SCSI 1013: Discrete Structure

CHAPTER 4

GRAPH THEORY

Part 3

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Shortest Path Problem

Shortest Path

- Let ***G*** be a weighted graph.
- Let ***u*** and ***v*** be two vertices in ***G***, and let ***P*** be a path in ***G*** from ***u*** to ***v***.
- The length of path ***P***, written ***L*(*P*)**, is the sum of the weights of all the edges on path ***P***.
- A **shortest path** from a vertex to another vertex is a path with the shortest length between the vertices.

Dijkstra's Shortest Path Algorithm

- 1. $S := \varnothing$
- 2. $N := V$
- 3. For all vertices, $u \in V$, $u \neq a$, $L(u) := \infty$
- 4. $L(a) := 0$

http://en.wikipedia.org/wiki/Dijkstra%27s_algorithm

- 5. While $z \notin S$ do,

  5.a :Let $v \in N$ be such that

  $L(v)=\min\{L(u) \mid u \in N\}$

  5.b : $S := S \in \{v\}$

  5.c : $N := N - \{v\}$

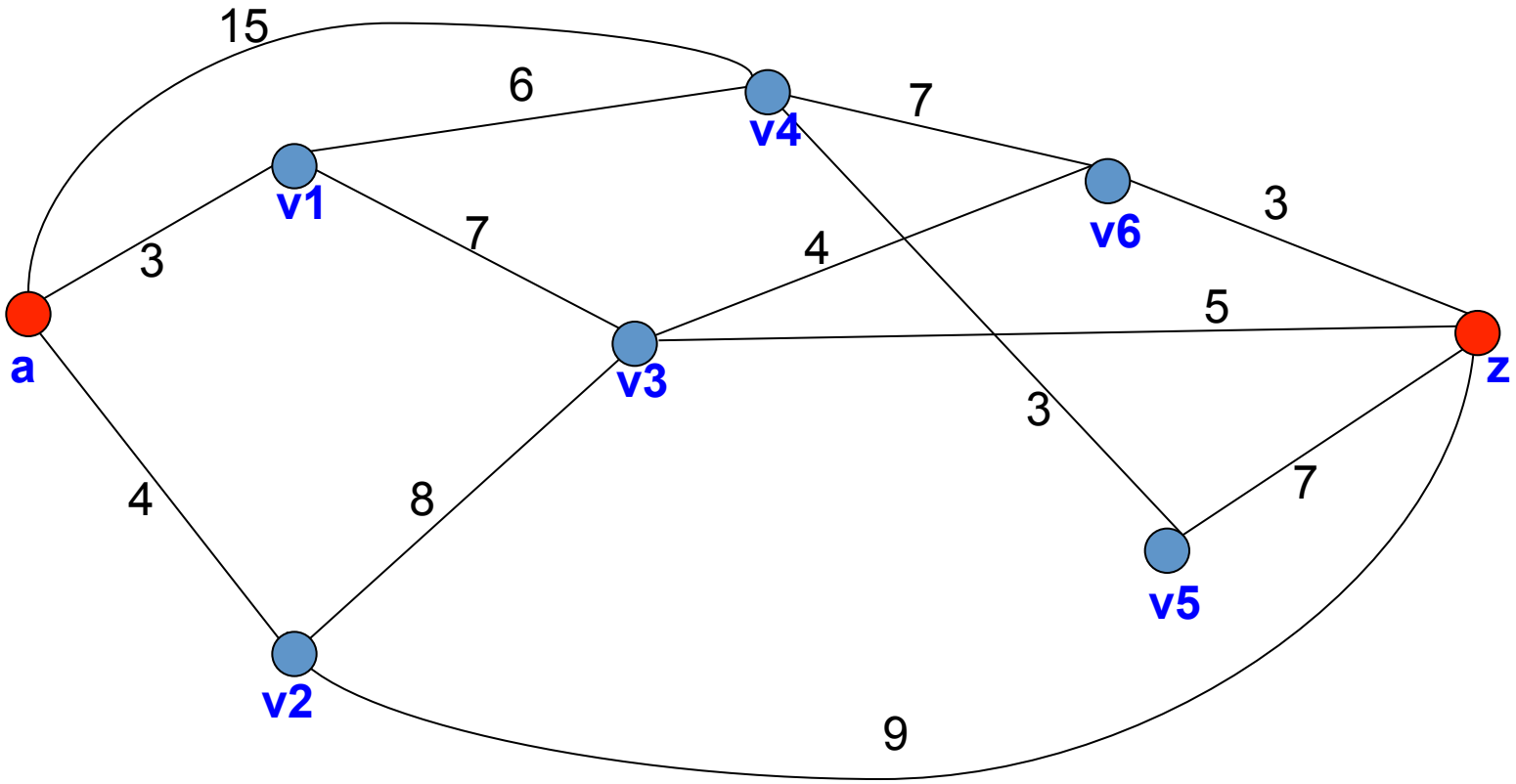
  5.d : For all $w \in N$ such that there is an edge from v to w

  5.d.1: If $L(v)+W[v,w] < L(w)$ then

  $L(w)=L(v)+W[v,w]$

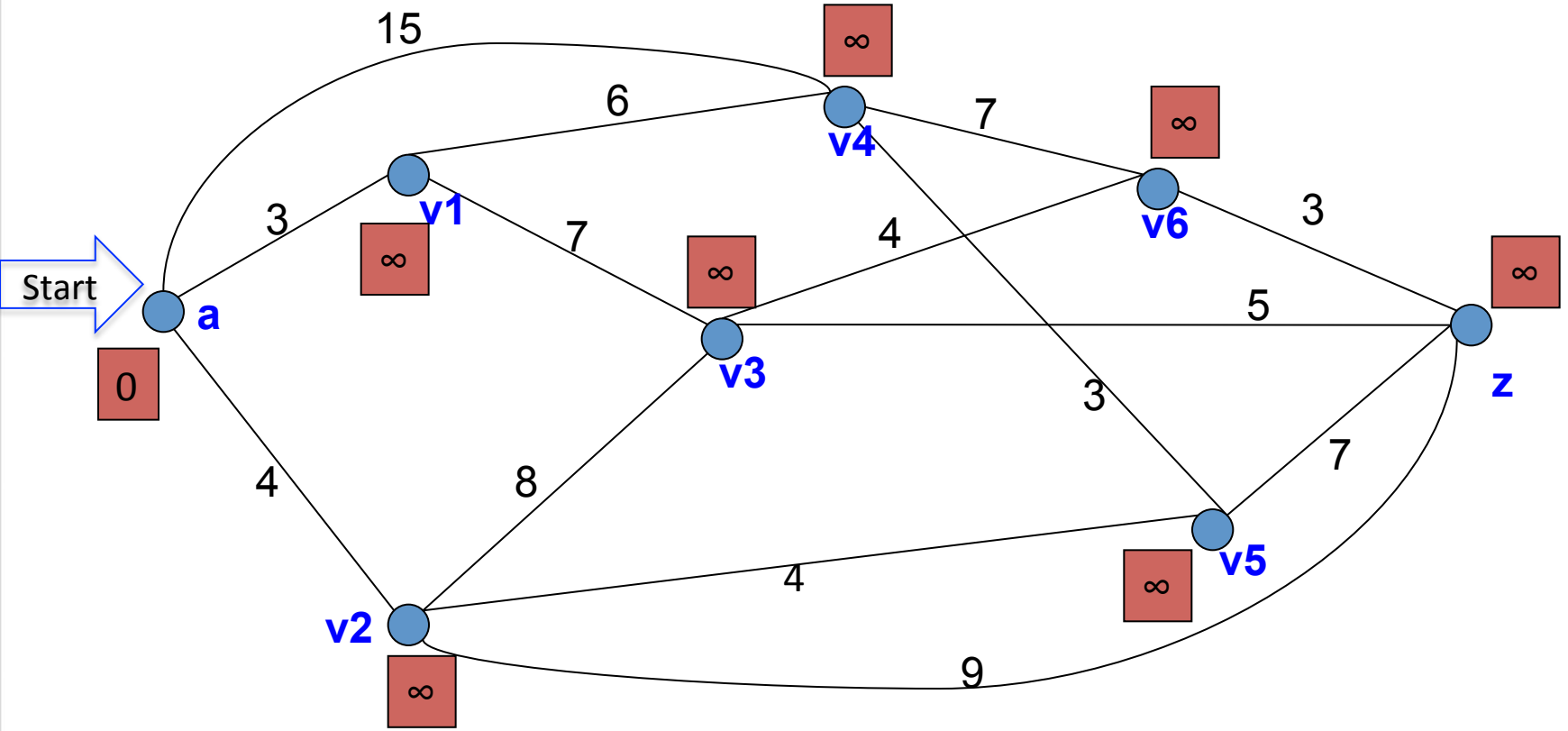
Example

What is the shortest path from **a** to **z**?



$$S = \varnothing$$

$$N = \{a, v1, v2, v3, v4, v5, v6, z\}$$



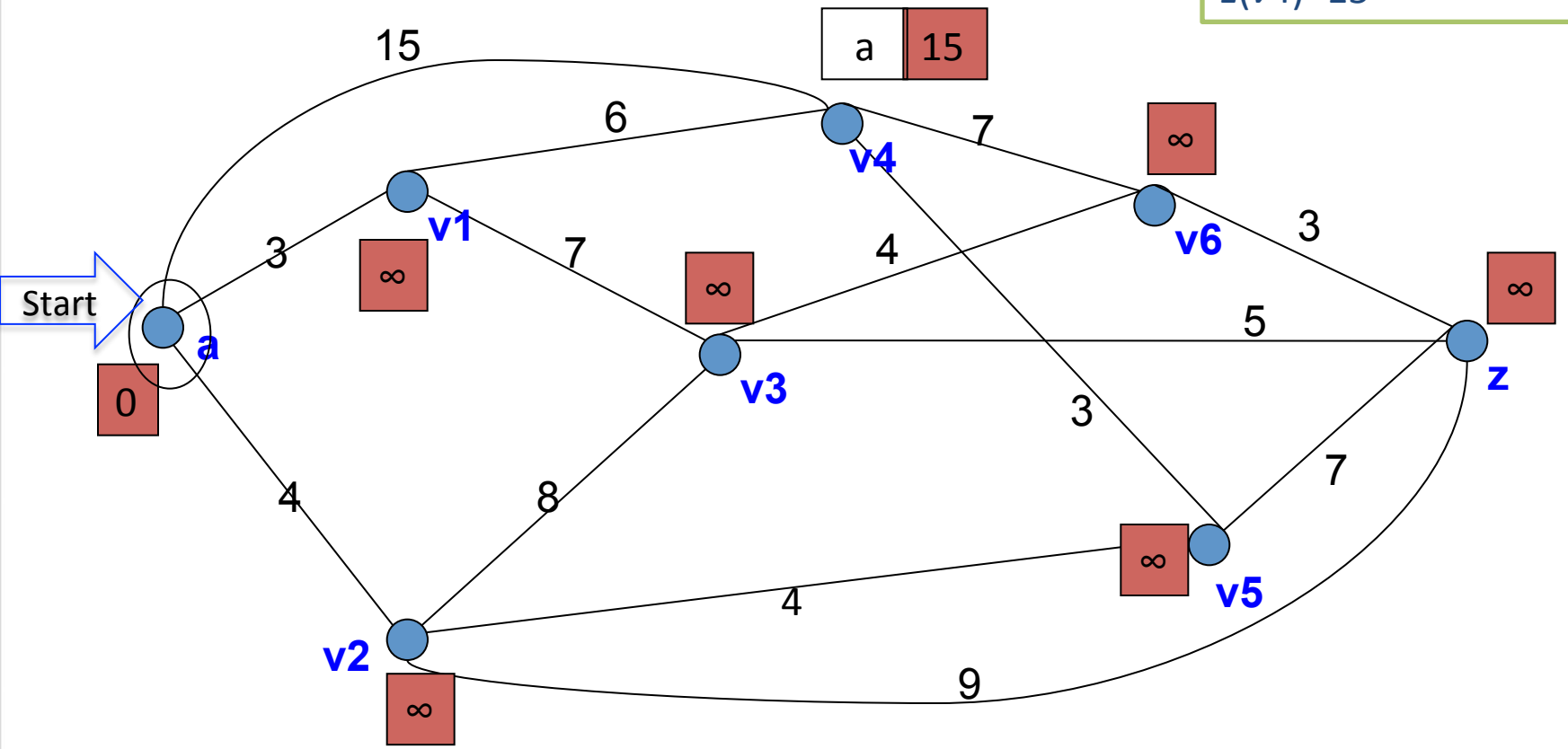
Djikstra Algorithm: Iterations Table

Iteration	S	N	L(a)	L(v1)	L(v2)	L(v3)	L(v4)	L(v5)	L(v6)	L(z)
0	{ }	{a, v1, v2, v3, v4, v5, v6, z}	0	∞	∞	∞	∞	∞	∞	∞

