CHAPTER 2

(Part 1)

RELATIONS

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Relations on Sets

- The most basic relation is " = " (e.g. $x = y$)
- **Binary relation** is a subset of the Cartesian product of two sets.
- Example: **A** and **B** are two sets. Define a relation *R* from **A** to **B**.

$$(x,y) \in \mathbf{A}\times\mathbf{B} \text{ and } R \subseteq \mathbf{A}\times\mathbf{B},$$
$$x\ R\ y \leftrightarrow (x,y) \in R.$$

> - If $(x,y) \in R$, where $x \in$ **A** and $y \in$ **B**, can be written as: $x\ R\ y$ => x is related to y.
> - If $(x,y) \notin R$, where $x \in$ **A** and $y \in$ **B**, can be written as: $x\ \not R\ y$ => x is not related to y.

Relations on Sets (cont'd)

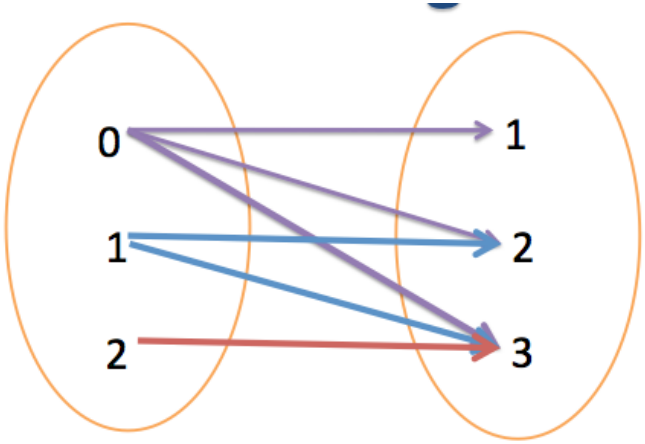
To show the relations, we can use one of the options:

a) **Use traditional notation:**

$0 < 1,\ 0 < 2,\ 0 < 3,\ 1 < 2,\ 1 < 3,\ 2 < 3$

b) **use set notation:** $R=\{(0,1),(0,2),(0,3),\ (1,2),(1,3),(2,3)\}$

c) use Arrow Diagrams:



$R=\{(0,1),(0,2),(0,3),\ (1,2),(1,3),(2,3)\}$

Example

Suppose that,

$\boldsymbol{A}$ = {Ipoh, Kangar, Segamat, Kemaman, Kuantan, Dungun}; and

$\boldsymbol{B}$ = {Terengganu, Johor, Perak, Perlis, Pahang}.

If $a \in A, b \in B$, and the relation between ***a*** and ***b*** is defined as "***a*** is a city in the state ***b***",

Then,

R = {(Ipoh, Perak), (Kangar, Perlis), (Segamat, Johor), (Kuantan, Pahang), (Kemaman, Terengganu), (Dungun, Terengganu)}.

Example

A and **B** are sets, where **A** = {1, 2, 3, 4}, ***B*** = {p, q, r}.

Define a relation *S* from **A** to **B** as follows: $S = \{(1, q), (2, r), (3, q), (4, p)\}$, and $S \subseteq \mathbf{A} \times \mathbf{B}$

Questions: i) Is 1 *S* q ? ii) Is 3 *S* p ?

Solution: i) Yes, ii) No.

Example

Define a relation L from **R** to **R** as follows: For all real numbers x and y,

$$x \; L \; y \Leftrightarrow x < y.$$

<u>Questions:</u>

i) Is 57 L 53 ?

ii) Is (-17) < -14 ?

iii) Is 143 L 143?

iv) Is (-35) L 1?

Example

Define a relation E from $\mathbf{Z}$ to $\mathbf{Z}$ as follows: For all $(m,n) \in \mathbf{Z}\times\mathbf{Z}$,

$$m \; E \; n \Leftrightarrow m - n \text{ is even.}$$

Questions:

i) Is 4 E 0?

ii) Is 2 E 6?

iii) Is 3 E (-3)?

iv) Is 5 E 2?

v) List 5 integers that are related by E to 1.

Exercise # 1

Let **A** = {3,4,5} and **B** = {2,4,6,8,10} and *R* be the relation from **A** *to* **B** defined by,

$$(a,b) \in R, \text{ if } 2a \leq b+1$$

Write the ordered pair of *R*.

