which of the relations f are functions from the set X to the set Y.

a) $X = \{ -2, -1, 0, 1, 2 \}$, $Y = \{ -3, 4, 5 \}$ and $f = \{ (-2,-3), (-1,-3), (0,4), (1,5), (2,-3) \}$

b) $X = \{ -2, -1, 0, 1, 2 \}$, $Y = \{ -3, 4, 5 \}$ and $f = \{ (-2,-3), (1,4), (2,5)\}$

c) $X = Y = \{ -3, -1, 0, 2 \}$ and $f = \{ (-3,-1), (-3,0), (-1,2), (0,2), (2,-1) \}$

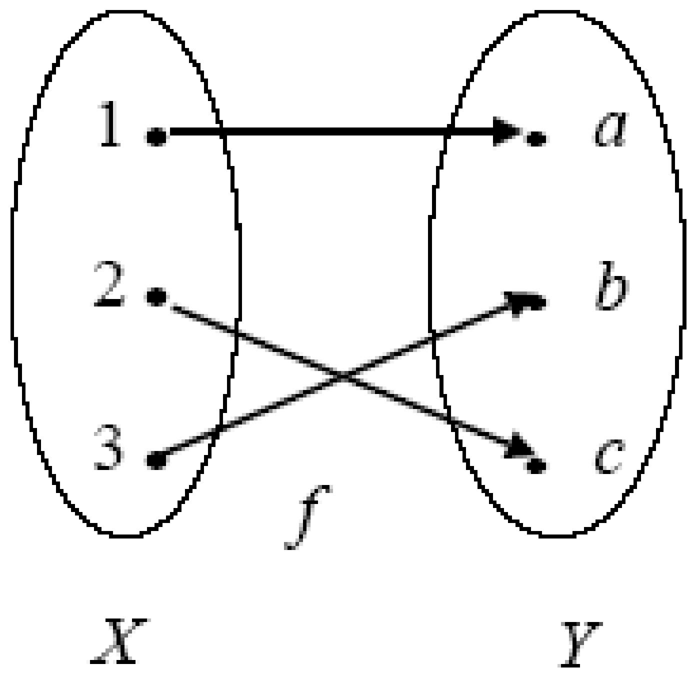
In case any of these relations are functions, determine if they are one-to-one, onto Y, and/or bijection.

Inverse function

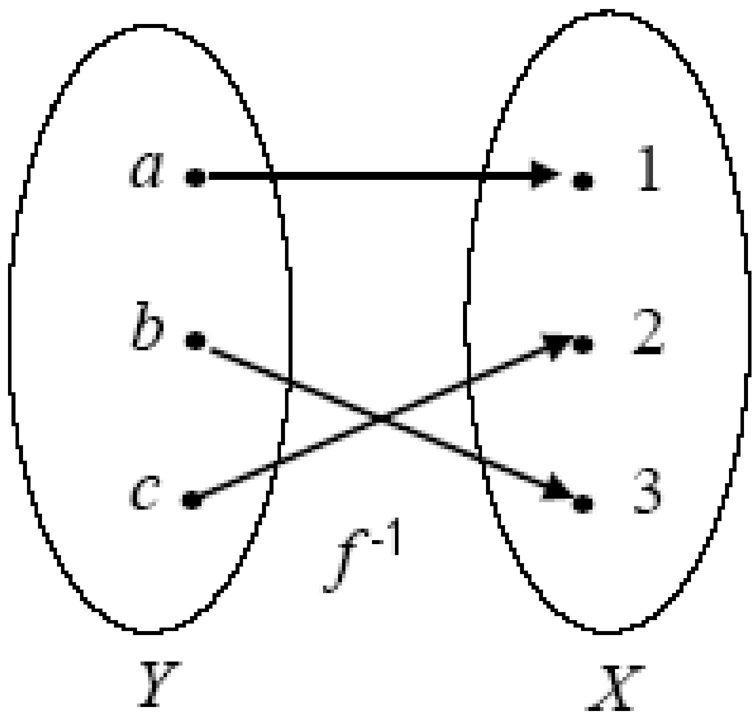
- Let $f: X \to Y$ be a function.
- The inverse relation $f^{-1} \subseteq Y \times X$ is a function from Y to X, if and only if f is both one-to-one and onto Y.

