CHAPTER 2

(Part 2)

FUNCTIONS

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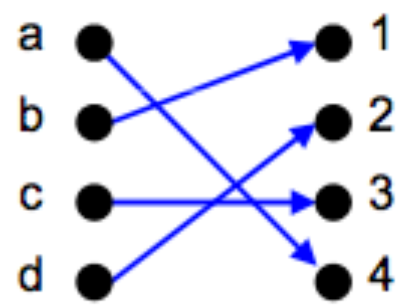
Definition

A **function** *f* from a set **X** to a set **Y**, denoted *f* :**X**→**Y**, is a relation from **X**, the **domain,** to **Y**, the **co-domain**, that satisfies two properties:

1) Every element in **X** is related to some element in **Y**,
2) No element in **X** is related to more than one element in **Y.**

Relations vs Functions

- Not all relations are functions
- But consider the following function:



- All functions are relations!

When to use which?

- A function is used when you need to obtain a SINGLE result for any element in the domain
  - Example: sin, cos, tan

- A relation is when there are multiple mappings between the domain and the co-domain
  - Example: students enrolled in multiple courses

Domain, Co-domain, Range

- A function from a set **X** to a set **Y** is denoted, f : **X** → **Y**
- The domain of f is the set **X**.
- The set **Y** is called the co-domain or target of f.
- The set $\{ y \mid (x,y) \in f \}$ is called the range.

Example

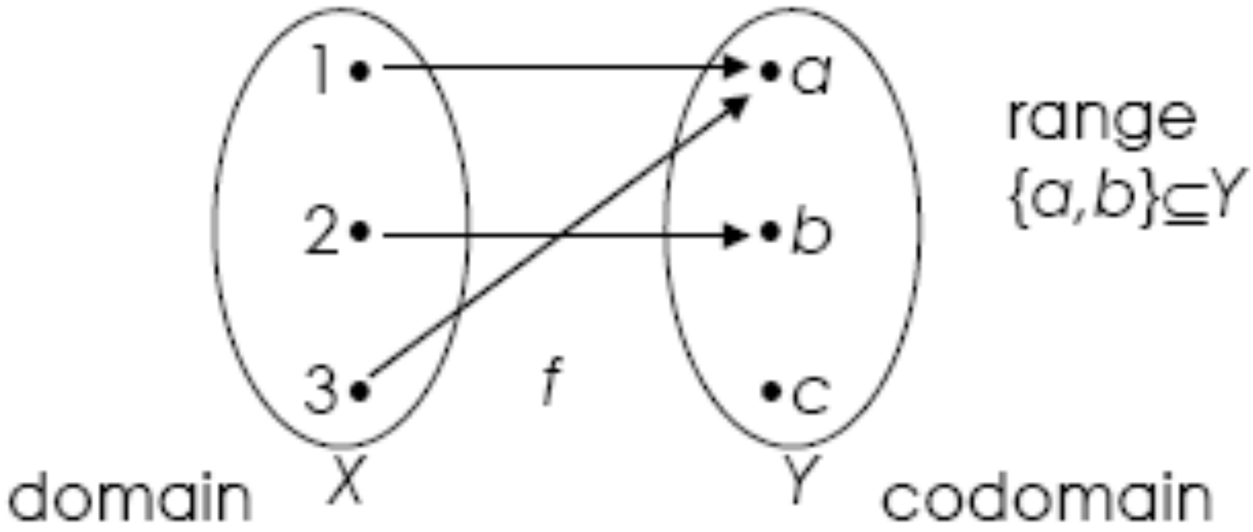
Given the relation, f = { (1,a), (2,b), (3,a) } from **X** = { 1, 2, 3 } to **Y**= { a, b, c } is a function from **X** to **Y**. State the domain and range.