Domain & Range

Let *R,* a relation from ***A*** to ***B.***

The **Domain** of relations *R* is the set of all **first** elements in ordered pairs (a,b) which is the element of *R*,

$$\{\, a \in A \mid (a,b) \in R \text{ for any } b \in B \,\}$$

The **Range** of relations *R* is the set of all **second** elements in ordered pairs (a,b) which is the element of *R*,

$$\{\, b \in B \mid (a,b) \in R \text{ for any } a \in A \,\}$$

Example

Let R be a relation on $\boldsymbol{X} = \{1, 2, 3, 4\}$ defined by $(x,y) \in R$ *if* $x \le y$, *and* $x,y \in \boldsymbol{X}$.

Then,
$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$

The **domain and range** of ***R*** are both equal to ***X***.

Example

Let $\boldsymbol{X} = \{2, 3, 4\}$ and $\boldsymbol{Y} = \{3, 4, 5, 6, 7\}$

If we **define a relation** *R* **from** $\boldsymbol{X}$ **to** $\boldsymbol{Y}$ **by,**

$(x,y) \in \boldsymbol{R}$ **if** y/x ***(****with zero* remainder).

We obtain,

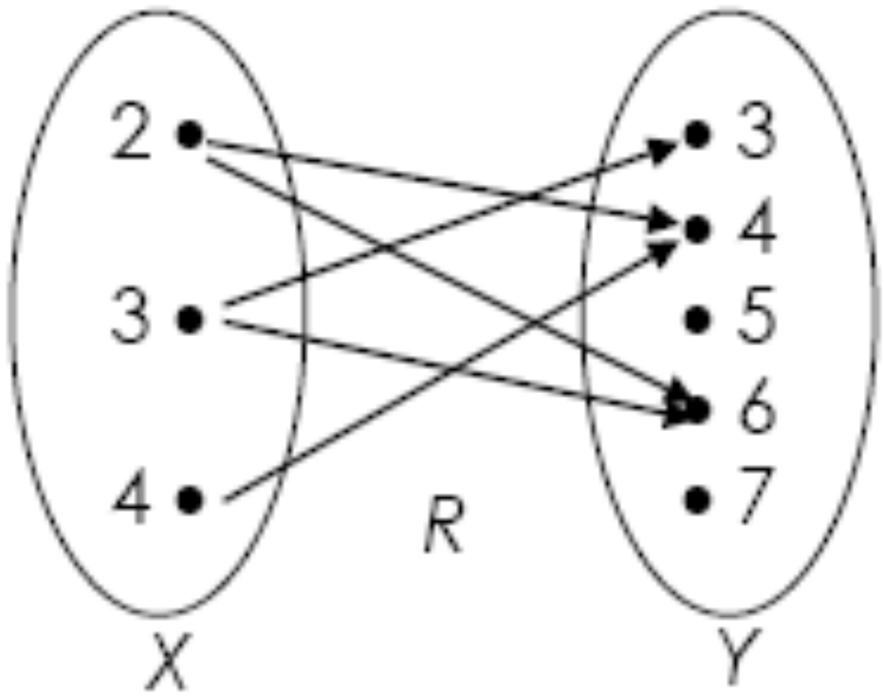
$R = \{ (2,4), (2,6), (3,3), (3,6), (4,4) \}$

The domain of R is $\{2,3,4\}$

The **range** of R is $\{3,4,6\}$

Example (cont'd)

$$R = \{ (2,4), (2,6), (3,3), (3,6), (4,4) \}$$



Arrow diagram

Exercise # 2

Find range and domain for:

(i) The relation $\boldsymbol{R}$ ={(1,2), (2,1), (3,3), (1,1), (2,2)} on $\boldsymbol{X}$ ={1, 2, 3}

(ii) The relation $\boldsymbol{R}$ on {1, 2, 3, 4} defined by $(x,y) \in R$ if $x^2 \leq y$

