CHAPTER 1

SET THEORY

[Part 1: Set & Subset]

Introduction

- Why are we studying sets
  - The concept of set is basic to all of mathematics and mathematical applications.
  - Serves as a basis of description of higher concept and mathematical reasoning
  - Set is fundamental in many areas of Computer Science.
- For us, sets are useful to understand the principles of *counting and probability theory*

Sets

- The concept of set is basic to all of mathematics and mathematical applications.
- A set is a well-defined collection of distinct objects.
- These objects are called members or elements of the set.

Sets

- Well-defined means that we can tell for certain whether an object is a member of the collection or not.
- If a set is finite and not too large, we can describe it by listing the elements in it.

Example

- A is a set of all positive integers less than 10,
  $A=\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- B is a set of first 5 positive odd integers,
  $B=\{1, 3, 5, 7, 9\}$
- C is a set of vowels, $C=\{a, e, i, o, u\}$

Defining Sets

- This can be done by:
  - Listing ALL elements of the set within braces.
  - Listing enough elements to show the pattern then an ellipsis.
  - Use set builder notation to define “rules” for determining membership in the set.

Example

1. Listing ALL elements. $A = \{1,\ 2,\ 3,\ 4\}$ explicitly
2. Demonstrating a pattern. $\mathbb{N} = \{1,\ 2,\ 3,\ \ldots\}$ implicitly
3. Using set builder notation. $P = \{x \mid x \in \mathbb{R} \text{ and } x \notin \mathbb{C}\}$ implicitly

Sets

- A set is determined by its elements and not by any particular order in which the element might be listed.
- **Example**,

  $A=\{1, 2, 3, 4\}$,

  A might just as well be specified as

  $\{2, 3, 4, 1\}$ or $\{4, 1, 3, 2\}$

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Sets

- The elements making up a set are assumed to be distinct, we may have duplicates in our list, only one occurrence of each element is in the set.
- **Example**

$\{a, b, c, a, c\} \longrightarrow \{a, b, c\}$

$\{1, 3, 3, 5, 1\} \longrightarrow \{1, 3, 5\}$

Sets

- Use uppercase letters A, B, C ... to denote sets, lowercase denote the elements of set.
- The symbol $\in$ stands for 'belongs to'
- The symbol $\notin$ stands for 'does not belong to'

Example

- $X=\{ a, b, c, d, e \}$, $b \in X$ and $m \notin X$

- $A=\{\{1\}, \{2\}, 3, 4\}$, $\{2\} \in A$ and $1 \notin A$

Sets

- If a set is a large finite set or an infinite set, we can describe it by listing a property necessary for memberships

- Let S be a set, the notation,

  $$A = \{x \mid x \in S, P(x)\} \text{ or } A = \{x \in S \mid P(x)\}$$

  means that A is the set of all elements x of S such that x satisfies the property P.

Example

- Let $A=\{1, 2, 3, 4, 5, 6\}$, we can also write A as,

  $A=\{x \mid x \in Z,\ 0 < x < 7\}$

  if Z denotes the set of integers.

- Let $B=\{x \mid x \in Z, x>0\}$, $B=\{1, 2, 3, 4, \ldots\}$

Example

The set of natural numbers: $\mathbb{N} = \{1, 2, 3, \ldots\}$

The set of integers: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

The set of positive integers: $\mathbb{Z}^{+} = \{1, 2, 3, \ldots\}$

The set of Rational Numbers (fractions): $\frac{1}{2}, \frac{2}{3}, \frac{5}{7}, etc \in \mathbb{Q}$

More formally: $\mathbb{Q} = \left\{ \frac{a}{b} \,\middle|\, a, b \in \mathbb{R},\ b \neq 0 \right\}$

The set of Irrational Numbers: $\sqrt{2}$, π, or e are irrational

The Real numbers $= \mathbb{R} =$ the union of the rational numbers with the irrational numbers

Some Symbols Used With Set Builder Notation

The standard form of notation for this is called "set builder notation".