Solution:

✓The domain of f is **X**

✓The range of f is {a, b}

- $f = \{ (1,a), (2,b), (3,a) \}$



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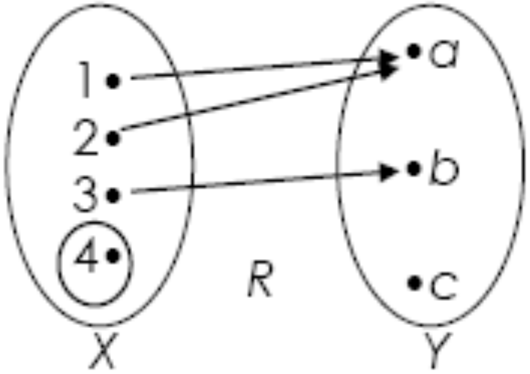
Example

- The relation, $R = \{(1,a), (2,a), (3,b)\}$ from $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c\}$ is NOT a function from X to Y.
- The domain of R, $\{1,2,3\}$ is not equal to X.

$R = \{(1,a), (2,a), (3,b)\}$



There is no arrow from 4

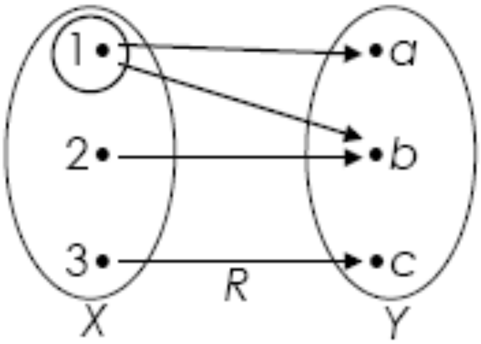
Example

- The relation, $R = \{(1,a), (2,b), (3,c), (1,b)\}$ from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$ is NOT a funtion from X to Y

- (1,a) and (1,b) in R but a ≠ b.

$R = \{(1,a), (2,b), (3,c), (1,b)\}$

There are 2 arrows from 1



Notation of function: $f(x)$

- For the function, $f = \{(1,a), (2,b), (3,a)\}$

- We may write:

$$f(1) = a,\ f(2) = b,\ f(3) = a$$

- Notation $f(x)$ is used to define a function.

Example

- Defined: $f = \{(x, x^2)|\ x \text{ is a real number}\}$
- $f(x) = x^2$
- $f(2) = 4,\ f(-3.5) = 12.25,\ f(0) = 0$

One-to-One Function

Let f be a function from a set **X** to a set **Y**.
f is **one-to-one** (or injective) if, and only if, for all elements x_1 and x_2 in **X**,

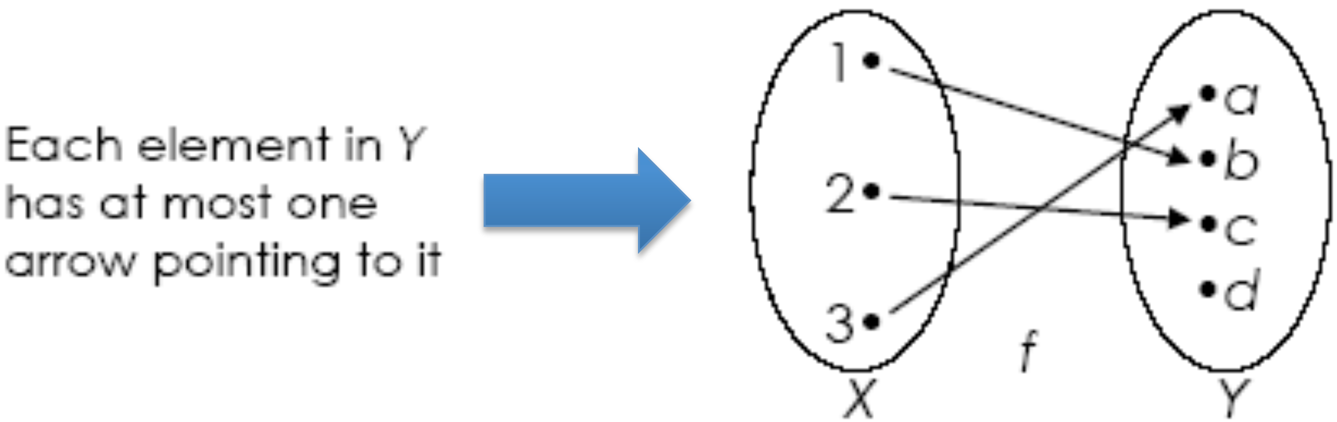
if $f(x_1) = f(x_2)$, then $x_1 = x_2$,
or, equivalently,
if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