$$S = \{a\}$$

$$N = \{v1, v2, v3, v4, v5, v6, z\}$$

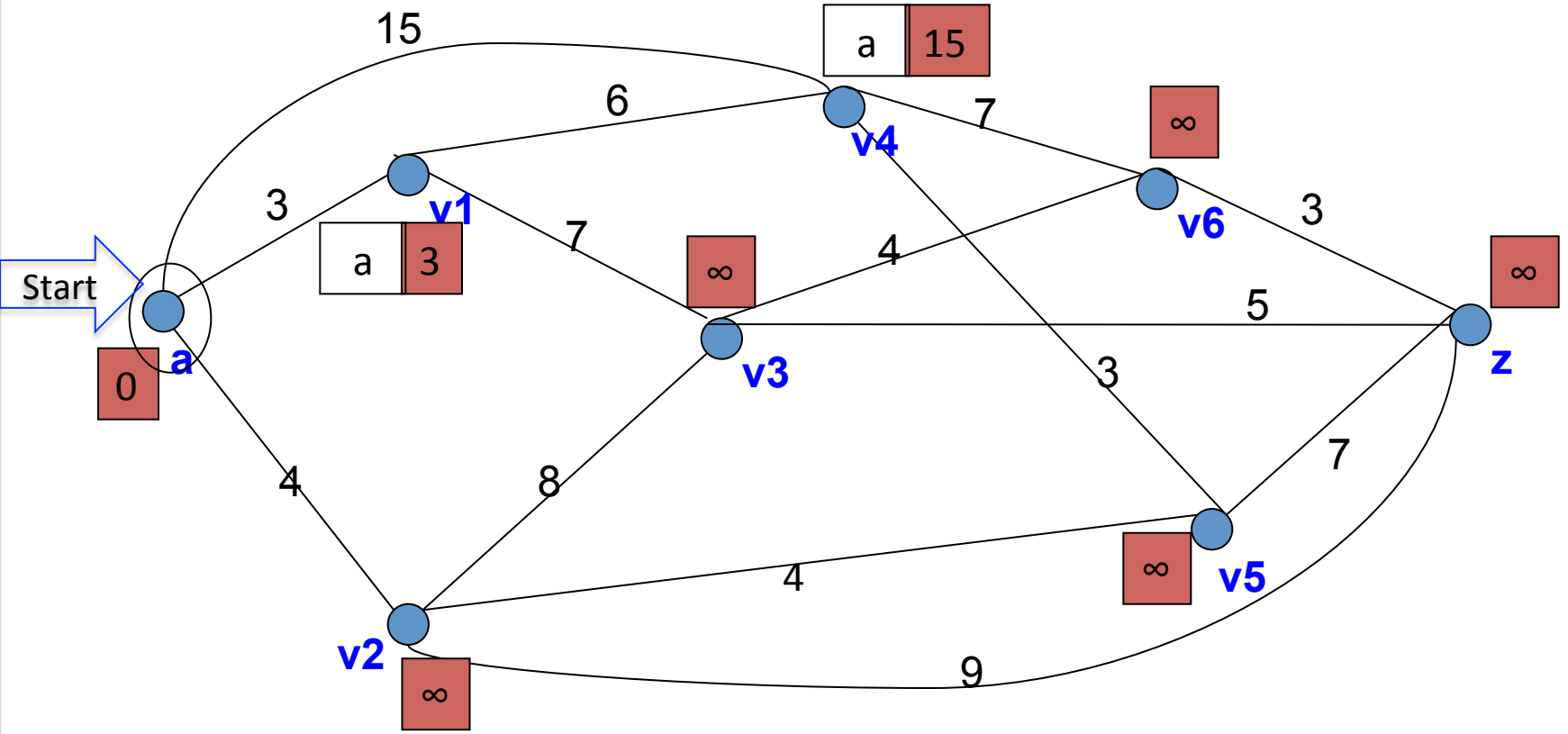
$L(a)+W[a,v4] < L(v4)$
$0+15=15 < \infty$
$L(v4)=15$



$S = \{a\}$

$N = \{v1, v2, v3, v4, v5, v6, z\}$

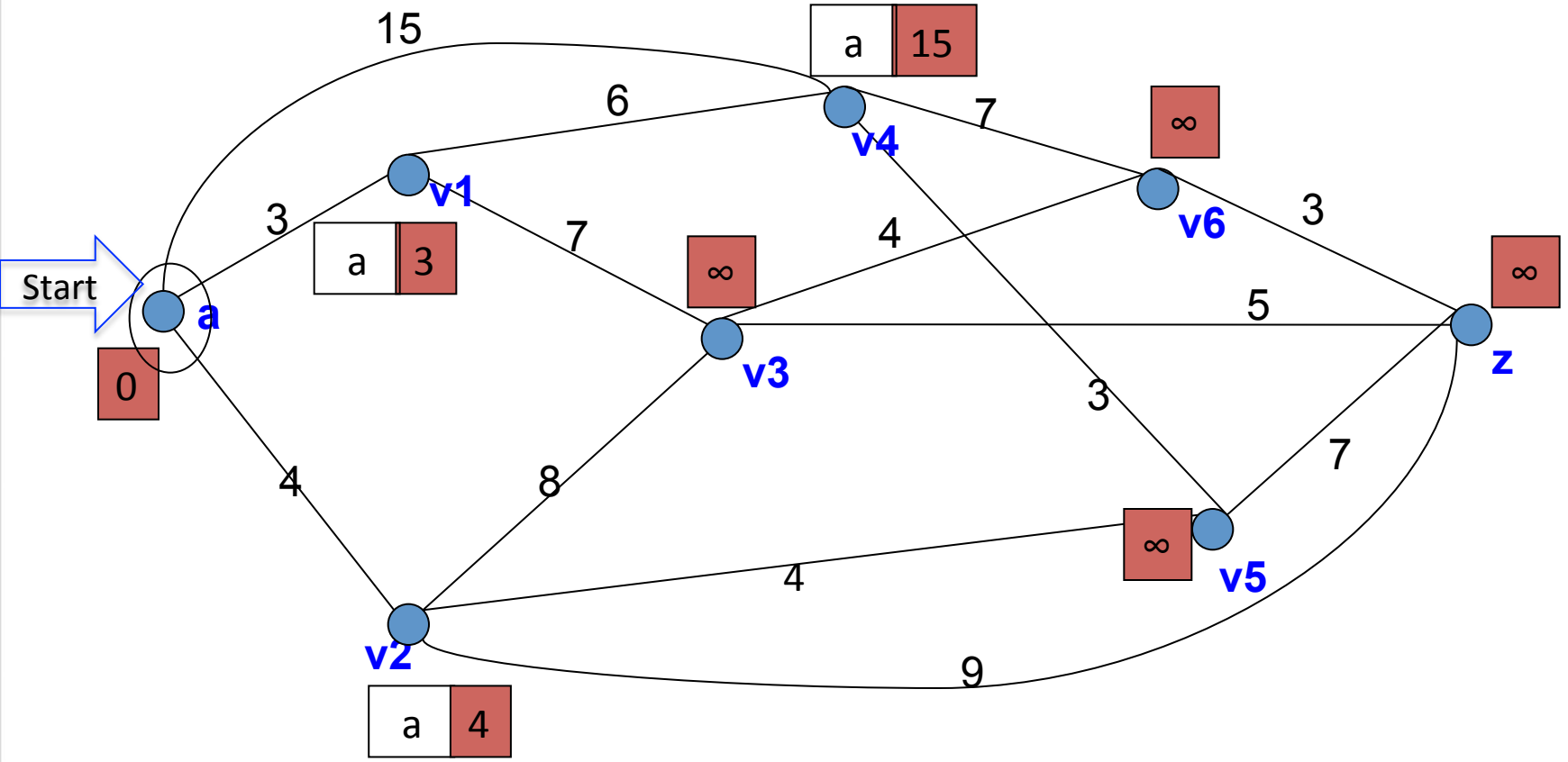
$L(a)+W[a,v1] < L(v1)$
$0+3= 3 < \infty$
$L(v1)=3$



$$S = \{a\}$$

$$N = \{v1, v2, v3, v4, v5, v6, z\}$$

$L(a)+W[a,v2] < L(v2)$
$0+4 = 4 < \infty$
$L(v2)=4$

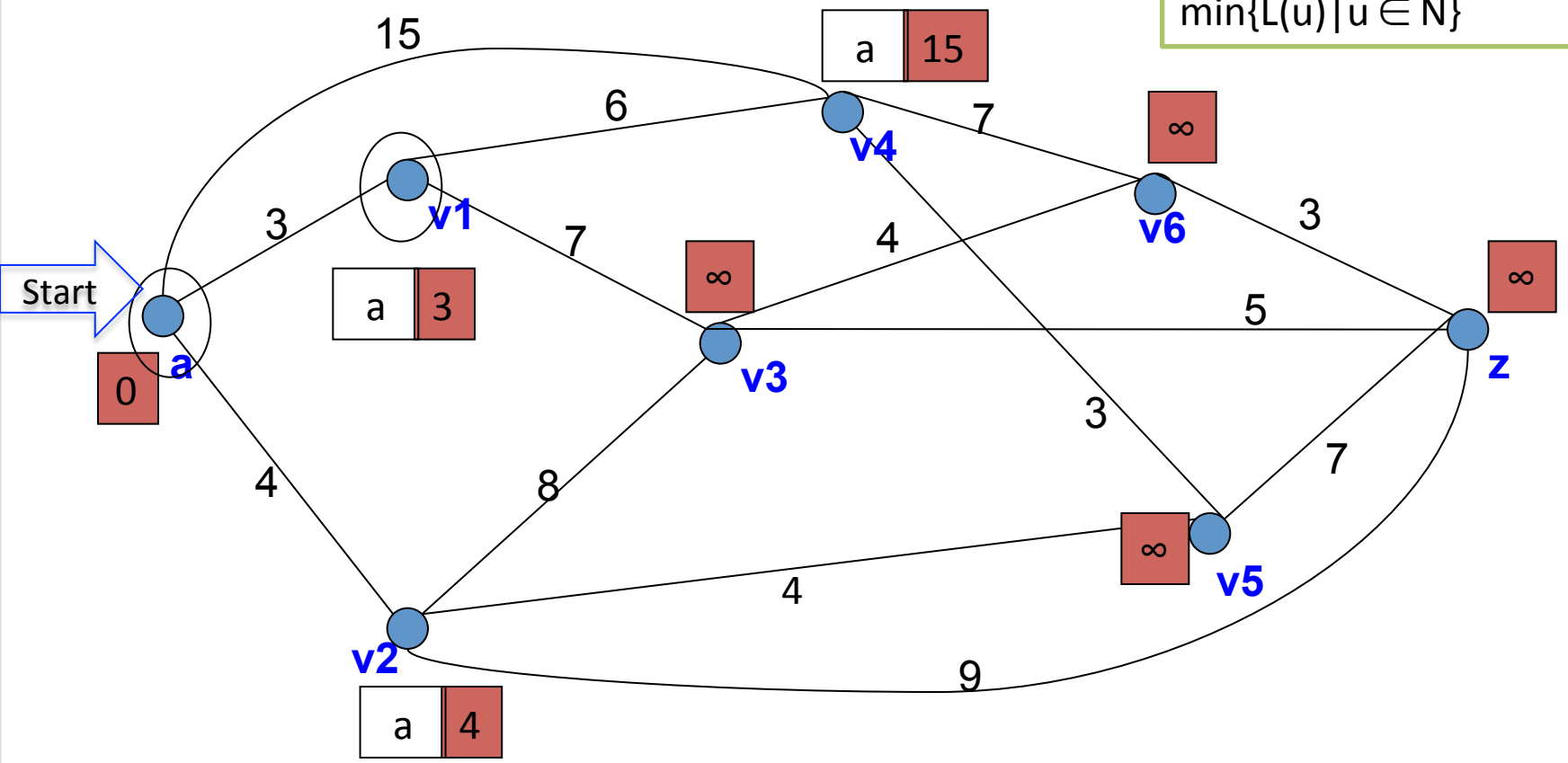


Iteration	*S*	*N*	*L*(a)	*L*(v1)	*L*(v2)	*L*(v3)	*L*(v4)	*L*(v5)	*L*(v6)	*L*(z)
0	{ }	{a, v1, v2, v3, v4, v5, v6, z}	0	∞	∞	∞	∞	∞	∞	∞
1	{a }	{v1, v2, v3, v4, v5, v6, z}	0	3	4	∞	15	∞	∞	∞