For instance, $\{x \mid x \text{ is an odd positive integer}\}$ represents the set $\{1, 3, 5, 7, 9, \ldots\}$

$\{x \mid x \text{ is an odd positive integer}\}$ is read as

"the set consisting of all x such that x is an odd positive integer".

The vertical bar, " | ", stands for "such that"

Other "short-hand" notation used in working with sets

"$\forall$" stands for "for every"

"$\exists$" stands for "there exists"

"$\bigcup$" stands for "union"

"$\bigcap$" stands for "intersection"

"$\subseteq$" stands for "is a subset of"

"$\subset$" stands for "is a (proper) subset of"

"$\not\subset$" stands for "is a not a (proper) subset of"

"$\varnothing$" stands for the "empty set"

"$\in$" stands for "is an element of"

"$\notin$" stands for "is not an element of"

"$\times$" stands for "cartesian cross product"

"$=$" stands for "is equal to"

Subset

- If every element of A is an element of B, we say that A is a subset of B and write $A \subseteq B$.

$A=B$, if $A \subseteq B$ and $B \subseteq A$

- The empty set ($\varnothing$) is a subset of every set.

Example

$A = \{1, 2, 3\}$

Subset of A,

$\varnothing$, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3} {1, 2, 3}

Note:

A is a subset of A

Exercise

Answer true or false

a) $\{x\} \subseteq \{x\}$

b) $\{x\} \in \{x\}$

c) $\{x\} \in \{x,\{x\}\}$

d) $\{x\} \subseteq \{x,\{x\}\}$

Proper Subsets

If $A \subseteq B$ and B contains an element that is not in A, then we say "A is a **proper subset** of B": $A \subset B$ or $B \supset A$.

Formally: $A \subseteq B$ means $\forall x\ [x \in A \rightarrow x \in B]$.

For all sets: $A \subseteq A$.

Note: If *A* is a subset of *B* and *A* does not equal B, we say that *A* is a proper subset of *B* (*A* $\subseteq$ *B* and *A*$\neq$*B* (*B* $\not\subseteq$ *A*))

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Example

- $A=\{1, 2, 3\}$

- Proper subset of A,

  $\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}$

Note:
A proper subset of a set *A* is a subset of *A* that is not equal to *A* ($\{1,2,3\} \not\subset A$)

Example

Let a set, B={1, 2, 3, 4, 5, 6} and the subset, A = {1, 2, 3}.

$\Rightarrow$ A is proper subset of B.

Example

$A=\{a,b,c,d,e,f,g,h\}$

$B=\{b,d,e\}$

$C=\{a,b,c,d,e\}$

$D=\{r,s,d,e\}$

- Proper subset of *A* ?? *B* and *C*

Empty Sets

The **empty set** $\varnothing$ or {} but not $\{\varnothing\}$
is the set without elements.

Note:

- Empty set has no elements
- Empty set is a subset of any set
- There is exactly one empty set
- Properties of empty set:
  $A \cup \varnothing = A$, $A \cap \varnothing = \varnothing$
  $A \cap A' = \varnothing$, $A \cup A' = U$
  $U' = \varnothing$, $\varnothing' = U$

Example

$\varnothing = \{x \mid x$ is a real number and $x^2 = -3\}$

$\varnothing = \{x \mid x$ is positive integer and $x^3 < 0\}$

Equal Sets

The sets A and B are **equal** (A=B) if and only if each element of A is an element of B and vice versa.

Formally: A=B means $\forall x\, [x \in A \leftrightarrow x \in B]$.

Example

$A=\{a, b, c\}$, $B=\{b, c, a\}$, $A=B$

$C=\{1, 2, 3, 4\}$

$D=\{x \mid x$ is a positive integer and $2x < 10\}$,

$C=D$

Exercise

Determine whether each pair of sets is equal

a) {1, 2, 2, 3}, {1, 3, 2}

b) $\{ x \mid x^2 + x = 2\}$, {1, -2}

c) $\{x \mid x$ is a real number and $0 < x \leq 2\}$, {1,2}

Equivalent Sets

Two sets, A and B, are equivalent if there exists a one-to-one correspondence between them.

When we say sets "have the same size", we mean that they are equivalent.