Symbolically,

$$f : X \to Y \text{ is one-to-one } \Leftrightarrow \forall x_1, x_2 \in X, \text{ if } f(x_1) = f(x_2) \text{ then } x_1 = x_2.$$

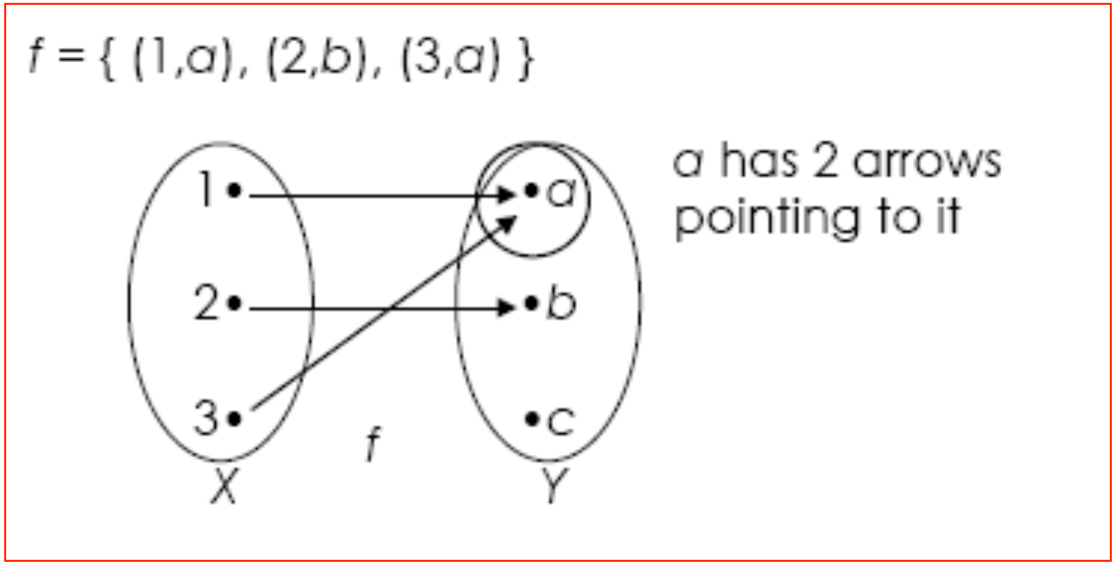
Example

- The function, f = { (1,*b*), (3,*a*), (2,*c*) } from **X** = { 1, 2, 3 } to **Y** = { *a, b, c, d* } is one-to-one.



- In terms of arrow diagrams, a one-to-one function can be thought of as a function that separates points. That is, it takes distinct points of the domain to distinct points of the co-domain.

- The function, $f = \{ (1,a), (2,b), (3,a) \}$ from **X** = { 1, 2, 3 } to **Y** = { *a, b, c*} is NOT one-to-one.
- $f(1) = f(3) = $ a



Example

Show that the function,

$$f(n) = 2n + 1$$

on the set of positive integers is one-to-one.

Solution:

- For all positive integer, n_1 and n_2 if $f(n_1) = f(n_2)$, then $n_1 = n_2$.

- Let, $f(n_1) = f(n_2)$, $f(n) = 2n+1$

  then $2n_1 + 1 = 2n_2 + 1$ (-1)

  $2n_1 = 2n_2$ (÷2)

  $n_1 = n_2$
- This shows that f is one-to-one.

Example

Show that the function,

$$f(n) = 2^n - n^2$$

on the set of positive integers is NOT one-to-one.

Solution:

- Need to find 2 positive integers, n_1 and n_2
  $n_1 \neq n_2$ with $f(n_1) = f(n_2)$.
- trial and error,
  $f(2) = f(4)$

$$f(n) = 2^n - n^2$$

$$n = 2 \Rightarrow 2^2 - 2^2 = 0$$

$$n = 4 \Rightarrow 2^4 - 4^2 = 0$$

- f is not one-to-one.

Onto Function

Let f be a function from a set **X** to a set **Y.**

f is **onto** (or **surjective**) if, and only if, given any element y in **Y**, it is possible to find an element x in **X** with the property that $y = f(x)$.

Symbolically,

$$f : X \to Y \text{ is onto } \Leftrightarrow \forall y \in Y, \exists x \in X \text{ such that } f(x) = y.$$

Example

Let **X** = { 1, 2, 3, 4 } and **Y** = { a, b, c }.