$$S = \{a\}$$

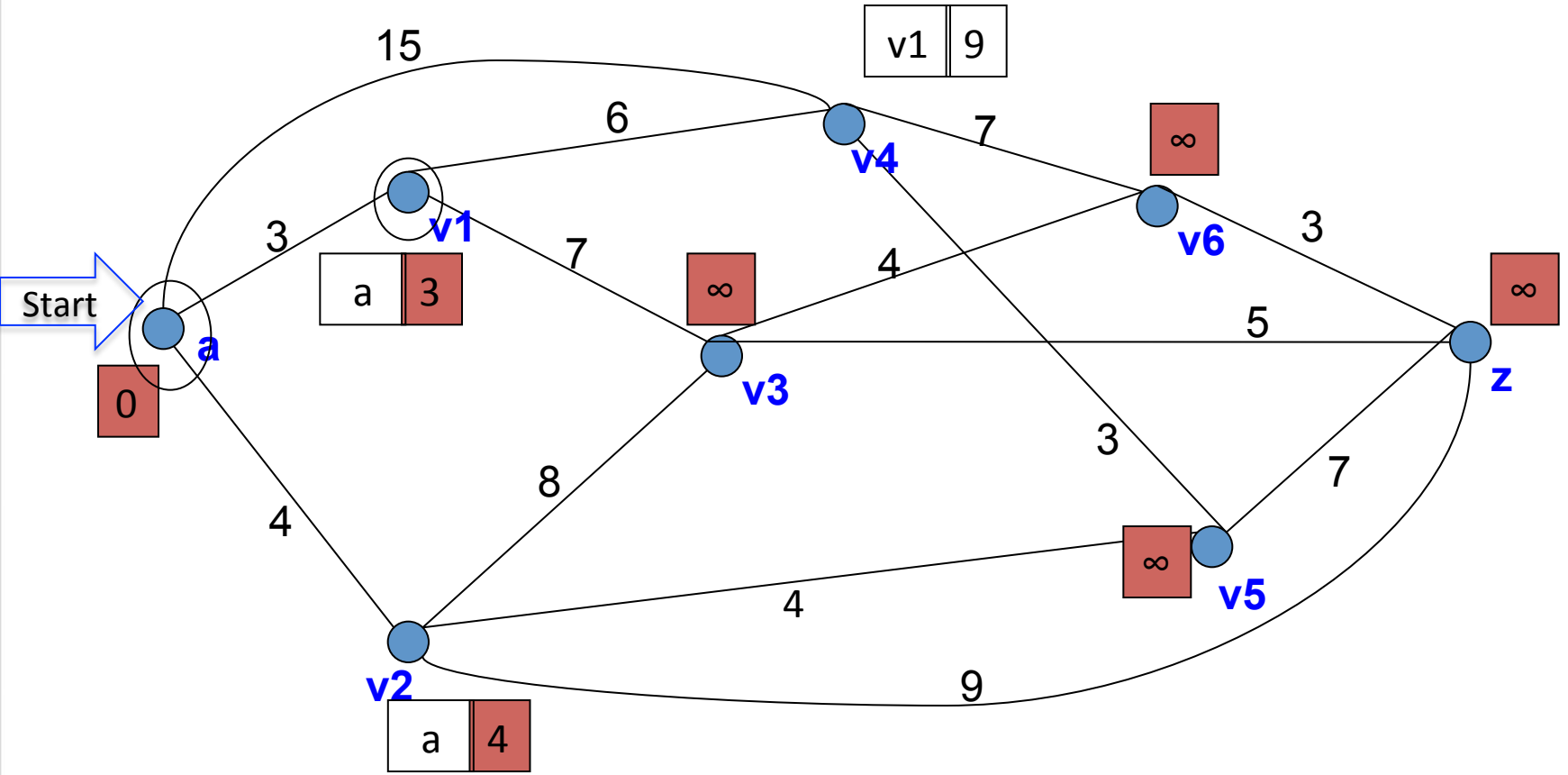
$$N = \{v1, v2, v3, v4, v5, v6, z\}$$

choose *v*1
because
$L(v1)=3 =$
$\min\{L(u) \mid u \in N\}$



$$S = \{a, v1\}$$
$$N = \{v2, v3, v4, v5, v6, z\}$$

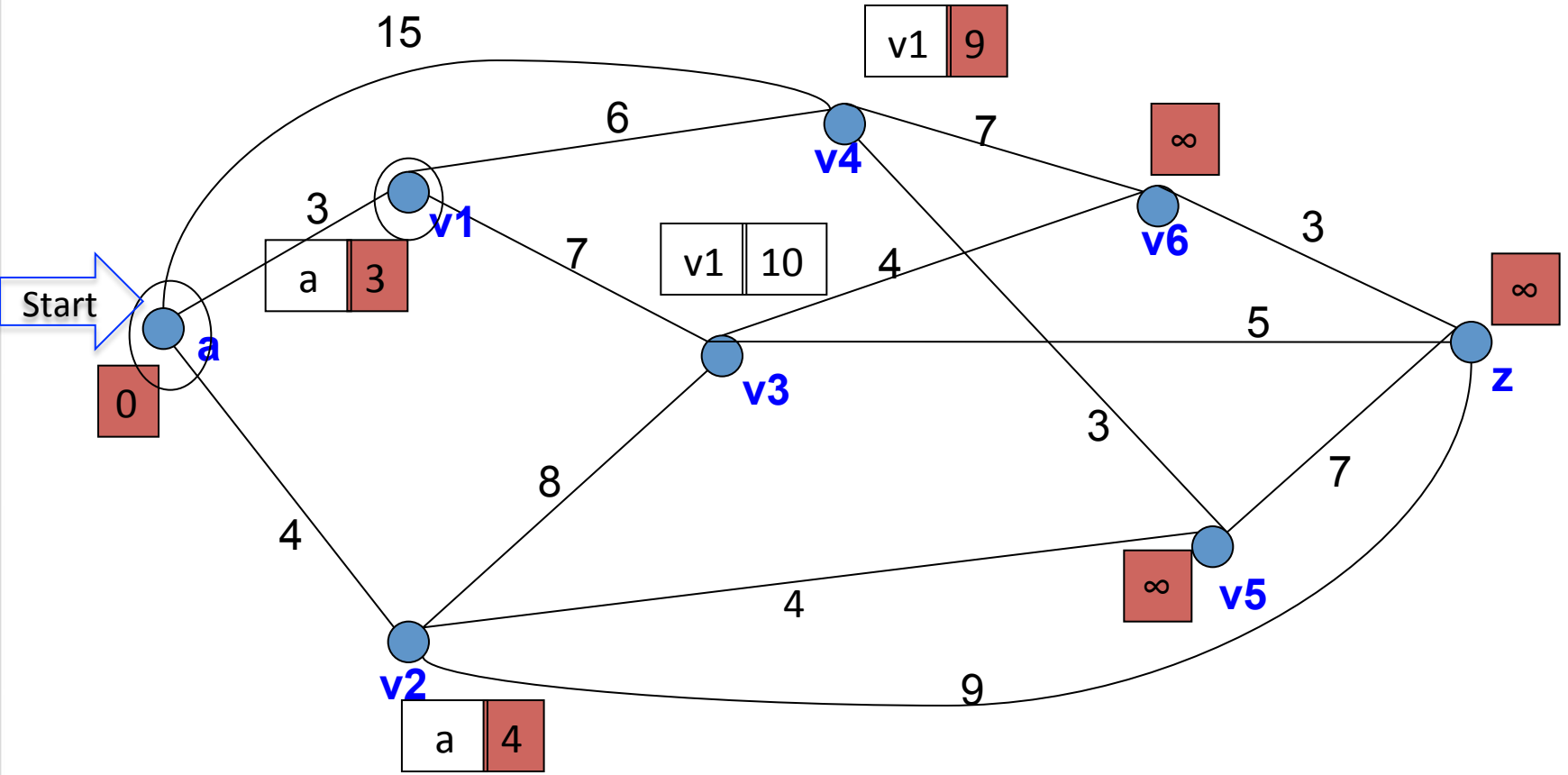
$L(v1)+W[v1,v4]<L(v4)$
$3+6= 9 < 15$
$L(v4)=9$



$S = \{a, v1\}$

$N = \{v2, v3, v4, v5, v6, z\}$

$L(v1)+W[v1,v3]<L(v3)$
$3+7= 10 < \infty$
$L(v4)=10$

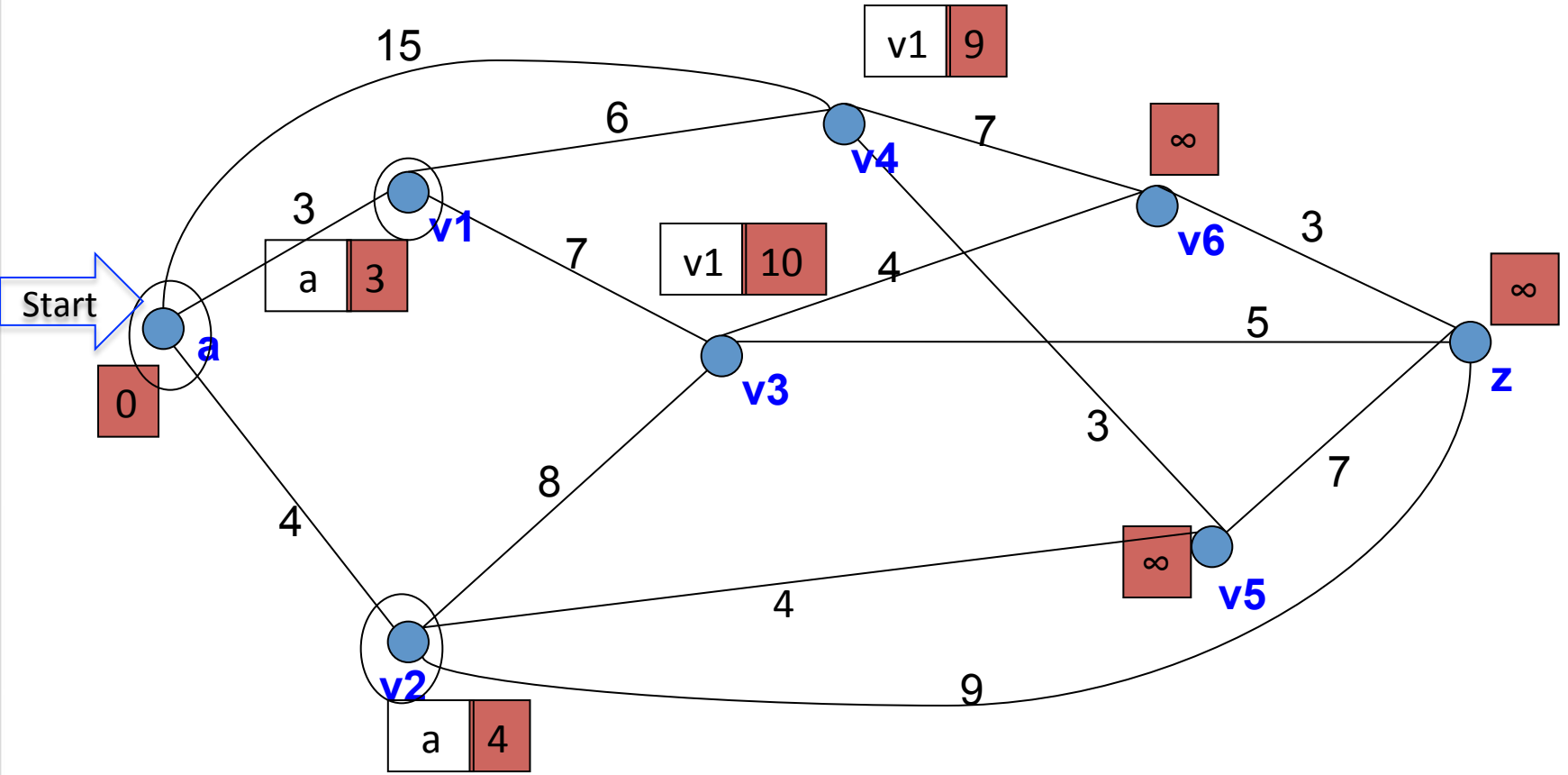


Iteration	*S*	*N*	*L*(a)	*L*(v1)	*L*(v2)	*L*(v3)	*L*(v4)	*L*(v5)	*L*(v6)	*L*(z)
0	{ }	{a, v1, v2, v3, v4, v5, v6, z}	0	∞	∞	∞	∞	∞	∞	∞
1	{a }	{v1, v2, v3, v4, v5, v6, z}	0	3	4	∞	15	∞	∞	∞
2	{a , v1}	{v2, v3, v4, v5, v6, z}	0	3	4	10	9	∞	∞	∞

