CHAPTER 1

Quantifiers & Proof Technique

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QUANTIFIERS

- Most of the statements in mathematics and computer science are not described properly by the propositions.
- Since most of the statements in mathematics and computer science use **variables**, the system of logic must be extended to include statements with the variables.

QUANTIFIERS (cont.)

- Let *P*(x) is a statement with variable x and **A** is a set.
- *P* a ***propositional function*** or also known as ***predicate*** if for each x in **A**, *P*(x) is a proposition.
- Set **A** is the ***domain of discourse*** of *P*.
- Domain of discourse -> the particular domain of the variable in a propositional function.

QUANTIFIERS (cont.)

- A predicate is a statement that contains variables.

- Example:

  $P(x) : x > 3$

  $Q(x,y) : x = y + 3$

  $R(x,y,z) : x + y = z$

Example

- $x^2 + 4x$ is an odd integer (domain of discourse is set of positive numbers).
- $x^2 - x - 6 = 0$ (domain of discourse is set of real numbers).
- UTM is rated as Research University in Malaysia (domain of discourse is set of research university in Malaysia).

QUANTIFIERS (cont.)

- A predicate becomes a proposition if the variable(s) contained is(are)
  - **Assigned specific value(s)**
  - **Quantified**
- $P(x) : x > 3$.

What are the truth values of $P(4)$ and $P(2)$?

- $Q(x,y) : x = y + 3$.

What are the truth values of $Q(1,2)$ and $Q(3,0)$?

QUANTIFIERS (cont.)

- Two types of quantifiers:
  - **Universal**
  - **Existential**

QUANTIFIERS (cont.)

- Let *A* be a propositional function with domain of discourse *B*. The statement

  for every x, *A*(x)

  is ***universally quantified statement***
- Symbol ∀ called a ***universal quantifier*** is used *"for every"*.
- Can be read as *"for all"*, *"for any"*.

QUANTIFIERS (cont.)

- The statement can be written as

$$\forall x \, A(x)$$

- Above statement is ***true*** if $A(x)$ is *true* for every x in B (*false* if $A(x)$ is ***false*** for at least one x in B).

- A value x in the domain of discourse that makes the statement $A(x)$ *false* is called a ***counterexample*** to the statement.

Example

- Let the universally quantified statement is

$$\forall x\ (x^2 \geq 0)$$

- Domain of discourse is the set of real numbers.
- This statement is *true* because *for every* real number x, it is true that the square of x is positive or zero.

Example

- Let the universally quantified statement is

$$\forall x\ (x^2 \leq 9)$$

- Domain of discourse is a set B = {1, 2, 3, 4}
- When x = 4, the statement produce *false* value.
- Thus, the above statement is *false* and the counterexample is 4.

QUANTIFIERS (cont.)

- Easy to prove a universally quantified statement is true or false if the domain of discourse is not too large.
- What happen if the domain of discourse contains a large number of elements?
- For example, a set of integer from 1 to 100, the set of positive integers, the set of real numbers or a set of students in Faculty of Computing. It will be hard to show that every element in the set is *true*.

Use existential quantifier!!

QUANTIFIERS (cont.)

- Let *A* be a propositional function with domain of discourse *B*. The statement

  There exist x, $A(x)$

  is ***existentially quantified statement***
- Symbol $\exists$ called an ***existential quantifier*** is used "*there exist*".
- Can be read as "*for some*", "*for at least one*".

QUANTIFIERS (cont.)

- The statement can be written as

$$\exists x\, A(x)$$

- Above statement is *true* if $A(x)$ is *true* for at least one x in B (*false* if every x in B makes the statement $A(x)$ *false*).

- Just find one x that makes $A(x)$ true!

Example

- Let the existentially quantified statement is

$$\exists x\left(\frac{x}{x^2+1}=\frac{2}{5}\right)$$

- Domain of discourse is the set of real numbers.
- Statement is *true* because it is possible to find at least one real number x to make the proposition *true*.