example

- $f = \{(1,a),(2,c),(3,b)\}$
- $f^{-1} = \{(a,1),(c,2),(b,3)\}$



example



example

- The function, $f(x) = 9x + 5$ for all $x \in R$ (R is the set of real numbers).
- This function is both one-to-one and onto.
- Hence, f^{-1} exists.

Let $(y, x) \in f^{-1}$, $f^{-1}(y) = x$

$(x,y) \in f$, $\quad y = 9x + 5$

$x = (y-5)/9$

$f^{-1}(y)=(y-5)/9$

exercise

- Find each inverse function.
- a) $f(x) = 4x + 2,\ x \in R$
- b) $f(x) = 3 + (1/x),\ x \in R$

Composition

- Suppose that g is a function from X to Y and f is a function from Y to Z.
- The composition of f with g,

$$f \circ g$$

is a function

$$(f \circ g)(x) = f(g(x))$$

from X to Z

example

- Given, $g = \{ (1,a), (2,a), (3,c) \}$

  a function from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$

  and

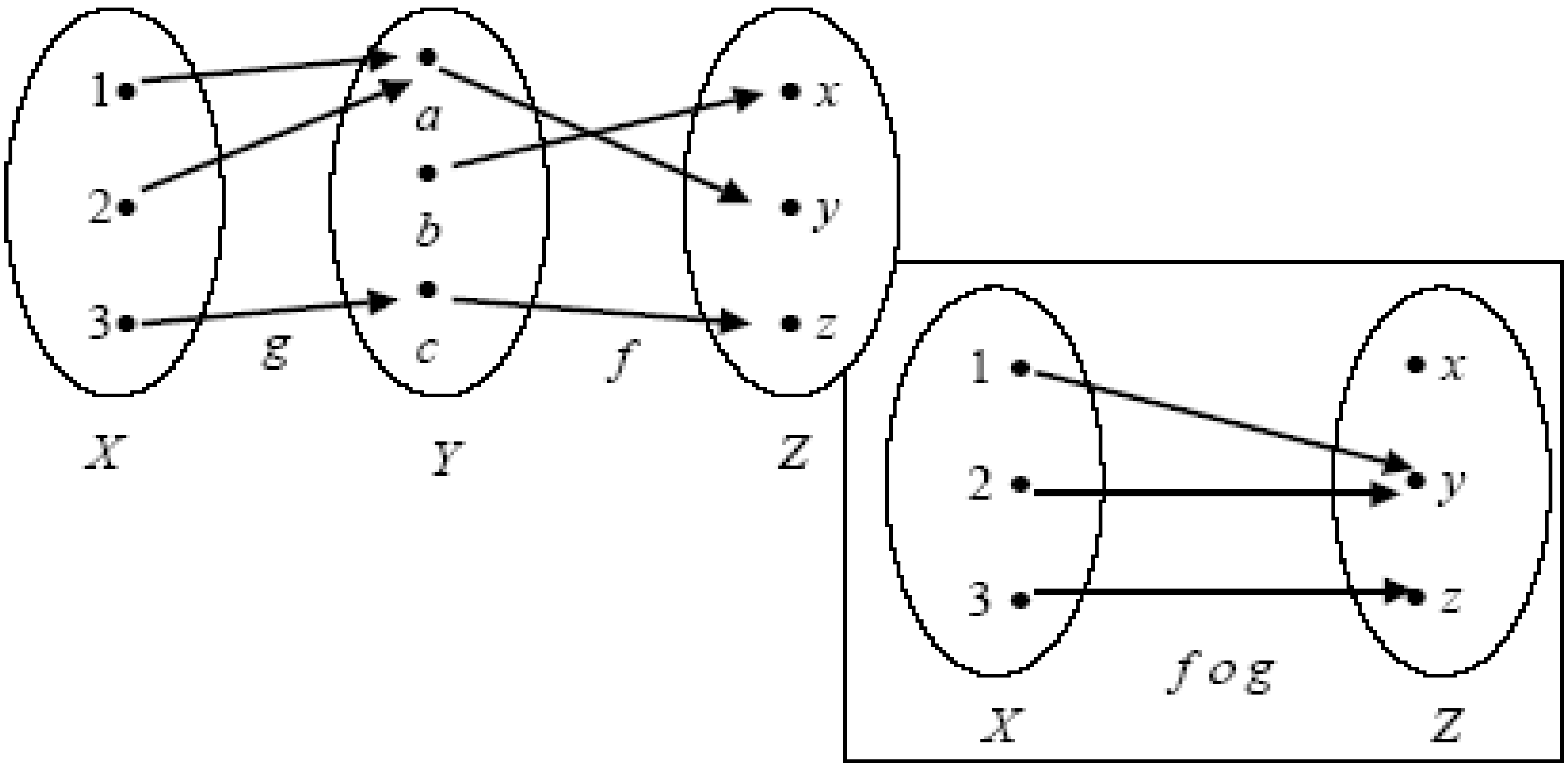
  $$f = \{ (a,y), (b,x), (c,z) \}$$

  a function from Y to $Z = \{ x, y, z \}$
- The composition function from X to Z is the

  function

  $$f \circ g = \{ (1,y), (2,y), (3,z) \}$$

example



example

- $f(x) = \log_3 x$ and $g(x) = x^4$
- $f(g(x)) = \log_3 (x^4)$
- $g(f(x)) = (\log_3 x)^4$

  Note: $f \circ g \neq g \circ f$

example

$$f(x) = \frac{1}{5}x \qquad g(x) = x^2 + 1$$

$$(g \circ f)(x) = g(f(x)) = g(\frac{x}{5})$$

$$= (\frac{x}{5})^2 + 1 = \frac{x^2}{25} + 1$$

example