$$S = \{a, v1\}$$

$$N = \{v2, v3, v4, v5, v6, z\}$$

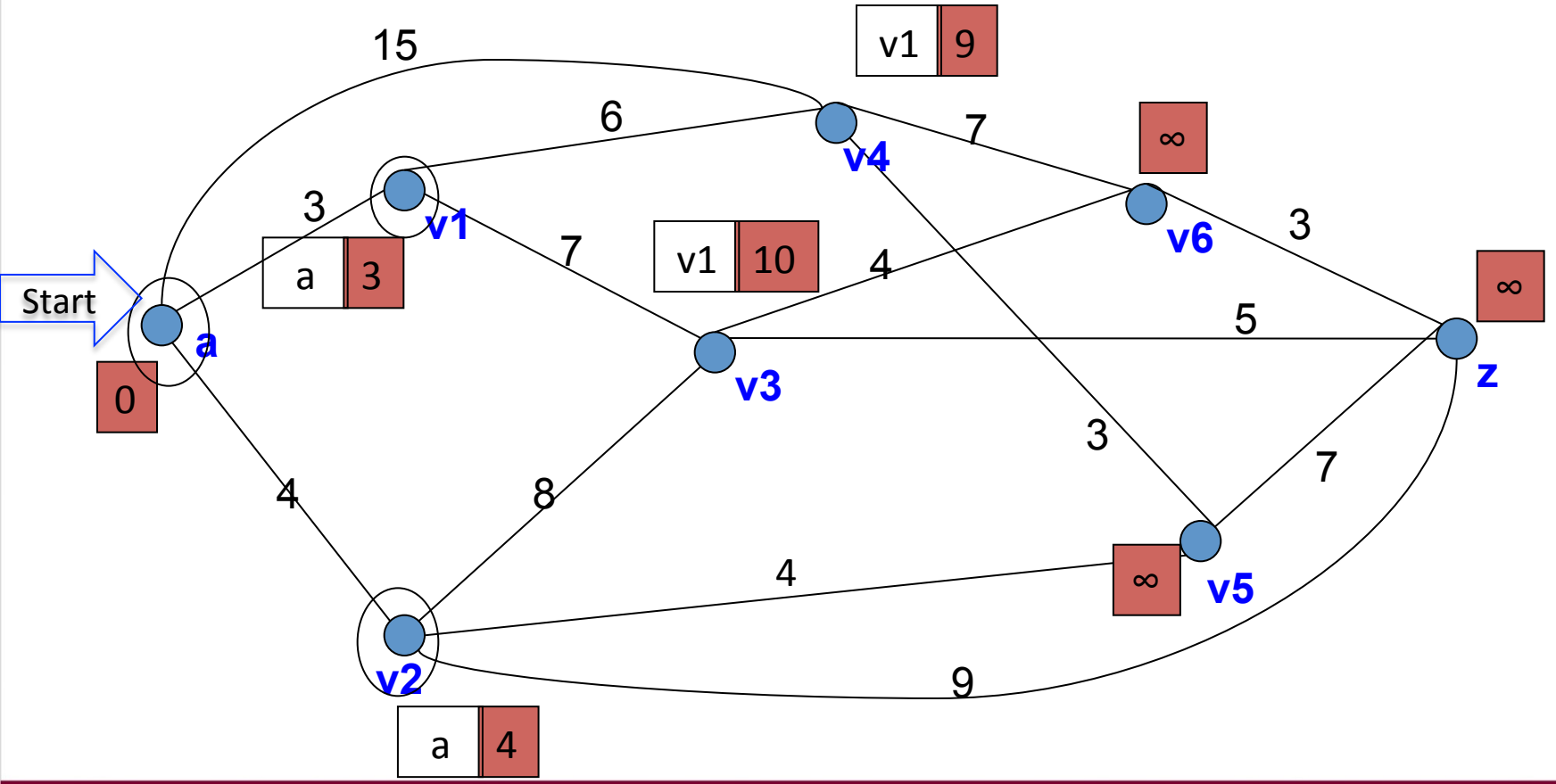
choose *v*2
because
$L(v2)=4 = \min\{L(u) \mid u \in N\}$



$S=\{a, v1, v2\}$

$N=\{v3, v4, v5, v6, z\}$

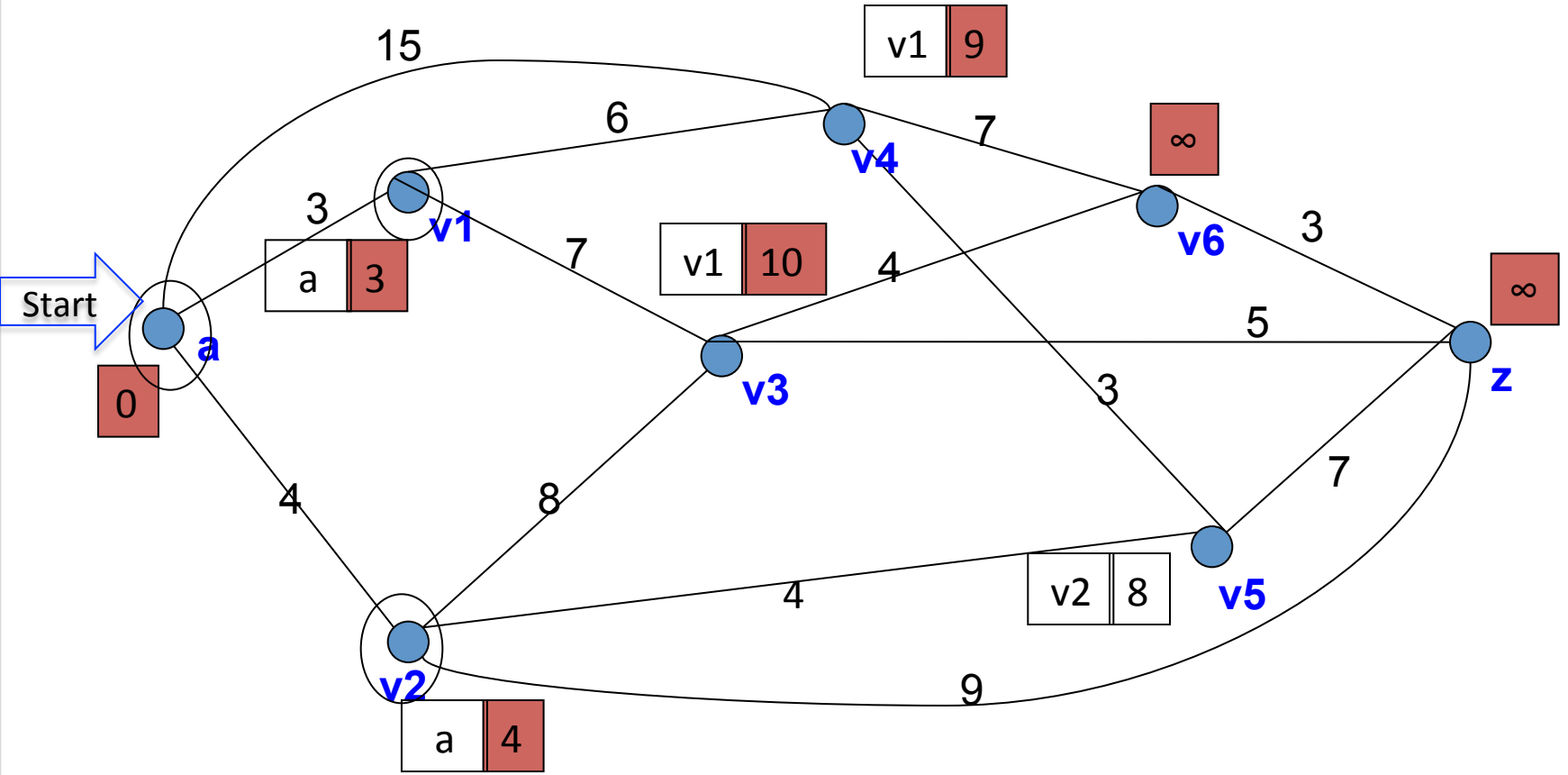
$L(v2)+W[v2,v3]<L(v3)$
$4+8=12>10$
$L(v3)$ remains the same.



$S = \{a, v1, v2\}$

$N = \{v3, v4, v5, v6, z\}$

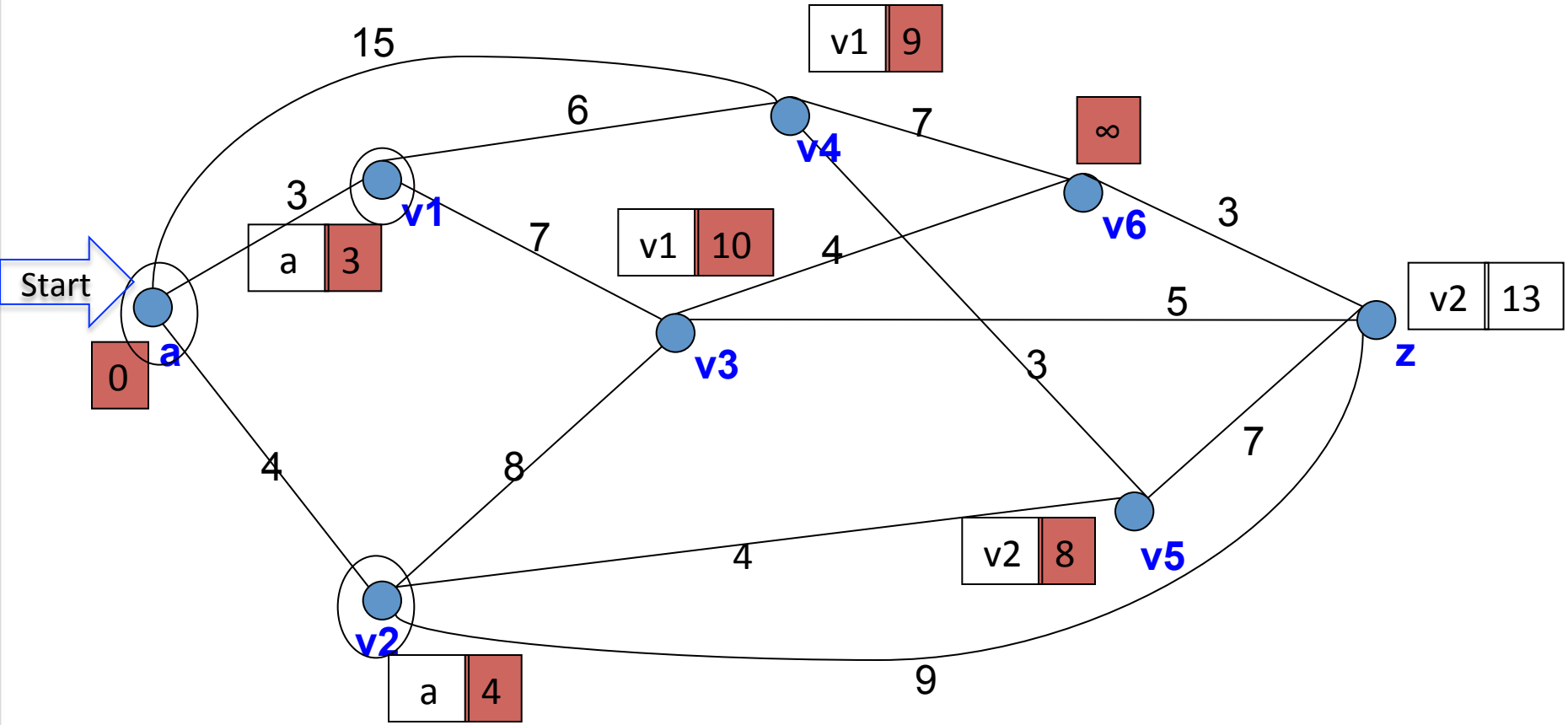
$L(v2)+W[v2,v5]<L(v5)$
$4+4= 8 < \infty$
$L(v5)=8$