- For example, if $x = 2$, we obtain the true proposition as below

$$\left(\frac{x}{x^2+1} = \frac{2}{5}\right) = \left(\frac{2}{2^2+1} = \frac{2}{5}\right)$$

Negation of Quantifiers

- Distributing a negation operator across a quantifier changes a universal to an existential and vice versa.

$$\neg (\forall x\, P(x)) ;\ \exists x \neg P(x)$$

$$\neg (\exists x\, P(x)) ;\ \forall x \neg P(x)$$

Example

- Let $P(x)$ = x is taking Discrete Structure course with the domain of discourse is the set of all students.
  - $\forall x\, P(x)$: All students are taking Discrete Structure course.
  - $\exists x\, P(x)$: There is some students who are taking Discrete Structure course.
  - $\neg \exists x\, P(x)$: No anyone of the students who are taking Discrete Structure course.
  - $\forall x \neg P(x)$: All students are not taking Discrete Structure course.

- $\neg \forall x\, P(x)$: Not All students are taking Discrete Structure course.

- $\exists x\, \neg P(x)$: There is some students who are not taking Discrete Structure course

PROOF TECHNIQUES

- **Mathematical systems consists:**
  - Axioms: assumed to be true.
  - Definitions: used to create new concepts.
  - Undefined terms: some terms that are not explicitly defined.
  - Theorem
- **Theorem**
  - Statement that can be shown to be true (under certain conditions)
  - Typically stated in one of three ways:
    - As Facts
    - As Implications
    - As Bi-implications

PROOF TECHNIQUES (cont.)

Direct Proof or Proof by Direct Method

- Proof of those theorems that can be expressed in the form $\forall x\ (P(x) \rightarrow Q(x))$, *D* is the domain of discourse.
- Select a particular, but arbitrarily chosen, member a of the domain *D.*
- Show that the statement $P(a) \rightarrow Q(a)$ is true. (Assume that $P(a)$ is true).
- Show that $Q(a)$ is true.
- By the rule of Universal Generalization (UG),

  $\forall x\ (P(x) \rightarrow Q(x))$ is true.

Example

- For all integer x, if x is odd, then x^2 is odd
- Or
  - *P(x) = is an odd integer*
  - *Q(x) = x^2 is an odd integer*
- $\forall x(P(x) \rightarrow Q(x))$, the domain of discourse is set *Z* of all integer.

- Can verify the theorem for certain value of x.

  $x=3$, $x^2=9$; odd

Example (cont.)

- a is an odd integer

$$\Rightarrow a = 2n + 1$$ → *for some integer **n***

$$\Rightarrow a^2 = (2n + 1)^2$$

$$\Rightarrow a^2 = 4n^2 + 4n + 1$$

$$\Rightarrow a^2 = 2(2n^2 + 2n) + 1$$

$$\Rightarrow a^2 = 2m + 1$$ → *where **m= 2n²** + 2n is an integer*

$$\Rightarrow a^2$$ → *is an odd integer*

PROOF TECHNIQUES (cont.)

- **Indirect Proof :**
  - The implication $p \rightarrow q$ is equivalent to the implication $(\sim q \rightarrow \sim p)$
  - Therefore, in order to show that $p \rightarrow q$ is true, one can also show that the implication $(\sim q \rightarrow \sim p)$ is true.
  - To show that $(\sim q \rightarrow \sim p)$ is true, assume that the negation of q is true and prove that the negation of p is true.

Example

P(n) : n^2+3 is an odd number

Q(n) : *n* is even number

$$\forall n(P(n) \rightarrow Q(n))$$

$$P(n) \rightarrow Q(n) \equiv \sim Q(n) \rightarrow \sim P(n)$$

- *~Q(n)* is true , *n* is not even, *n* is odd so *n=2k+1*

$$\begin{aligned} n^2 + 3 &= (2k+1)^2 + 3 \\ &= 4k^2 + 4k + 1 + 3 \\ &= 4k^2 + 4k + 4 \\ &= 2(2k^2 + 2k + 2) \end{aligned}$$

Example (cont.)

$$t = 2k^2 + 2k + 2$$ → *t is integer*

$$n^2 + 3 = 2t$$

n^2+3 is an even integer, thus *~P(n)* is true

PROOF TECHNIQUES (cont.)

Proof by Contradiction

Assume that the conclusion is not true and then arrive at a contradiction.

Example

Prove that there are infinitely many prime numbers.

Proof:

- Assume there are not infinitely many prime numbers, therefore they are can be listed, i.e. $p_1, p_2, \ldots, p_n$
- Consider the number $q = p_1 \times p_2 \times \ldots \times p_n + 1$.
- q is either prime or not divisible, but not listed above. Therefore, q is a prime. However, it was not listed.
- Contradiction! Therefore, there are infinitely many primes numbers.