Directed Graph

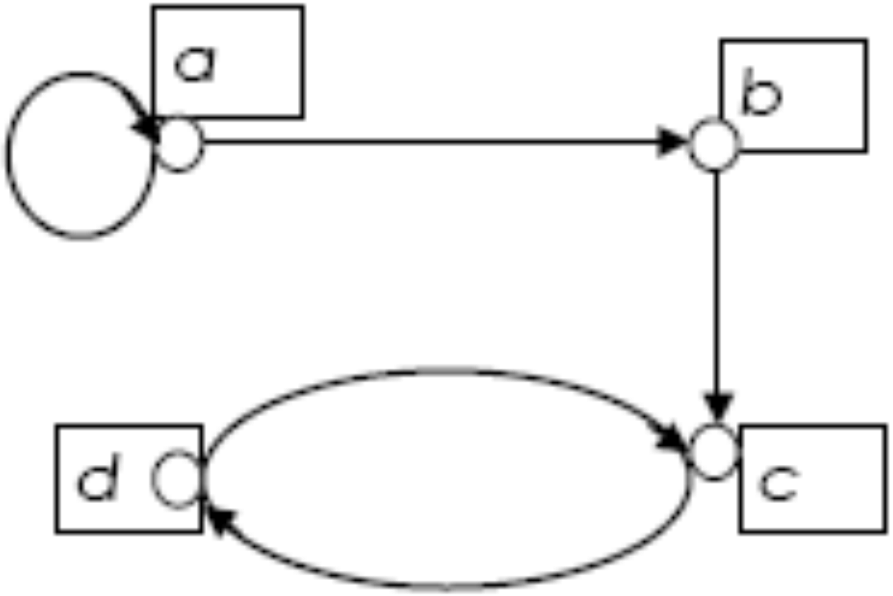
An informative way to **picture a relation** on a set is to draw its directed graph (digraph).

- ❖ Let R be a relation on a finite set A.
- ❖ Draw dots (vertices) to represent the elements of A.
- ❖ If the element $(a,b) \in R$, draw an arrow (called a directed edge) from a to b.