$S = \{a, v1, v2\}$

$N = \{v3, v4, v5, v6, z\}$

$L(v2)+W[v2, z]<L(z)$
$4+9= 13 < \infty$
$L(z)=13$

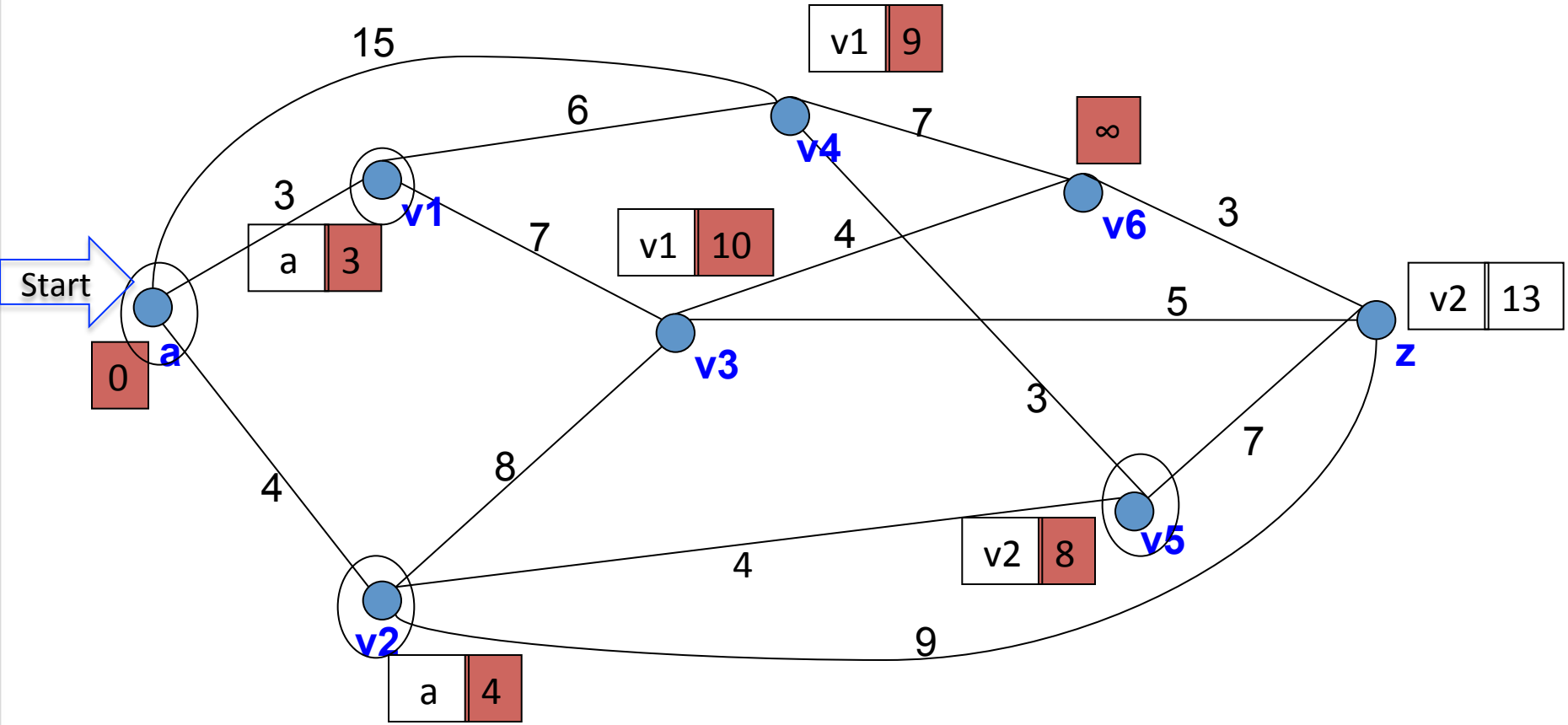


Iteration	S	N	L(a)	L(v1)	L(v2)	L(v3)	L(v4)	L(v5)	L(v6)	L(z)
0	{ }	{a, v1, v2, v3, v4, v5, v6, z}	0	∞	∞	∞	∞	∞	∞	∞
1	{a }	{v1, v2, v3, v4, v5, v6, z}	0	3	4	∞	15	∞	∞	∞
2	{a , v1}	{v2, v3, v4, v5, v6, z}	0	**3**	4	10	9	∞	∞	∞
3	{a , v1,v2}	{v3, v4, v5, v6, z}	0	**3**	**4**	10	9	8	∞	13

$S = \{a, v1, v2\}$

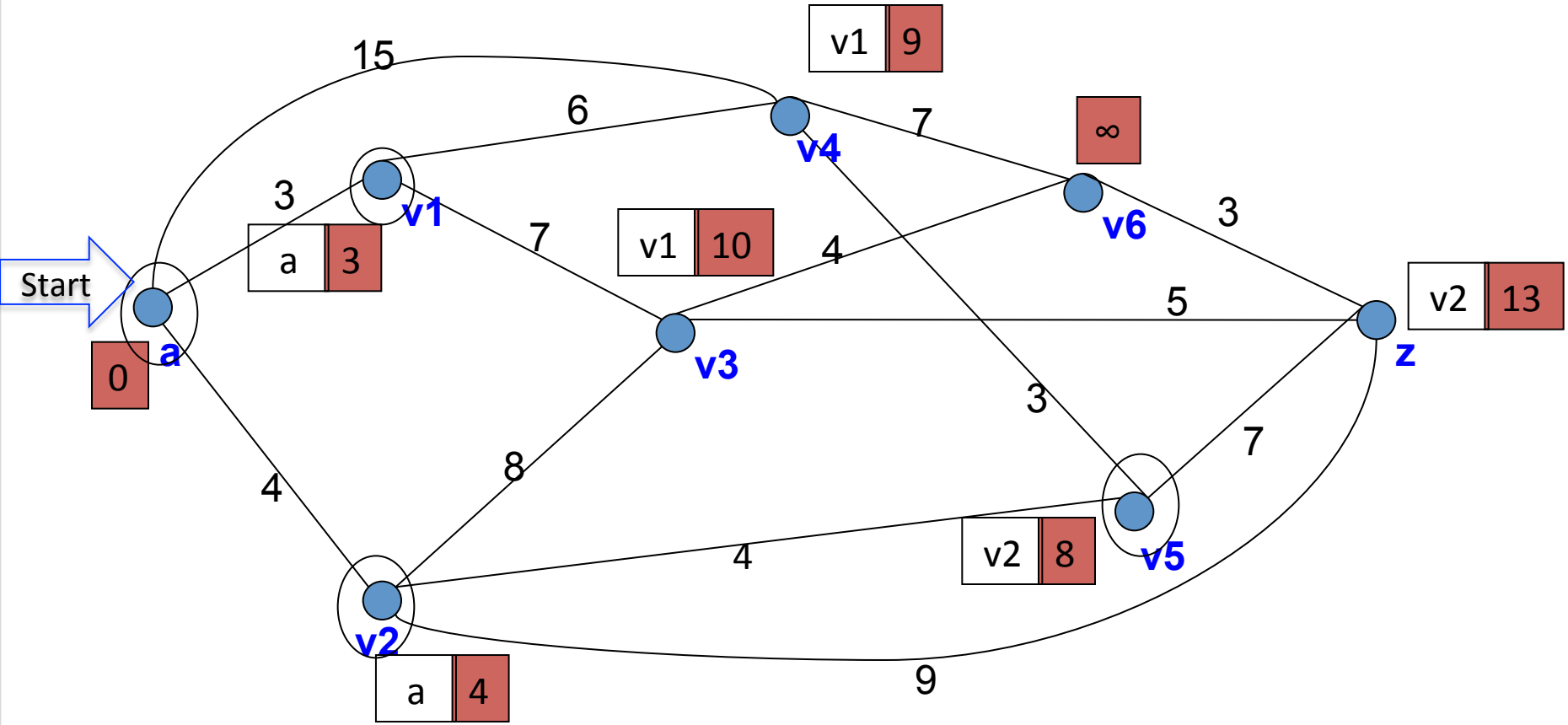
$N = \{v3, v4, v5, v6, z\}$

choose $v5$
because
$L(v5)=8 = \min\{L(u) \mid u \in N\}$



$$S = \{a, v1, v2, v5\}$$
$$N = \{v3, v4, v6, z\}$$

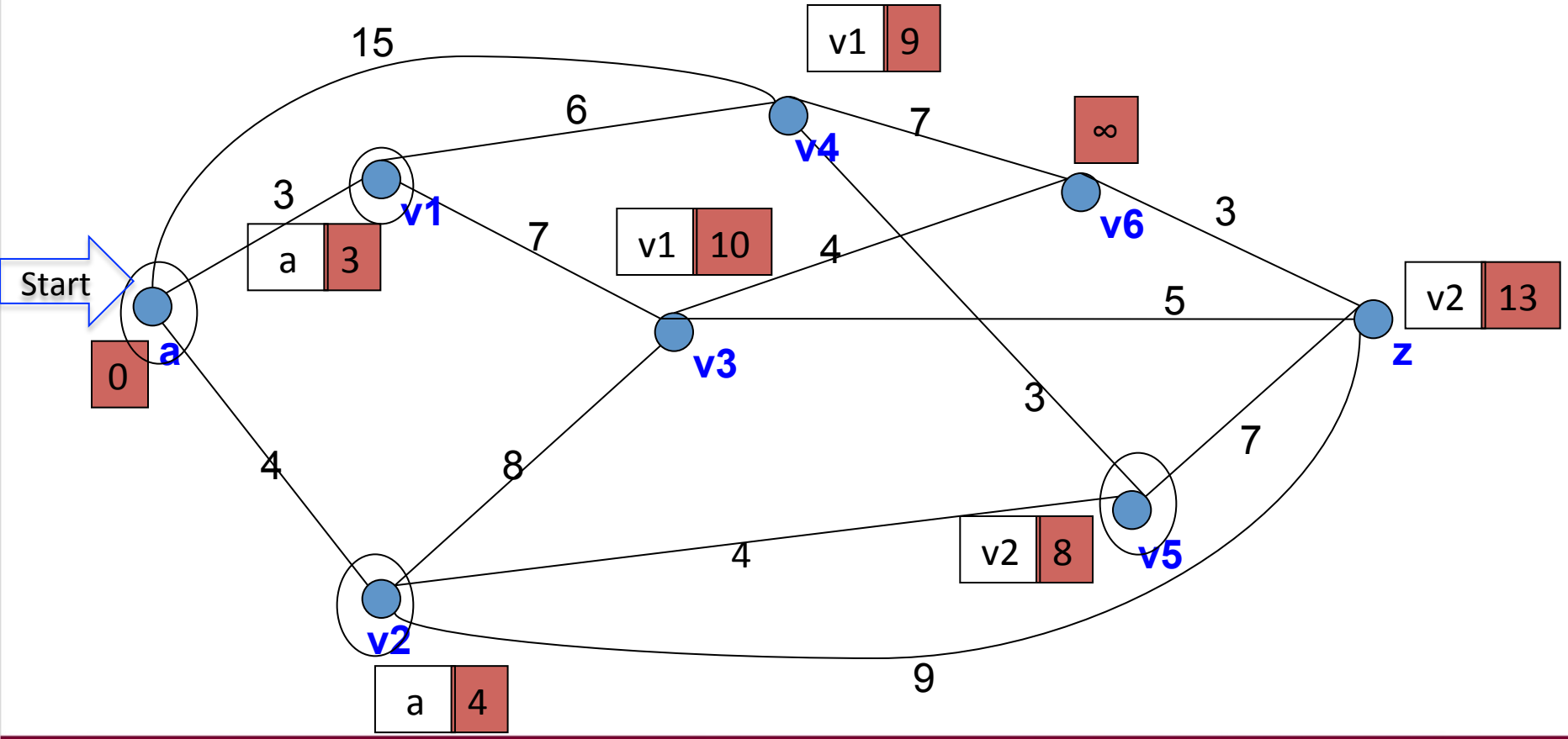
$L(v5)+W[v5, v4]<L(v4)$
$8+3= 11 > 9$
$L(v4)$ remains the same