Define h: **X** → **Y** as follows:

$h(1) = \text{c}, h(2) = \text{a}, h(3) = \text{c}, h(4) = \text{b}.$

Define k: **X** → **Y** as follows:

$k(1) = \text{c}, k(2) = \text{b}, k(3) = \text{b}, k(4) = \text{c}.$

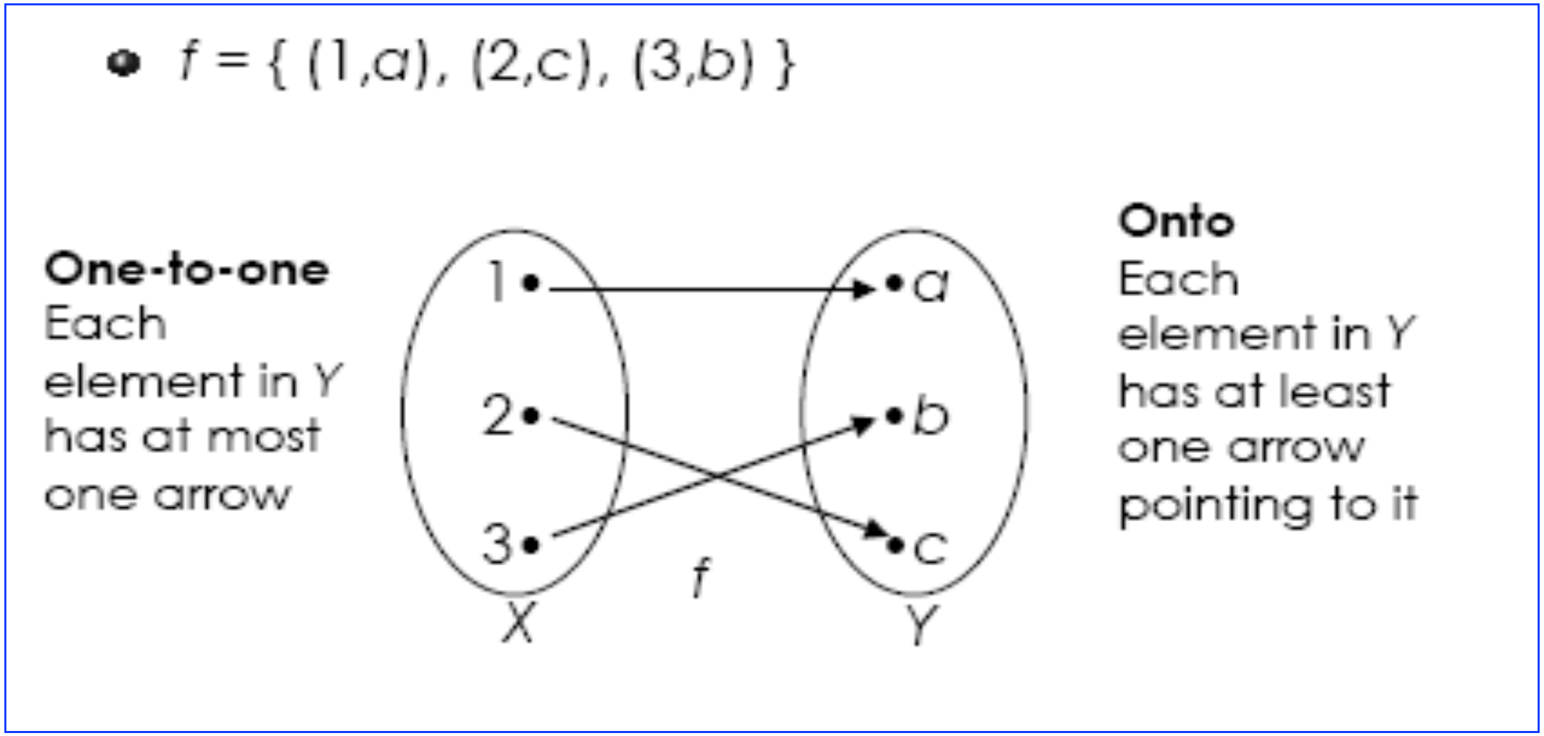
Is either h or k onto?