Example

The relation R on $A = \{a, b, c, d\}$,

$R = \{(a, a), (a, b), (c, d), (d, c), (b, c)\}$



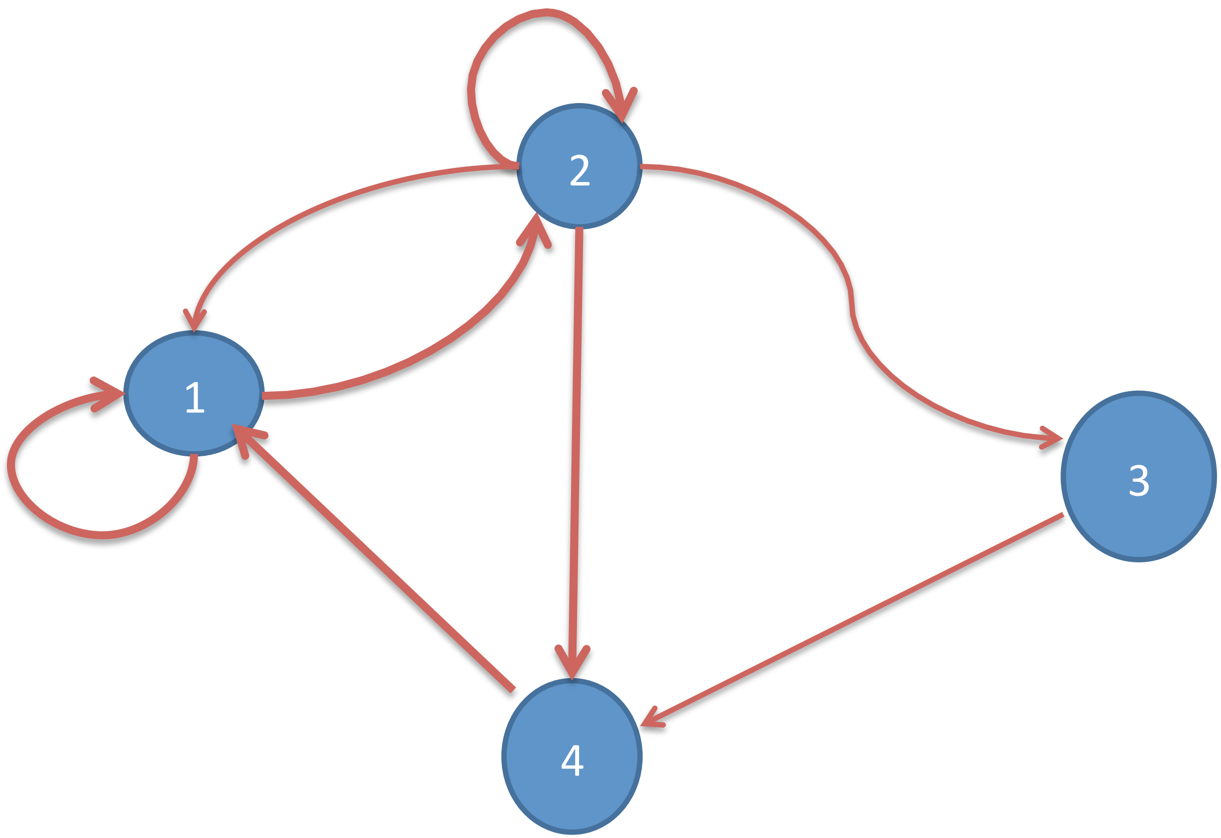
Example

Let

$\boldsymbol{A} = \{ 1,2,3,4\}$, and

$\boldsymbol{R} = \{ (1,1), (1,2), (2,1), (2,2), (2,3), (2,4), (3,4) , (4,1)\}$

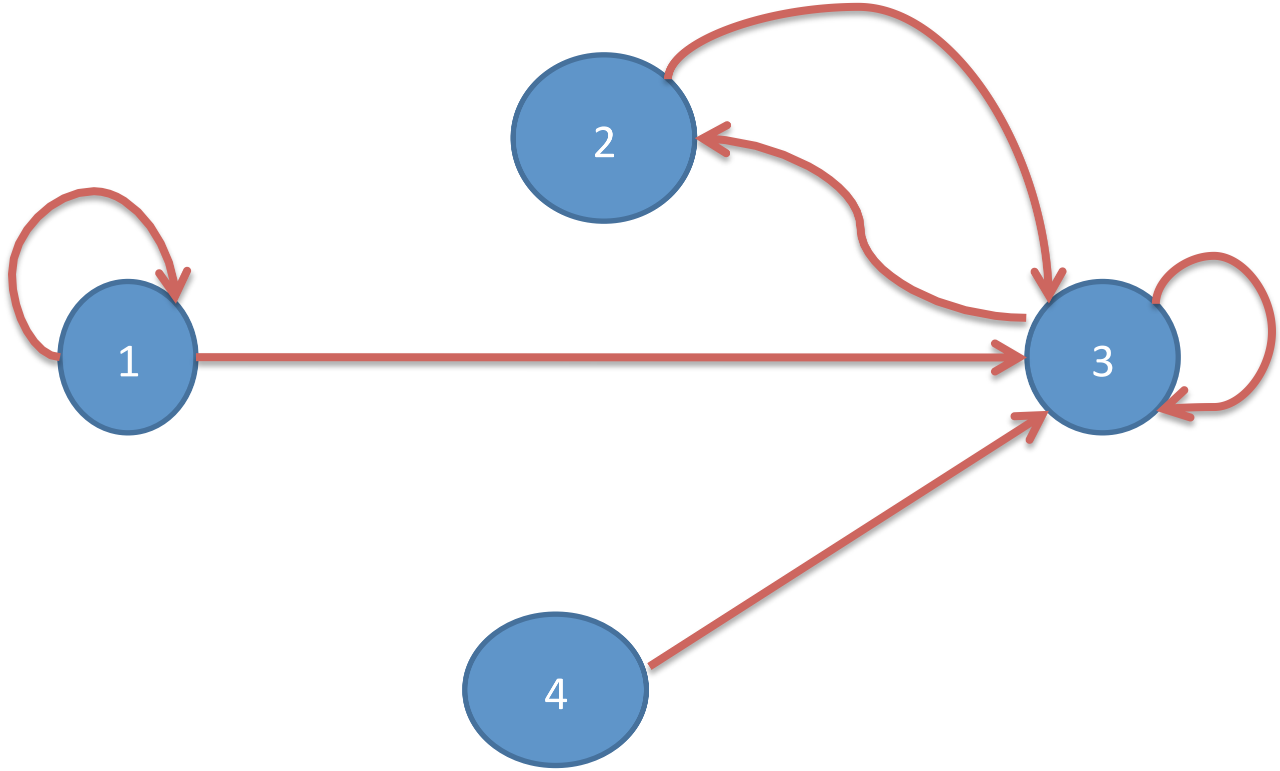
Draw the digraph of ***R***.