$S = \{a, v1, v2, v5\}$

$N = \{v3, v4, v6, z\}$

$L(v5)+W[v5, z] < L(z)$
$8+7= 15 > 13$
$L(z)$ remains the same

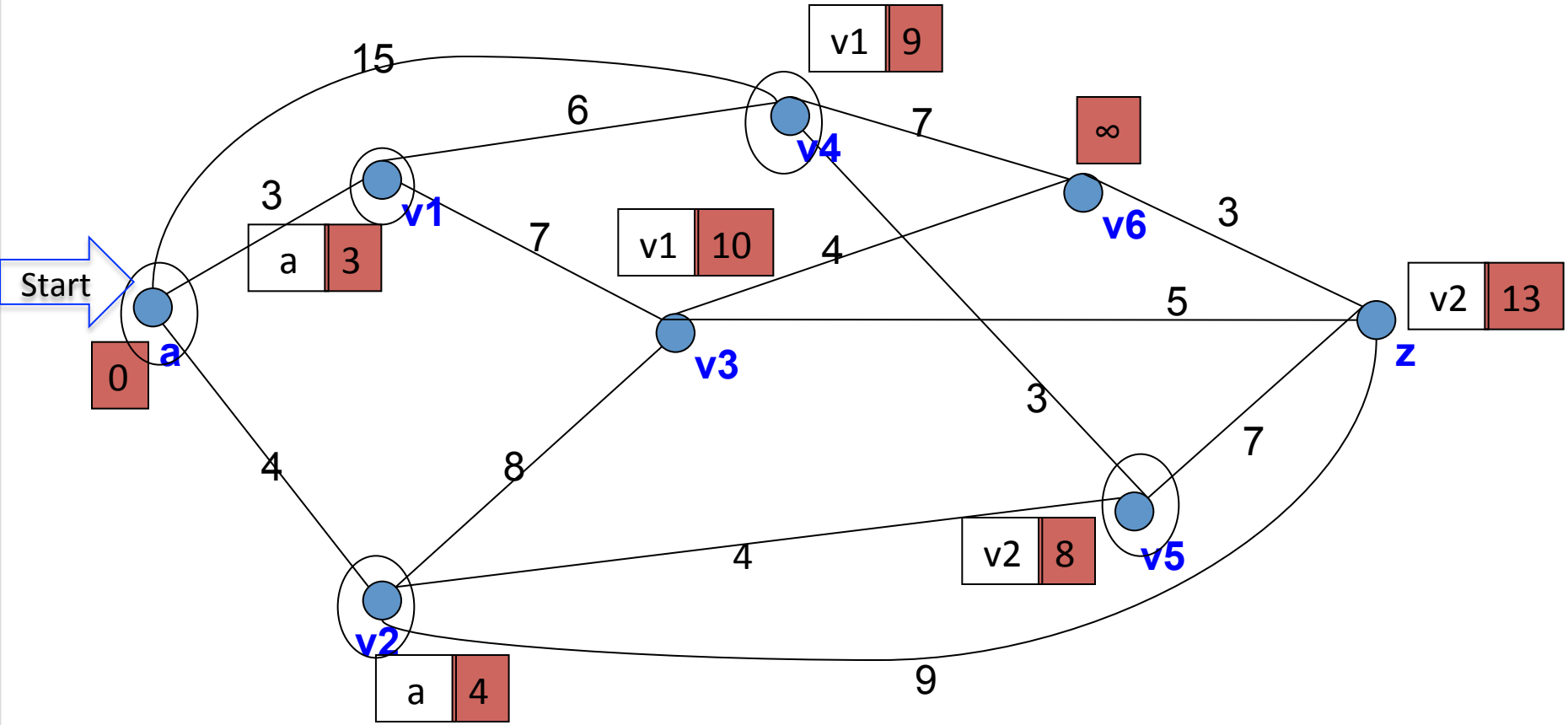


Iteration	S	N	L(a)	L(v1)	L(v2)	L(v3)	L(v4)	L(v5)	L(v6)	L(z)
0	{ }	{a, v1, v2, v3, v4, v5, v6, z}	0	∞	∞	∞	∞	∞	∞	∞
1	{a }	{v1, v2, v3, v4, v5, v6, z}	0	3	4	∞	15	∞	∞	∞
2	{a , v1}	{v2, v3, v4, v5, v6, z}	0	3	4	10	9	∞	∞	∞
3	{a , v1,v2}	{v3, v4, v5, v6, z}	0	3	4	10	9	8	∞	13
4	{a , v1,v2,v5}	{v3, v4, v6, z}	0	3	4	10	9	8	∞	13

$S = \{a, v1, v2, v5\}$

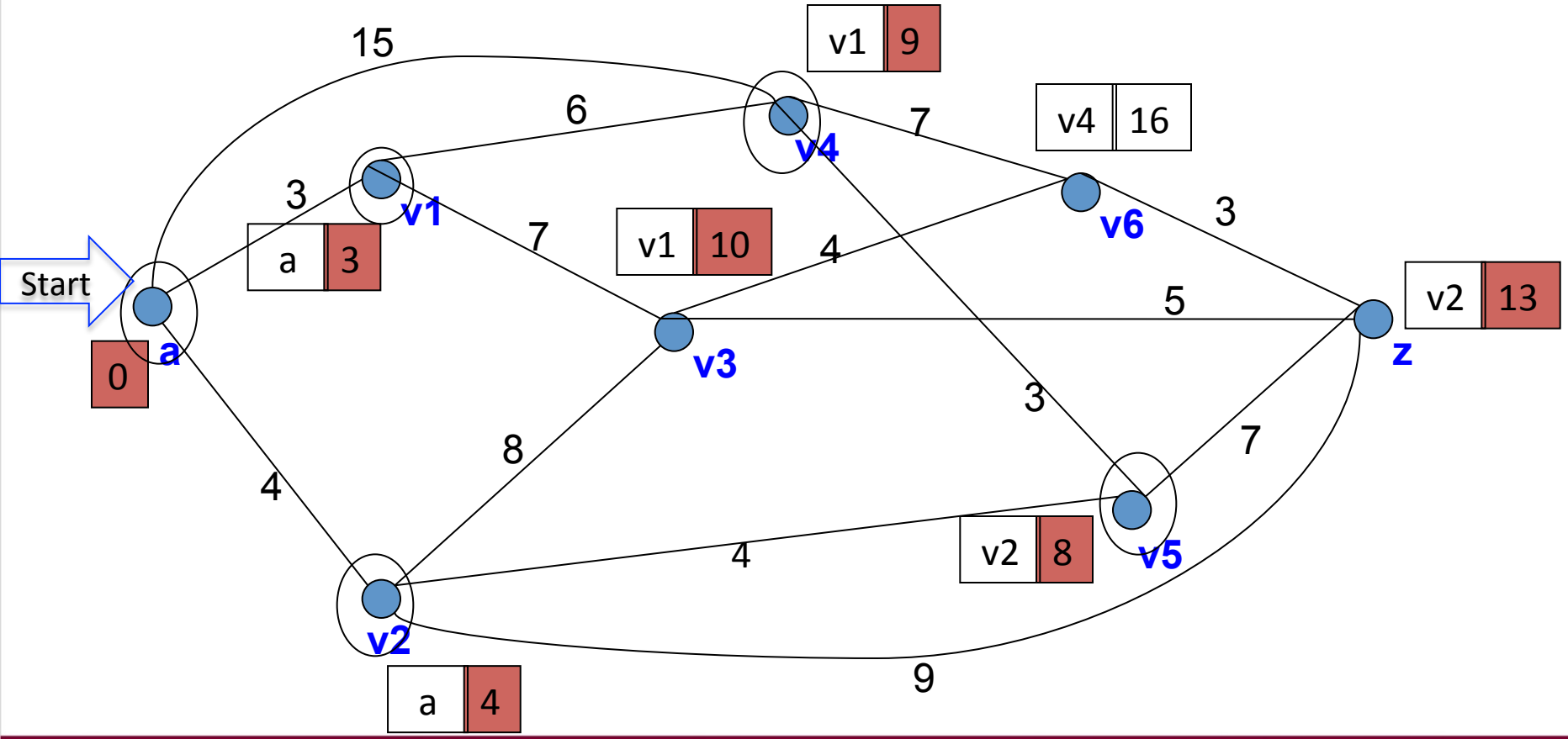
$N = \{v3, v4, v6, z\}$

choose *v*4
because
$L(v4)=9 = \min\{L(u) \mid u \in N\}$



$$S = \{a, v1, v2, v5, v4\}$$
$$N = \{v3, v6, z\}$$

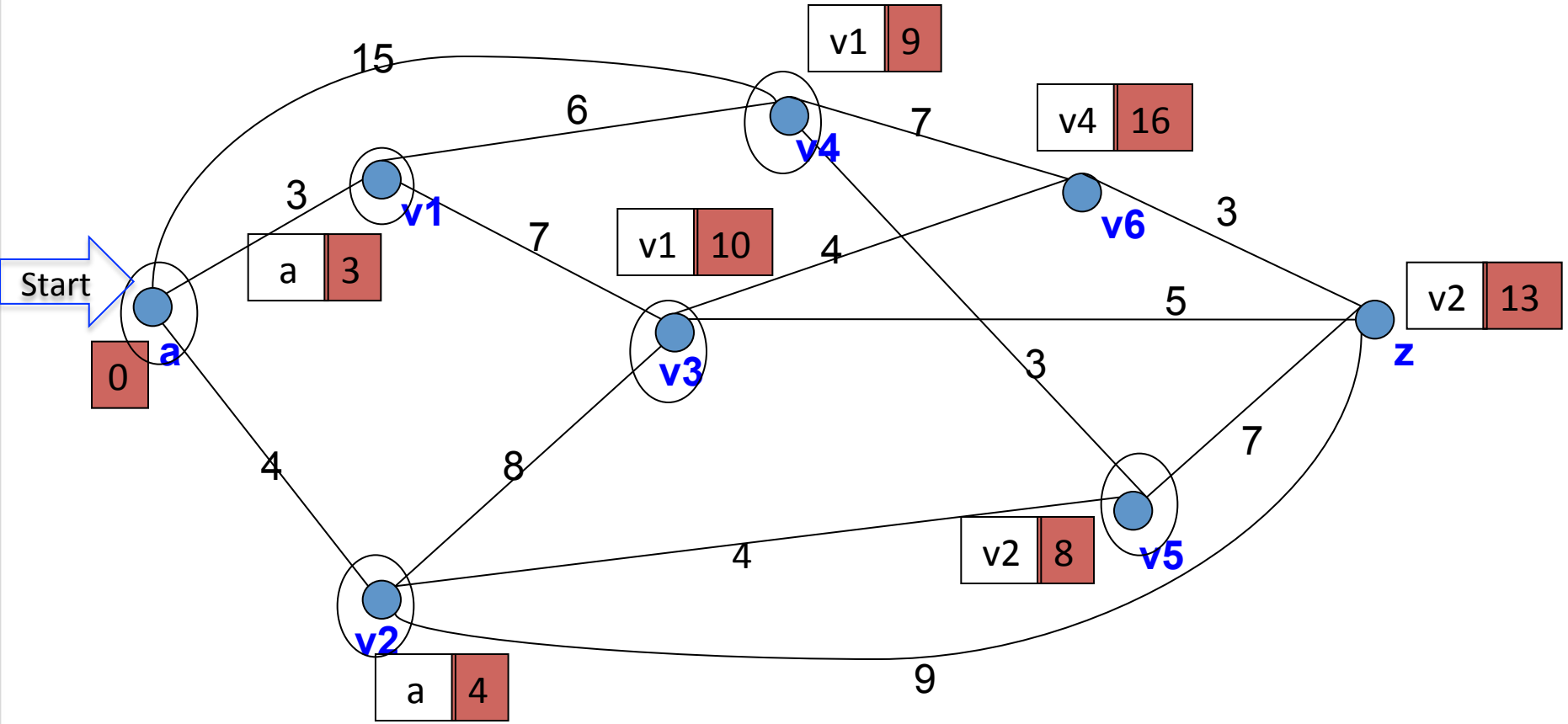
$L(v4)+W[v4, v6]<L(v6)$
$9+7= 16 < \infty$
$L(v6) = 16$