h is onto because each of the three elements of the co-domain of h is the image of some element of the domain of h.

k is not onto because $a \neq k(x)$ for any x in $\{1,2,3,4\}$

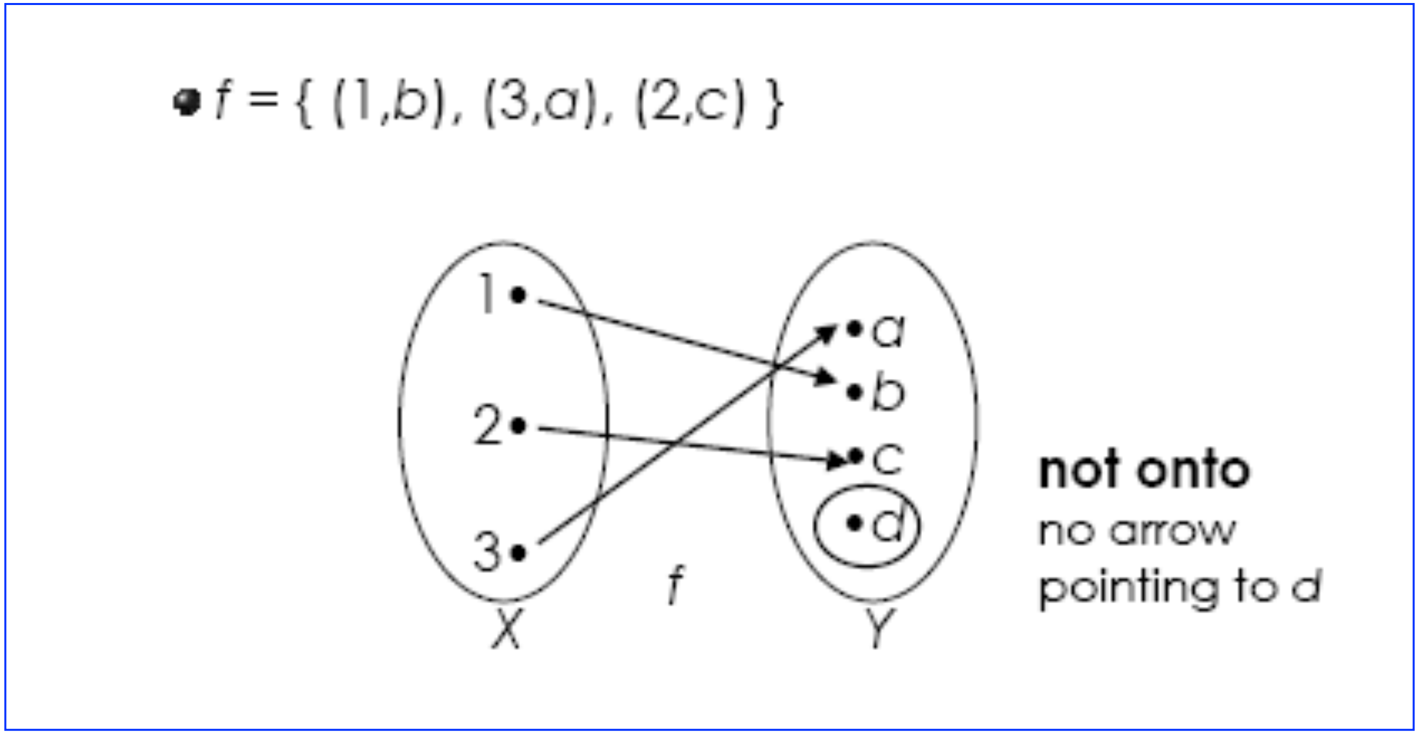
Example

- The function, f = { (1,a), (2,c), (3,b) } from **X** = { 1, 2, 3 } to **Y** = { a, b, c } is one-to-one and onto **Y**.