Digraph

Example

Find the relation determined by digraph below.



Since a_i ***R*** a_j if and only if there is an edge from a_i to a_j, so

R = { (1,1), (1,3), (2,3), (3,2), (3,3), (4,3)}

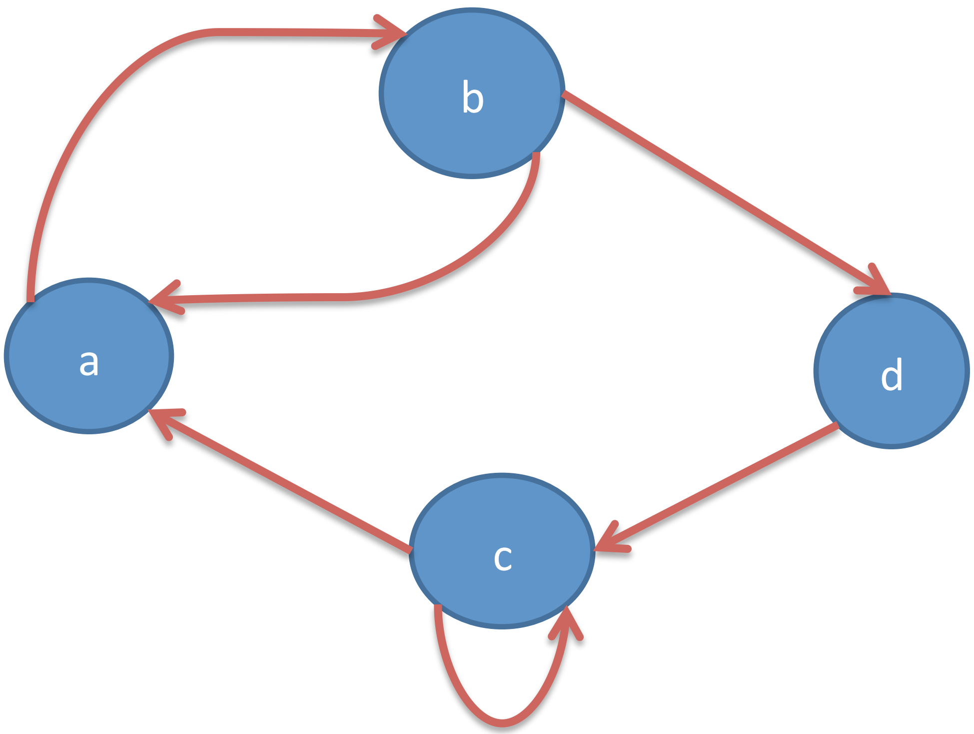
Exercise #3

Draw the **digraph** of the relation.

(i) ***R***={(a,6), (b,2), (a,1), (c,1)}

(ii) The relation ***R*** on {1, 2, 3, 4} defined by $(x,y) \in$ ***R*** if $x^2 \geq y$

iii) Based on the digraph below, write the relation as a set of ordered pair.



Representing Relations Using Matrices

A relation is defined to be a set of pairs, but it can also be represented in other ways, e.g., by **matrix**.

- A matrix is simply a rectangular array of numbers.
- If a matrix has n rows and m columns, we call it an n x m matrix.
- In a matrix M, the number at the intersection of its i-th row and j-th column is written as M_{ij}.

A matrix is a convenient way to represent a relation *R* from ***A*** to ***B***.

- Label the rows with the elements of ***A*** (in some arbitrary order).
- Label the columns with the elements of ***B*** (in some arbitrary order).

Let $A=\{a_1, a_2, \ldots, a_n\}$ and $B=\{b_1, b_2, \ldots, b_p\}$ be finite nonempty sets.

Let R be a relation from A into B.