$S = \{a, v1, v2, v5, v4\}$

$N = \{v3, v6, z\}$

choose *v*3
because
$L(v3)=10 = \min\{L(u) \mid u \in N\}$

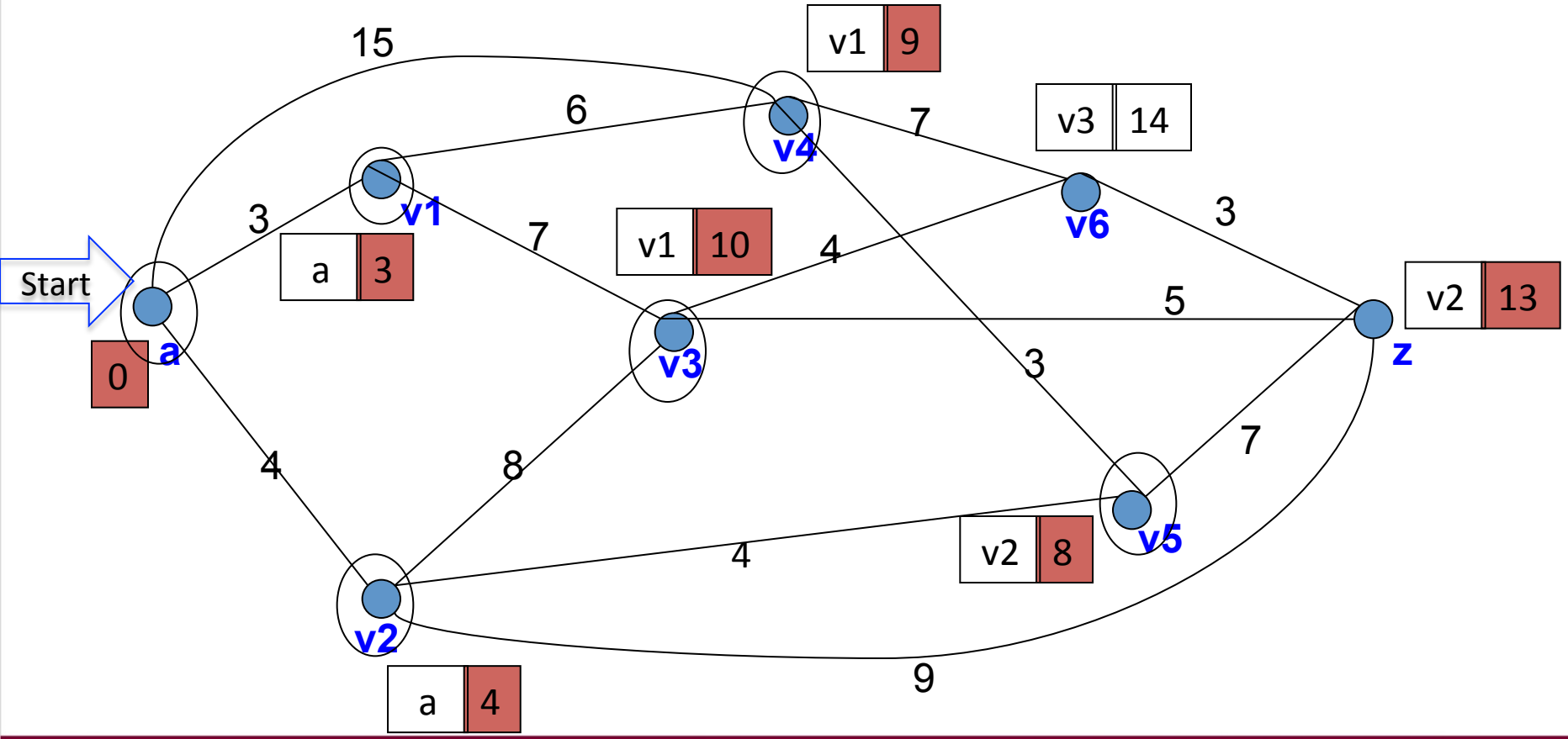


Iteration	*S*	*N*	*L*(a)	*L*(v1)	*L*(v2)	*L*(v3)	*L*(v4)	*L*(v5)	*L*(v6)	*L*(z)
0	{ }	{a, v1, v2, v3, v4, v5, v6, z}	0	∞	∞	∞	∞	∞	∞	∞
1	{a }	{v1, v2, v3, v4, v5, v6, z}	0	3	4	∞	15	∞	∞	∞
2	{a , v1}	{v2, v3, v4, v5, v6, z}	0	**3**	4	10	9	∞	∞	∞
3	{a , v1,v2}	{v3, v4, v5, v6, z}	0	**3**	**4**	10	9	8	∞	13
4	{a , v1,v2,v5}	{v3, v4, v6, z}	0	**3**	**4**	10	9	**8**	∞	13
5	{a , v1,v2,v5, v4}	{v3, v6, z}	0	**3**	**4**	10	**9**	**8**	16	13

$$S = \{a, v1, v2, v5, v4, v3\}$$

$$N = \{v6, z\}$$

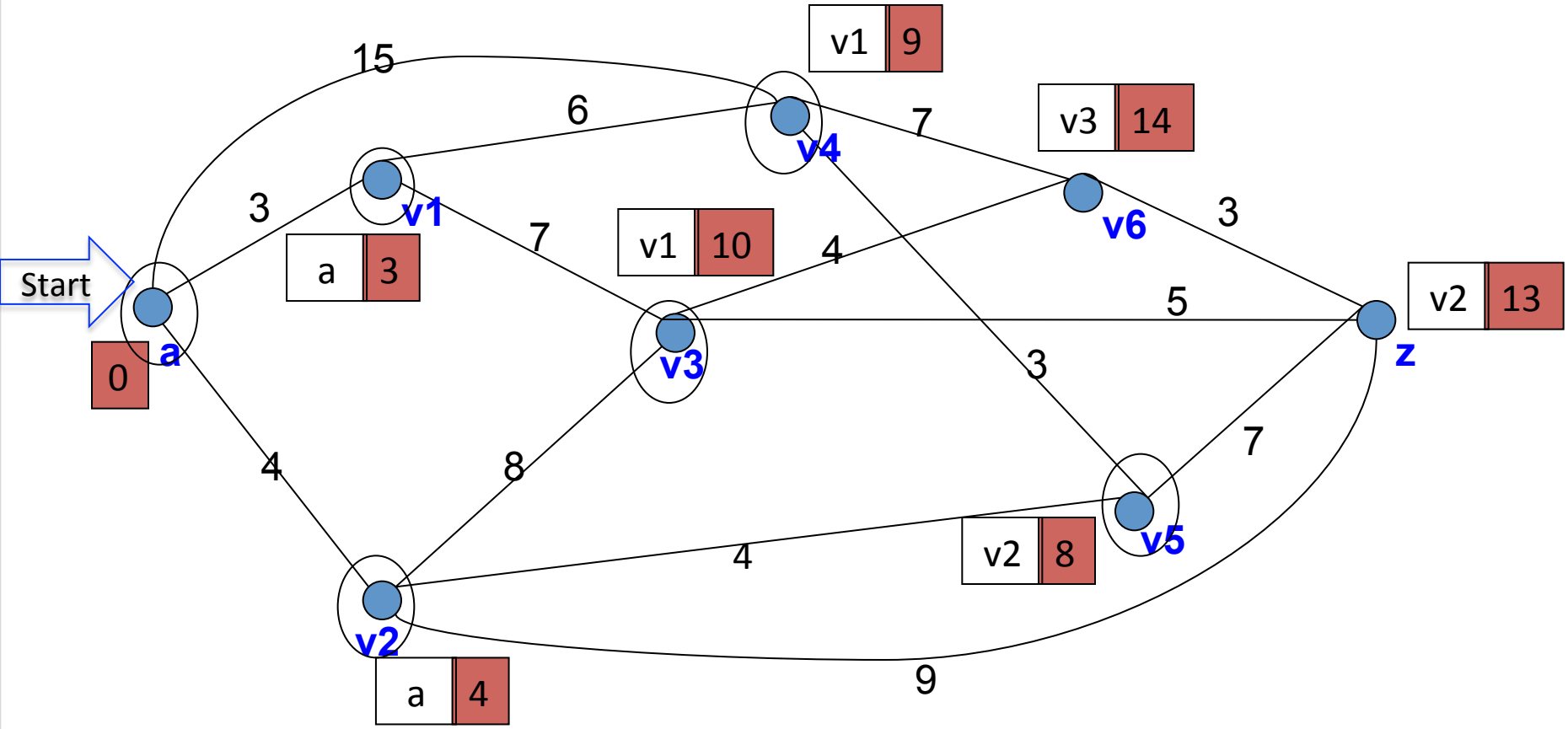
$L(v3)+W[v3, v6]<L(v6)$
$10+4= 14 < 16$
$L(v6) = 14$



$S = \{a, v1, v2, v5, v4, v3\}$

$N = \{v6, z\}$

$L(v3)+W[v3, z] < L(z)$
$10+5= 15 > 13$
$L(z)$ remains the same

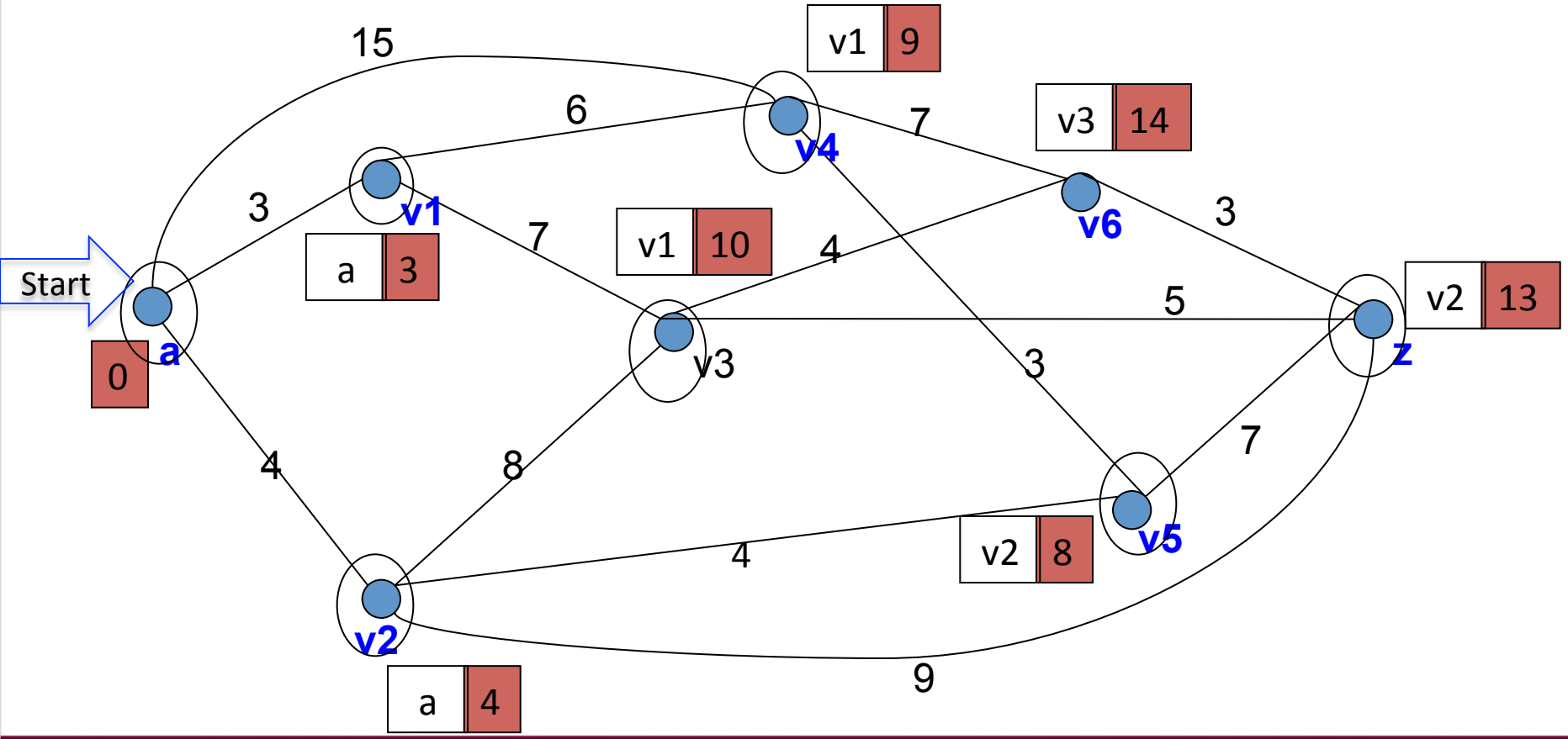


Iteration	*S*	*N*	*L*(a)	*L*(v1)	*L*(v2)	*L*(v3)	*L*(v4)	*L*(v5)	*L*(v6)	*L*(z)
0	{ }	{a, v1, v2, v3, v4, v5, v6, z}	0	∞	∞	∞	∞	∞	∞	∞
1	{a }	{v1, v2, v3, v4, v5, v6, z}	0	3	4	∞	15	∞	∞	∞
2	{a , v1}	{v2, v3, v4, v5, v6, z}	0	**3**	4	10	9	∞	∞	∞
3	{a , v1,v2}	{v3, v4, v5, v6, z}	0	**3**	**4**	10	9	8	∞	13
4	{a , v1,v2,v5}	{v3, v4, v6, z}	0	**3**	**4**	10	9	**8**	∞	13
5	{a , v1,v2,v5, v4}	{v3, v6, z}	0	**3**	**4**	10	**9**	**8**	16	13
6	{a , v1,v2,v5, v4, v3}	{v6, z}	0	**3**	**4**	**10**	**9**	**8**	14	13