Example

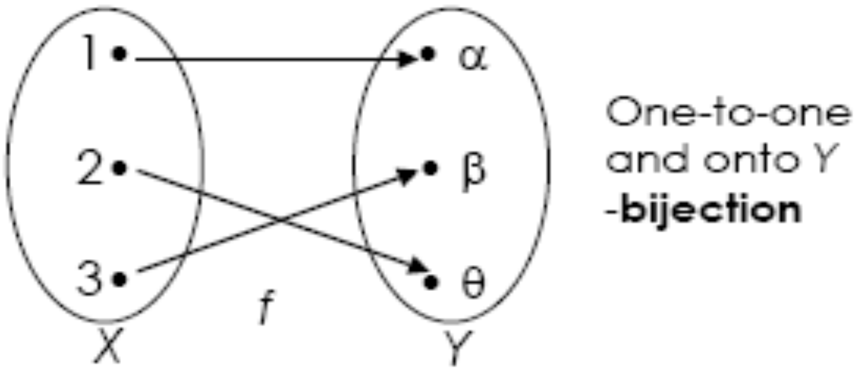
- The function, f = { (1,*b*), (3,*a*), (2,*c*) } is not onto **Y** = {*a, b, c, d*}



Bijection Function

- A function, f is called one-to-one correspondence (or bijective/bijection) if f is both one-to-one and onto.
- Example:

$f = \{ (1,\alpha), (2,\theta), (3,\beta) \}$



Exercise # 1

Determine which of the relations f are functions from the set **X** to the set **Y**. In case any of these relations are functions, determine if they are one-to-one, onto **Y**, and/or bijection.

a) **X** = { -2, -1, 0, 1, 2 } , **Y** = { -3, 4, 5 } and
f = { (-2,-3), (-1,-3), (0,4), (1,5), (2,-3) }

b) **X** = { -2, -1, 0, 1, 2 } , **Y** = { -3, 4, 5 } and
f = { (-2,-3), (1,4), (2,5)}

c) **X** = **Y**= { -3, -1, 0, 2 } and
f = { (-3,-1), (-3,0), (-1,2), (0,2), (2,-1) }

Exercise #2

Let **X** = {1, 2, 3 }, **Y** = { 1, 2, 3, 4} and **Z** = {1, 2}.

a) Define a function f : X → Y that is one-to-one but not onto.

b) Define a function g :X → Z that is onto but not one-to-one.

c) Define a function h :X → X that is neither one-to-one nor onto.

Inverse Function

If f is a one-to-one correspondence from a set **X** to a set **Y**, then there is a function from **Y** to **X** that "undoes" the action of f (it sends each element of **Y** back to the element of **X** that it came from). This function is called the inverse function for f.

Theorem

Suppose f: **X** → **Y** is one-to-one correspondence; that is, suppose f is one-to-one and onto. Then there is a function f^{-1}:**Y** → **X** that is defined as follows:

Given any element y in **Y**,
$f^{-1}(y)$ = that unique element x in **X** such that $f(x)$ equals y.