Let $M_R = [m_{ij}]_{nxp}$ be the Boolean nxp matrix, where

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{otherwise} \end{cases}$$

$$M_R = \begin{bmatrix} m_{11} & m_{12} & .. & .. & m_{1p} \\ m_{21} & m_{22} & .. & .. & m_{2p} \\ \vdots & \vdots & .. & .. & \vdots \\ \vdots & \vdots & .. & .. & \vdots \\ m_{n1} & m_{n2} & .. & .. & m_{np} \end{bmatrix}$$

Example

Let **A** = {1, 3, 5} and **B** = {1,2}. Let *R* be a relation from **A** to **B** and *R* = {(1,1), (3,2), (5,1)}. Then the matrix represent *R* is:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Example

- The relation,
  $R = \{ (1,b),(1,d),(2,c),(3,c),(3,b),(4,a) \}$
  from, $X = \{ 1, 2, 3, 4 \}$ to $Y = \{ a, b, c, d \}$

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array} \qquad \begin{array}{c} \\ 2 \\ 3 \\ 4 \\ 1 \end{array} \begin{array}{c} \begin{array}{cccc} d & b & a & c \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

Example

The matrix of the relation R from { 2, 3, 4 } to { 5, 6, 7, 8 } defined by

$x \, R \, y$ if x divides y

$$\begin{array}{c c} & \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{array}{c} 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Example

Let A={ a, b, c, d }

Let R be a relation on A.

R = { (a,a),(b,b),(c,c),(d,d),(b,c),(c,b) }

$$
\begin{array}{cc}
 & \begin{array}{cccc} a & b & c & d \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{array}
$$

Example

Let $\boldsymbol{A}$ = {1, 2, 3,4} and R is a relation on $\boldsymbol{A}$. Suppose R = {(1,2), (1,3), (1,4), (2,3), (2,4) , (3,4)}

i) What is R (represent)?

ii) What is the matrix for R?

In Degree and Out Degree

If R is a relation on a set $\boldsymbol{A}$ and $a \in \boldsymbol{A}$, then the in-degree of a (relative to relation R) is the number of $b \in \boldsymbol{A}$ such that $(b,a) \in R$.

The out degree of a is the number of $b \in \boldsymbol{A}$ such that $(a,b) \in R$.

Meaning that, in terms of the digraph of $\boldsymbol{R}$, is that
the in-degree of a vertex is
"the number of edges terminating at the vertex"
The out-degree of a vertex is
" the number of edges leaving the vertex"