Example

Set A: {A, B, C, D, E} and **Set** B: {1, 2, 3, 4, 5}

Note:

- An **equivalent set** is simply a **set** with an **equal** number of elements.
- The **sets** do not have to have the same exact elements, just the same number of elements.

Finite Sets

A set A is **finite**

if it is empty

or

if there is a natural number n such that set A is equivalent to $\{1, 2, 3, \ldots n\}$.

Example

$A = \{1, 2, 3, 4\}$

$B = \{x \mid x \text{ is an integer}, 1 \leq x \leq 4\}$

Note:

There exists a nonnegative integer n such that *A* has *n* elements (*A* is called a finite set with *n* elements)

Infinite Sets

A set *A* is infinite

if there is NOT a natural number *n* such that set *A* is equivalent to {1, 2, 3, . . . *n*}.

Infinite sets are uncountable.

Are all infinite sets equivalent?

A set is infinite if it is equivalent to a proper subset of itself!

Example

$Z = \{x \mid x \text{ is an integer}\}$

or $Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

$S = \{x \mid x \text{ is a real number and } 1 \leq x \leq 4\}$

Example

$C = \{5, 6, 7, 8, 9, 10\}$ ***(finite set)***

$B = \{x \mid x \text{ is an integer}, 10 < x < 20\}$ ***(finite set)***

$D = \{x \mid x \text{ is an integer}, x > 0\}$ ***(infinite set)***

Universal Set

- Sometimes we are dealing with sets all of which are subsets of a set *U*.
- This set *U* is called a universal set or a universe.
- The set *U* must be explicitly given or inferred from the context.

Universal Set

Typically we consider a set A
a part of a **universal set** $\mathcal{U}$,
which consists of all possible elements.
To be entirely correct we should say

$$\forall x \in \mathcal{U}\ [x \in A \leftrightarrow x \in B]$$

instead of

$$\forall x\ [x \in A \leftrightarrow x \in B] \text{ for } A=B.$$

Note that $\{ x \mid 0<x<5\}$ is can be ambiguous.
Compare $\{ x \mid 0<x<5,\ x \in N \}$ with $\{ x \mid 0<x<5,\ x \in Q \}$

Example

- The sets $A=\{1,2,3\}$, $B=\{2,4,6,8\}$ and $C=\{5,7\}$
- One may choose $U=\{1,2,3,4,5,6,7,8\}$ as a universal set.
- Any superset of U can also be considered a universal set for these sets A, B, and C.

  For example, $U=\{x \mid x \text{ is a positive integer}\}$

Cardinality of Set

- Let S be a finite set with n distinct elements, where $n \geq 0$.
- Then we write $|S|=n$ and say that the cardinality (or the number of elements) of S is n.
- **Example**

  $A = \{1, 2, 3\}$, $|A|=3$

  $B = \{a,b,c,d,e,f,g\}$, $|B|=7$

Exercise

- If *M* is finite, determine the $|M|$
  - If $M = \{1, 2, 3, 4\}$
  - If $M = \{4, 4, 4\}$
  - If $M = \{\}$
  - If $M = \{ø,\{ø\}, \{ø, \{ø\}\}\}$

Power Set

- The set of all subsets of a set A, denoted $P(A)$, is called the power set of A.

$$P(A) = \{X \mid X \subseteq A\}$$

- If $|A| = n$, then $|P(A)| = 2^n$

Example

- $A=\{1,2,3\}$

- The power set of A,
  $P(A)=\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

- Notice that $|A| = 3$, and $|P(A)| = 2^3 = 8$

Exercise

- Let $X = \{1, 2, 2, \{1\}, a\}$
- Find:
  - $|X|$
  - Proper subset of X
  - Power set of X

Exercise

- List the member of *P*({a, b, c, d}). Which are proper subset of {a, b, c, d}?

Summary

How to Think of Sets

The elements of a set do not have an ordering,
hence {a,b,c} = {b,c,a}

The elements of a set do not have multitudes,
hence {a,a,a} = {a,a} = {a}

All that matters is: "Is x an element of A or not?"

The size of A is thus the number of *different* elements