CHAPTER 2

(Part 3)

SEQUENCE, RECURRENCE RELATIONS & RECURSIVE ALGORITHM

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SEQUENCE & RECURRENCE RELATIONS

Sequence

- A sequence is a discrete structure used to represent an ordered list.

- A sequence is a function from a subset of the set integers to a set **S**. We use the notation a_n to denote the image of the integer n. We call a_n a term of the sequence.

Example

Consider the sequence $\{a_n\}$, where

$$a_n = \frac{1}{n}$$

The list of the terms of this sequence, beginning with a_1, namely

$$a_1, a_2, a_3, a_4, \ldots.$$

start with

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots.$$

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Recurrance Relations

- A recurrence relations (or recurrence) for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of sequence, namely, a_0, a_1, a_2,…,a_{n-1}, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer. A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.

- The information called the initial condition(s) for the recurrence must be provided to give enough information about the equation to get started.

Simple Recurrence Relation

- The simplest form of a **recurrence relation** is the case where the next term depends only on the immediately previous term.

- For example, given an initial condition, $a_1 = 3$, the list of terms $a_1, a_2, a_3, \ldots,$ begin with 3, 8, 13, 18, 23, . . ., is generated from a recurrence relation defined by

$$a_n = a_{n-1} + 5, \qquad n \geq 2$$

n-th Term of a Sequence

- Recurrence relation can be used to compute any n-th term of the sequence.
- Example: Given the initial condition, a_1:

$$a_n = a_{n-1} + 5, \quad n \geq 2$$

3, 8, 13, 18, 23, **28,** $a_{n-1} + 5$, . . .

$$a_2 = a_1 + 5, \quad 3 + 5 = 8$$
$$a_3 = a_2 + 5, \quad 8 + 5 = 13$$
$$a_4 = a_3 + 5, \quad 13 + 5 = 18$$
$$\vdots$$
$$a_6 = a_5 + 5, \quad 23 + 5 = 28$$

Example

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1, 2, 3, \ldots.,$ and suppose that $a_0 = 2$. What are a_1, a_2, and a_3?

Solution:

We see from the recurrence relation that,

$a_1 = a_0 + 3 = 2+3 = 5$. It then follows that,

$a_2 = a_1 + 3 = 5+3 = 8$ and

$a_3 = a_2 + 3 = 8+3 = 11$.

Example

Consider the following sequence:

$$3, 9, 27, 81, 243, \ldots$$

The above sequence shows a pattern:

$$3^1,\ 3^2,\ 3^3,\ 3^4,\ 3^5, \ldots$$
$$a_1,\ a_2,\ a_3,\ a_4,\ a_5, \ldots$$

Recurrence relation is defined by:

$$a_n = 3^n, n \geq 1$$

Example

Given initial condition, $a_0 = 1$ and recurrence relation:

$$a_n = 1 + 2a_{n-1} \quad , n \geq 1$$

First few sequence are:

$$a_1 = 1 + 2\,(1) = 3$$
$$a_2 = 1 + 2(3) = 7$$
$$a_3 = 1 + 2(7) = 15$$

1, 3, 7, 15, 31, 63, …

Example

Given initial conditions, $a_0 = 1$, $a_1 = 2$ and recurrence relation:

$$\boldsymbol{a_n = 3(a_{n-1} + a_{n-2}),\ n \geq 2}$$

First few sequence are:

$$a_2 = 3(2 + 1) = 9$$
$$a_3 = 3(9 + 2) = 33$$
$$a_4 = 3(33 + 9) = 126$$

1, 2, 9, 33, 126, 477, 1809, 6858, 26001,…

Example

For a photo shoot, the staff at a company have been arranged such that there are 10 people in the front row and each row has 7 more people than in the row in front of it. Find the recurrence relation and compute number of staff in the first 5 rows.

Example 4 -Solution

Notice that the difference between the number of people in successive rows is a constant amount.

This means that the n_{th} term of this sequence can be found using:

$$\boldsymbol{a_n} = \boldsymbol{a_{n-1}} + 7,\ n \geq 2 \text{ with } \boldsymbol{a_1 = 10}$$

Number of staff in the first 5 rows:

$\boldsymbol{a_1 = 10}$,

$\boldsymbol{a_2 = a_1} + 7 \Rightarrow 10 + 7 = 17,$

$\boldsymbol{a_3 = a_2} + 7 \Rightarrow 17 + 7 = 24,$

$\boldsymbol{a_4 = a_3} + 7 \Rightarrow 24 + 7 = 31,$

$\boldsymbol{a_5 = a_4} + 7 \Rightarrow 31 + 7 = 38.$

10, 17, 24, 31, 38

Example

Find a recurrence relation and initial condition for

1, 5, 17, 53, 161, 485, …

However the original sequence is not.

1(3)=**3**, 5(3)=**15**,17(3)=**51**, . . .

1, 5, 17, 53, 161, 485, …

It appears that we always end up with 2 less than the next term.

So, the recurrence relation is defined by:

Example

A depositor deposits RM 10,000 in a savings account at a bank yielding 5% per year with interest compounded annually. How much money will be in the account after 30 years? Let P_n denote the amount in the account after n years.

Example - Solution

Derive the following **recurrence relation**:

Where, $\boldsymbol{P_n}$ = Current balance and $\boldsymbol{P_{n-1}}$ = Previous year balance and 0.05 is the compounding interest.

Initial condition, $P_0 = 10{,}000$. Then,

now we can use this formula to calculate n_{th} term without iteration.