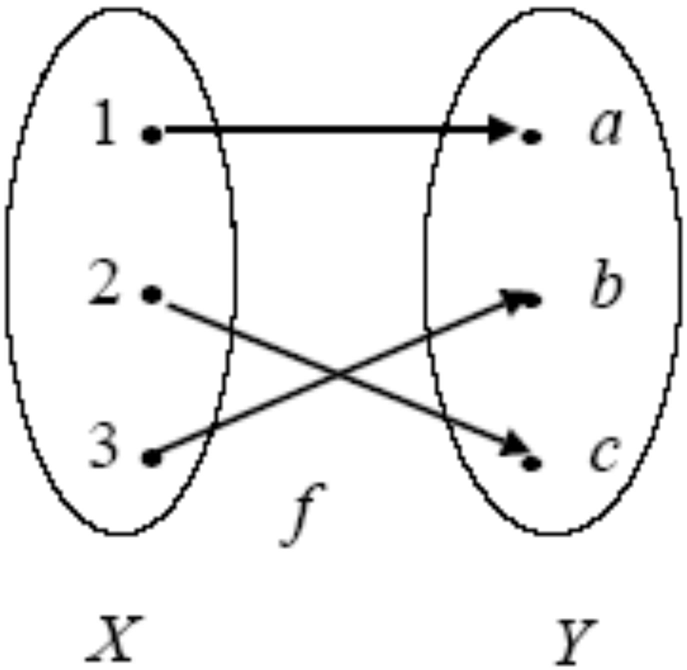
In other words,

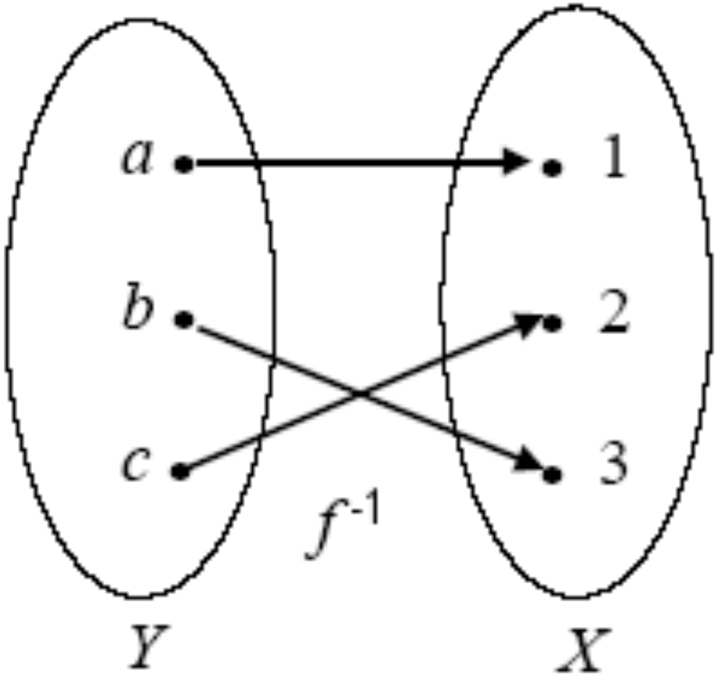
$$f^{-1}(y) = x \Leftrightarrow y = f(x)$$

Example

$f = \{(1,a),(2,c),(3,b)\}$

$f^{-1} = \{(a,1),(c,2),(b,3)\}$





Example

The function, f : **R** → **R** defined by the formula

$$f(x) = 4x - 1 \quad \text{for all } x \in \mathbf{R} \text{ (real number)}$$

This function is both one-to-one and onto. Find the inverse function.

Solution:

$$f(x) = y$$

$$\Leftrightarrow 4x - 1 = y$$

$$\Leftrightarrow x = \frac{y+1}{4}$$

$$\text{Hence } f^{-1}(y) = \frac{y+1}{4}$$

Exercise

- Find each inverse function.

a) $f(x) = 4x + 2, \; x \in R$

b) $f(x) = 3 + (1/x), \; x \in R$

Composition

- Suppose that g is a function from **X** to **Y** and f is a function from **Y** to **Z**.

- The composition of f with g,

$$f \circ g$$

is a function

$$(f \circ g)(x) = f(g(x))$$

from **X** to **Z**.

- Composition sometimes allows us to decompose complicated functions into simpler functions.