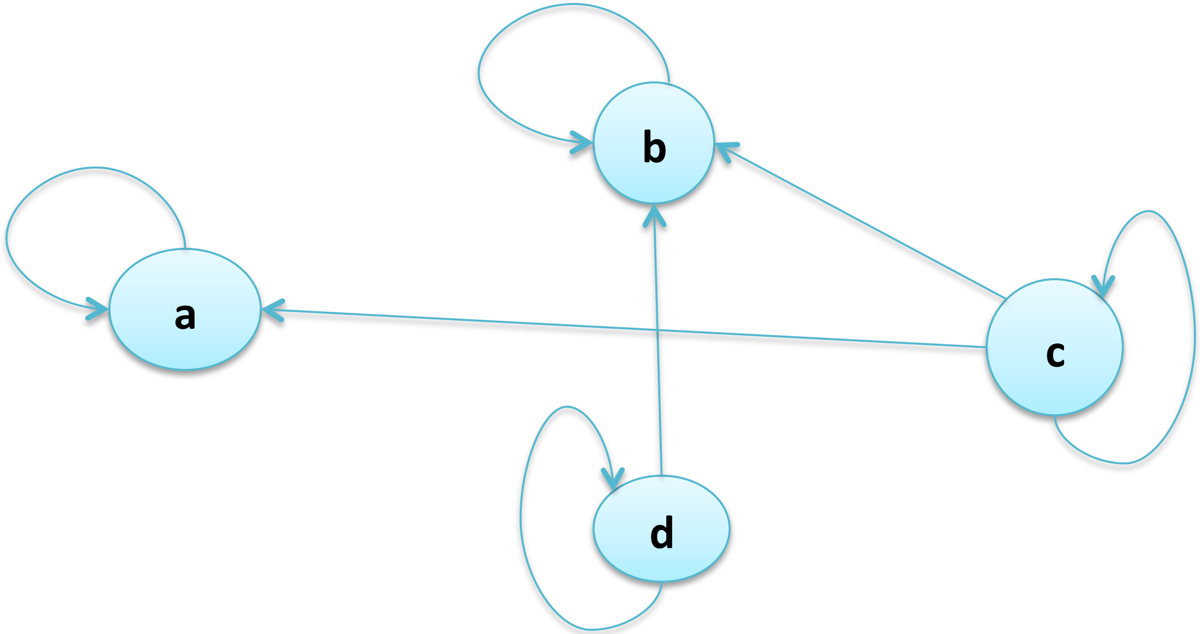
Example

Let $\boldsymbol{A} = \{a, b, c, d\}$, and let R be the relation on $\boldsymbol{A}$ that has the matrix (see below). Construct the digraph of R, and list in-degrees and out-degrees of all vertices.

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Solution

	a	b	c	d
In-degree	2	3	1	1
Out-degree	1	1	3	2



Exercise #4

Let $\boldsymbol{A}$ = {1, 4, 5} and let R be given by the digraph shown below. Find M_R and R. Then, list the in-degree and the out-degree.

Example

An airline services the five cities c_1, c_2, c_3, c_4 and c_5. Table below gives the cost (in dollars) of going from c_i to c_j. Thus the cost of going from c_1 to c_3 is \$100, while the cost of going from c_4 to c_2 is \$200

To from	c_1	c_2	c_3	c_4	c_5
c_1		140	100	150	200
c_2	190		200	160	220
c_3	110	180		190	250
c_4	190	200	120		150
c_5	200	100	200	150	

If the relation ***R*** on the set of cities $A = \{c_1, c_2, c_3, c_4, c_5\} : c_i\, \boldsymbol{R}\, c_j$ if and only if the cost of going from c_i to c_j is defined and less than or equal to \$180. Find ***R***.

Solution:

R = $\{(c_1,c_2), (c_1,c_3), (c_1,c_4), (c_2,c_4),$
$(c_3, c_1), (c_3,c_2), (c_4,c_3), (c_4,c_5), (c_4,c_5),$
$(c_5,c_2), (c_5,c_4)\}$

Characteristics of Relations

Reflexive Relations

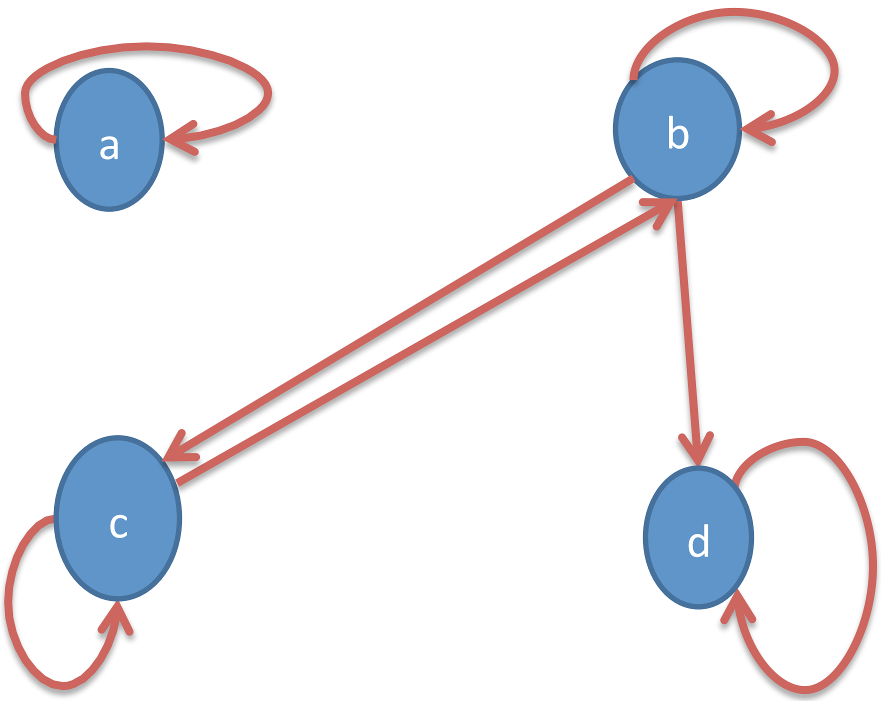
- Let R be a relation on a set $\boldsymbol{A}$**.** Then R is called **reflexive** if $(x,x) \in R$ for every $x \in \boldsymbol{A}$ that is, if xRx for all $x \in \boldsymbol{A}$.