Let us use this formula to find P_{30} under the initial condition $P_0 = 10{,}000$:

After 30 years, the account contains RM 43,219.42.

Exercise # 1

Consider the following sequence:

1, 5, 9, 13, 17

Find the recurrence relation that defines the above sequence.

Exercise #2

A basketball is dropped onto the ground from a height of 15 feet. On each bounce, the ball reaches a maximum height 55% of its previous maximum height.

a)Write a recursive formula, a_n , that completely defines the height reached on the n_{th} bounce, where the first term in the sequence is the height reached on the ball's first bounce.

b)How high does the basketball reach after the 4$^{\text{th}}$ bounce? Give your answer to two decimal places.

Exercise #2 - Solution

Recursion

- Recursion is a powerful, elegant and natural way to solve a large class of problems that relate to sequence/recurrence relation.
- A recursive procedure is a procedure that invokes itself.
- A recursive algorithm is an algorithm that contains a recursive procedure.

Example

Factorial problem

- If $n \geq 1$,

  $n! = n\,(n-1) \Rightarrow 2 \times 1$

  and $0! = 1$

- Notice that, if $n \geq 2$, n factorial can be written as,

  $n! = n(n-1)(n-2) \Rightarrow 2 \times 1 \times 0$

  $= n(n-1)!$

Example

- $n=5$

- 5!

$$5! = 5.\ 4.\ 3.\ 2.\ 1 = 5.\ 4!$$

$$4! = 4.\ 3!$$

$$3! = 3.\ 2!$$

Recursive Algorithm for Factorial

- Input: n, integer ≥ 0
- Output: $n!$
- Factorial (n) {

```
    if (n=0)
        return 1
    return n*factorial(n-1)
}
```

Example

- Fibonacci sequence, f_n

$$f_1 = 1$$
$$f_2 = 1$$
$$f_n = f_{n-1} + f_{n-2}, \text{ for } n \geq 3$$

1, 1, 2, 3, 5, 8, 13, ...

Recursive algorithm for Fibonacci Sequence :

- Input: *n*
- Output: *f* (*n*)
- *f*(*n*) {

```
    if (n=1 or n=2)
        return 1
    return f (n-1) + f (n-2)
}
```

Example

Consider the following arithmetic sequence:

1, 3, 5, 7, 9, ………

Suppose a_n is the term sequence. The generating rule is

$$a_n = a_{n-1} + 2, \text{ for } n \geq 1.$$

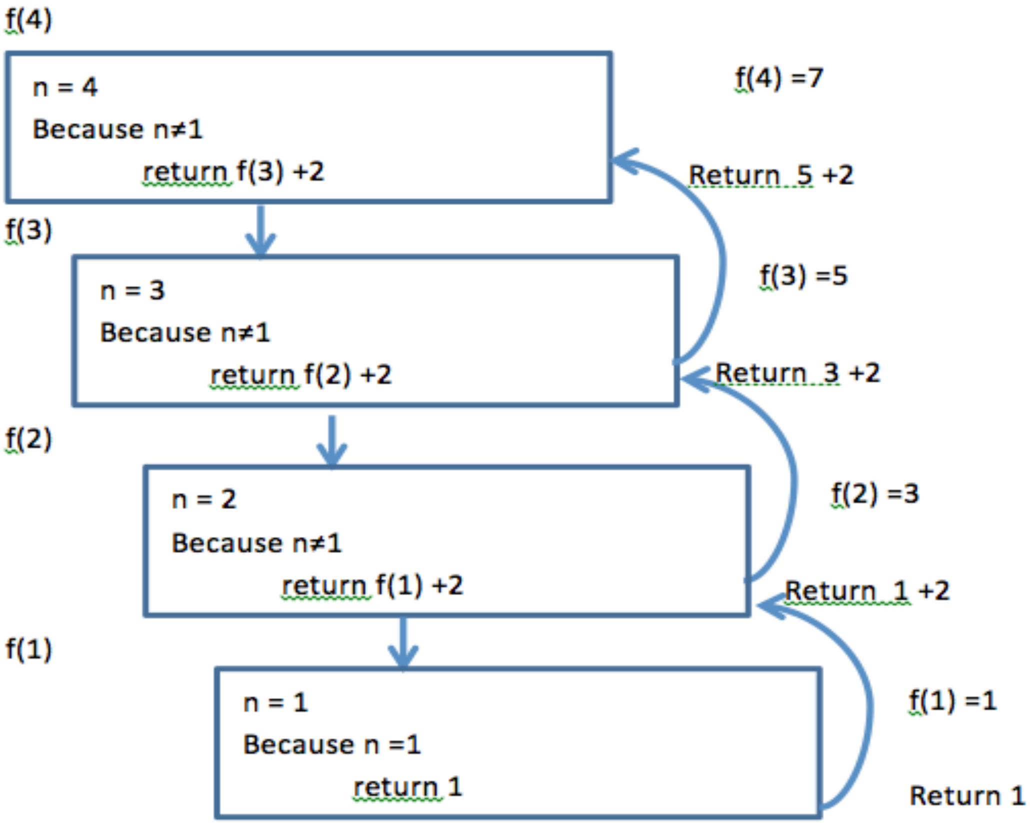
The relevant recursive algorithm can be written as

```
f (n)
 { if (n =1)
      return 1
  return f (n-1) + 2
 }
```

Use the above recursive algorithm to trace $n = 4$.

Example- Solution

Trace the output if n = 4



Answer = 7