- example $f(x) = \sqrt{\sin 2x}$

$$g(x) = \sqrt{x} \qquad h(x) = \sin x \qquad w(x) = 2x$$

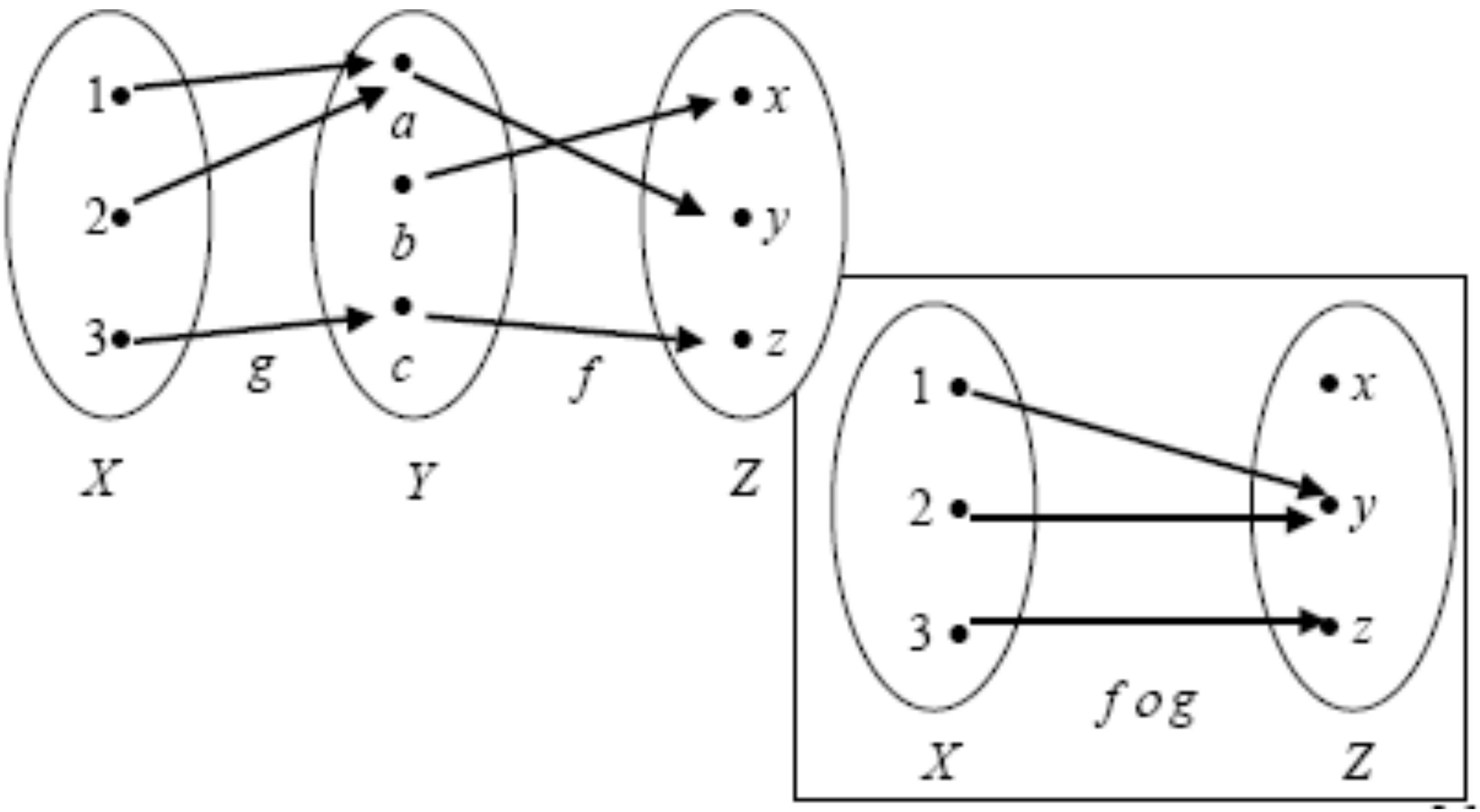
$$f(x) = g(h(w(x)))$$

Example

- Given, $g = \{ (1,a), (2,a), (3,c) \}$
  a function from **X** = {1, 2, 3} to **Y** = {*a, b, c*}
  and,
  $f = \{ (a,y), (b,x), (c,z) \}$
  a function from **Y** to **Z** = { *x, y, z* }.

- The composition function from **X** to **Z** is the function

$$f \circ g = \{ (1,y), (2,y), (3,z) \}$$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f o g
X
Z

Example

Let, $f(x) = \log_3 x$, and $g(x) = x^4$.

Find:

a) $f \circ g$ b) $g \circ f$

Solution:

a) $f \circ g = f(g(x)) = \log_3(x^4)$

b) $g \circ f = g(f(x)) = (\log_3 x)^4$

$\therefore$ *Note* : $f \circ g \neq g \circ f$

Example

Define, $f : \mathbf{Z} \rightarrow \mathbf{Z}$ and $g: \mathbf{Z} \rightarrow \mathbf{Z}$ by the rules $f(a) = 7a$ and $g(a) = a \bmod 5$ for all integers a.

Find:

a) $(g \circ f)(0)$

b) $(g \circ f)(1)$

c) $(g \circ f)(2)$

d) $(g \circ f)(3)$

e) $(g \circ f)(4)$

Exercise

Define, $f : \mathbf{Z} \rightarrow \mathbf{Z}$ and $g : \mathbf{Z} \rightarrow \mathbf{Z}$ by the rules $f(n) = n^3$, $g(n) = n-1$ for all integers n.

Find the compositions of the following:

a) $f \circ f$

b) $g \circ g$

c) $f \circ g$

d) $g \circ f$

e) Is $f \circ g = g \circ f$?