- Composition sometimes allows us to decompose complicated functions into simpler functions.

example $f(x) = \sqrt{\sin 2x}$

$g(x) = \sqrt{x}$ $h(x) = \sin x$ $w(x) = 2x$

$$f(x) = g(h(w(x)))$$

exercise

- Let f and g be functions from the positive integers to the positive integers defined by the equations, $f(n) = n^2$, $g(n) = 2^n$
- Find the compositions
  - a) $f \circ f$
  - b) $g \circ g$
  - c) $f \circ g$
  - d) $g \circ f$

Recursive Algorithms

- A recursive procedure is a procedure that invokes itself.
- A recursive algorithm is an algorithm that contains a recursive procedure.
- Recursion is a powerful, elegant and natural way to solve a large class of problems.

example

Factorial problem

- If $n \geq 1$,

  $n! = n\ (n-1)\ \ldots.2.\ 1$

  and $0! = 1$

- Notice that, if $n \geq 2$, n factorial can be

  written,

  $n! = n\ (n-1)(n-2)\ \ldots.2.\ 1$

  $= n.\ (n-1)!$

example

- n=5
- 5!

  5! = 5. 4. 3. 2. 1 = 5. 4!

  4! = 4. 3!

  3! = 3. 2!

Algorithm

- Input: *n*, integer ≥ 0
- Output: *n*!
- Factorial (*n*) {

  if (*n*=0)

  return 1

  return *n**factorial(*n*-1)

  }

example

- Fibonacci sequence, f_n

  $f_1 = 1$

  $f_2 = 1$

  $f_n = f_{n-1} + f_{n-2}, n \geq 3$

  1, 1, 2, 3, 5, 8, 13, ….

Algorithm

- Input: n
- Output: $f(n)$
- $f(n)$ {

```
if (n=1 or n=2)
return 1
return f (n-1) + f (n-2)
}
```