$S = \{a, v1, v2, v5, v4, v3\}$

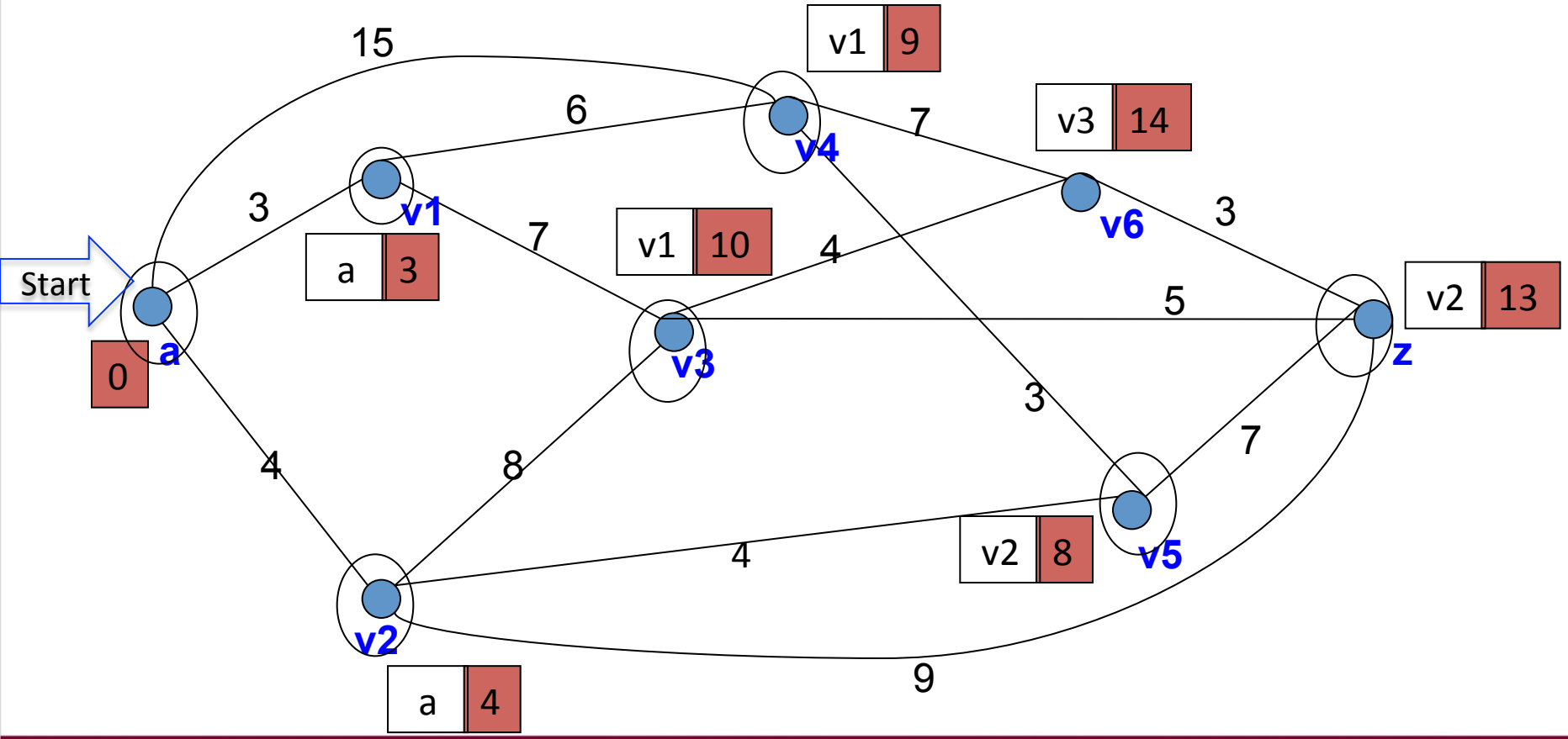
$N = \{v6, z\}$

choose *z*
because
$L(z)=13 = \min\{L(u) \mid u \in N\}$



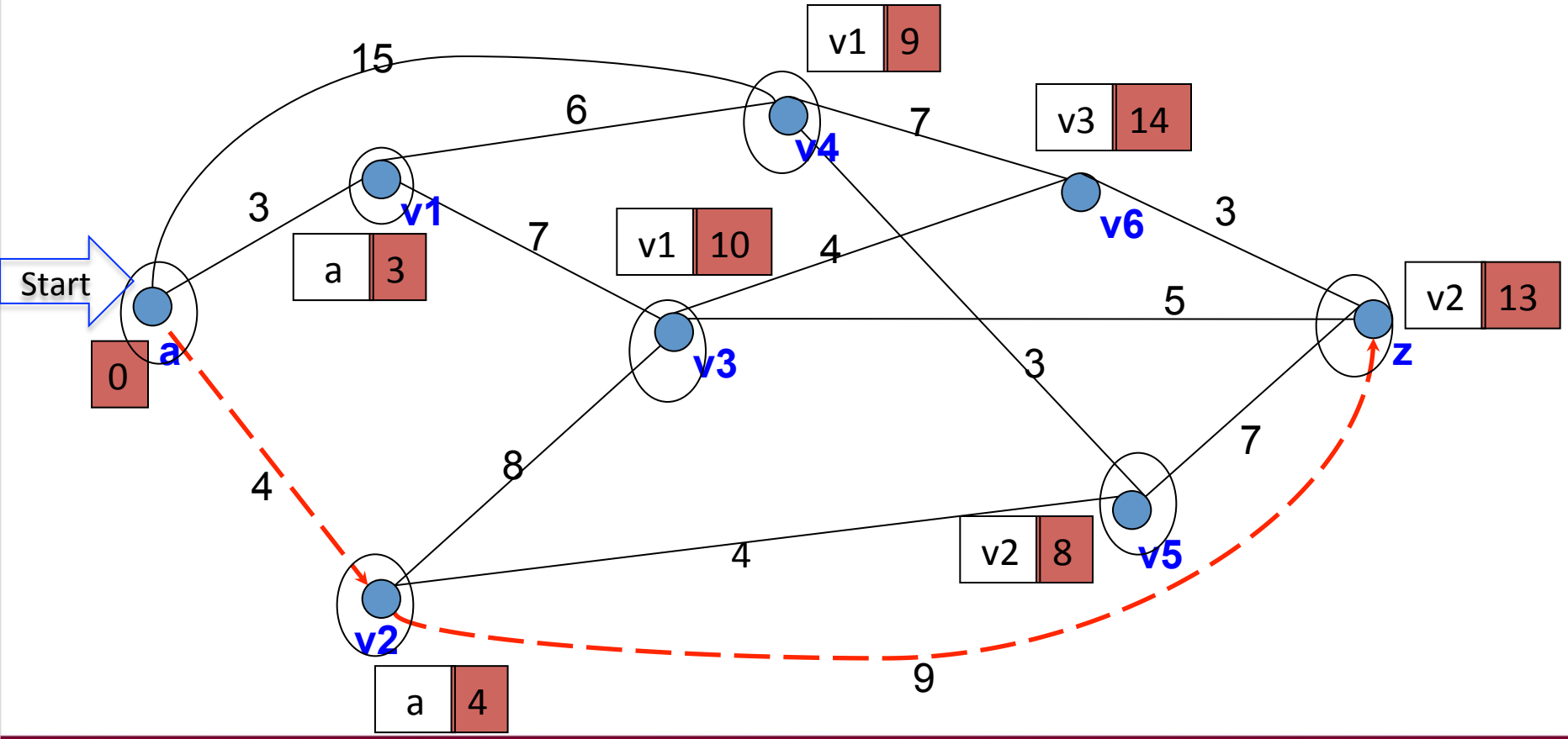
$$S = \{a,\ v1,\ v2,\ v5, v4, v3, z\}$$
$$N = \{v6\}$$

The loop terminates because $z \in S$



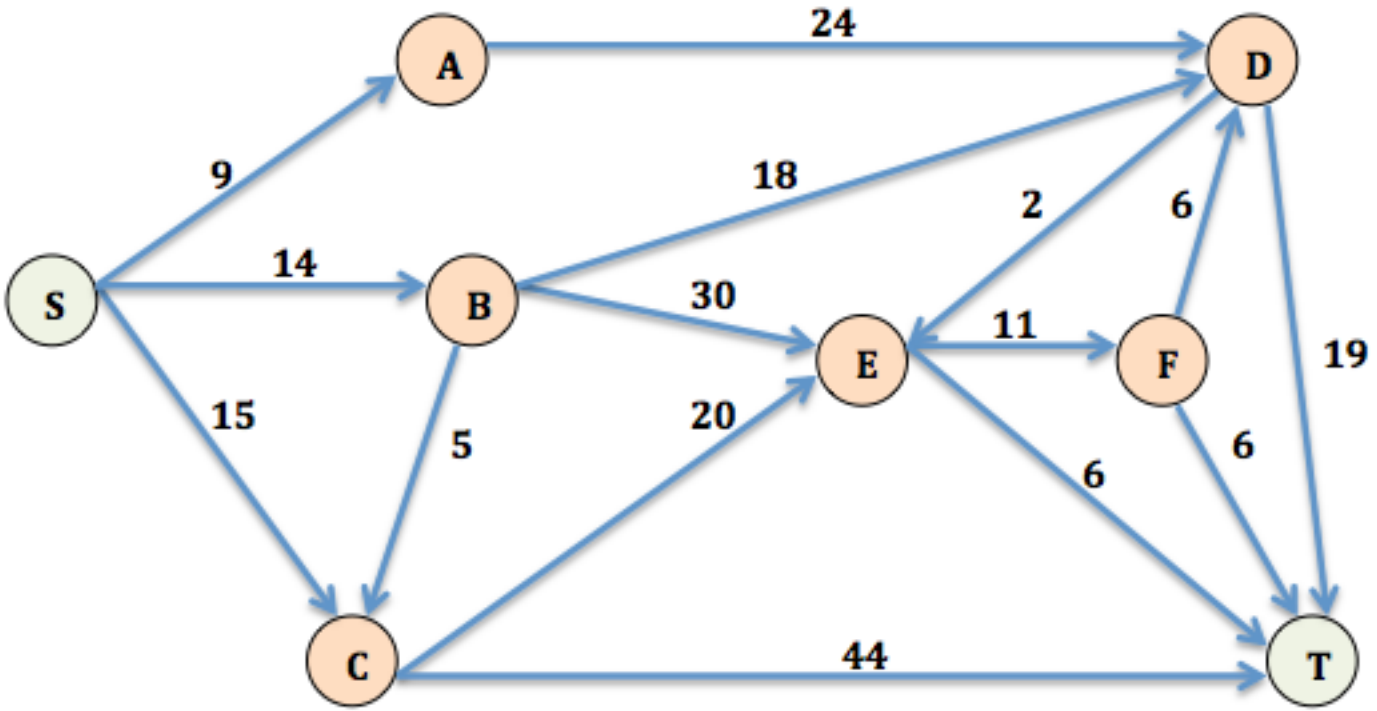
Iteration	*S*	*N*	*L*(a)	*L*(v1)	*L*(v2)	*L*(v3)	*L*(v4)	*L*(v5)	*L*(v6)	*L*(z)
0	{ }	{a, v1, v2, v3, v4, v5, v6, z}	0	∞	∞	∞	∞	∞	∞	∞
1	{a }	{v1, v2, v3, v4, v5, v6, z}	0	3	4	∞	15	∞	∞	∞
2	{a , v1}	{v2, v3, v4, v5, v6, z}	0	**3**	4	10	9	∞	∞	∞
3	{a , v1,v2}	{v3, v4, v5, v6, z}	0	**3**	**4**	10	9	8	∞	13
4	{a , v1,v2,v5}	{v3, v4, v6, z}	0	**3**	**4**	10	9	**8**	∞	13
5	{a , v1,v2,v5, v4}	{v3, v6, z}	0	**3**	**4**	10	**9**	**8**	16	13
6	{a , v1,v2,v5, v4, v3}	{v6, z}	0	**3**	**4**	**10**	**9**	**8**	14	13
7	{a , v1,v2,v5, v4, v3, z}	{v6}	0	**3**	**4**	**10**	**9**	**8**	14	**13**

Shortest path from ***a*** to **z** is **a** --> **v2** --> **z**, with the shortest length is 13.



Exercise

Q: Given a weighted digraph, find the shortest path from **S** to **T**, using Djikstra Algorithm.



Note: Weights are arbitrary numbers (i.e., not necessarily distances).