- The matrix of reflexive relation must have the value **1** on its diagonal.

- Digraph R will have a loop at every vertex.

Reflexive Relations

Digraph



Matrix

$$
\begin{array}{c|cccc}
 & a & b & c & d \\
\hline
a & 1 & 0 & 0 & 0 \\
b & 0 & 1 & 1 & 0 \\
c & 0 & 1 & 1 & 0 \\
d & 0 & 0 & 0 & 1
\end{array}
$$

Irreflexive Relations

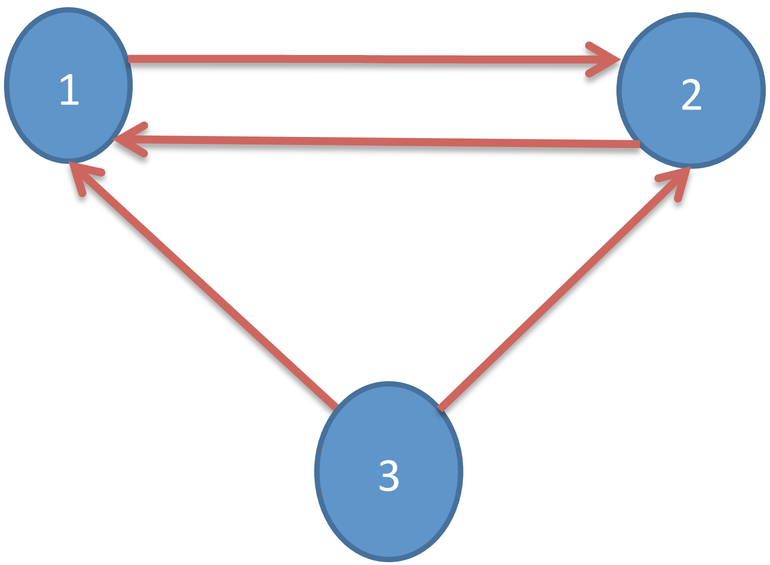
- Let *R* be a relation on a set ***A.*** Then *R* is called **irreflexive** if $(x,x) \notin R$ for every $x \in$ ***A.***

- The matrix of reflexive relation must have the value **0** on its diagonal.

- Digraph *R* will not have any loop at every vertex.

Irreflexive Relations

Digraph



Matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

Not Reflexive Relations

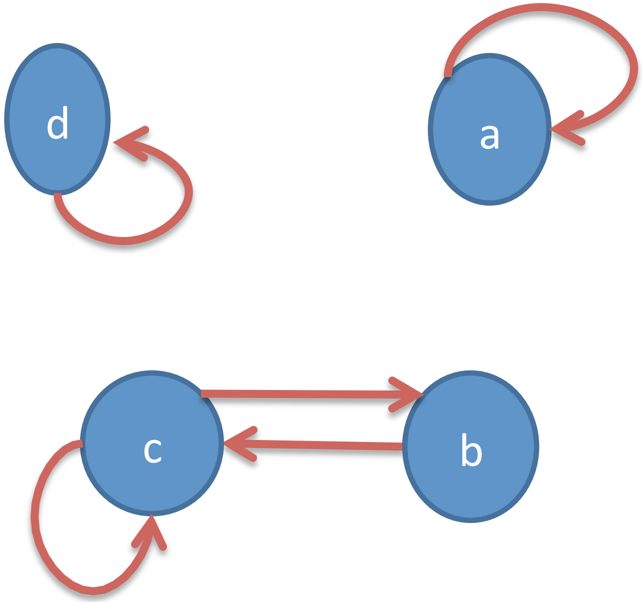
- Let R be a relation on a set $\boldsymbol{A}$**.** Then R is called **not reflexive** if there exist $(x,x) \notin R$ for $x \in \boldsymbol{A}$.

- The matrix of reflexive relation will have the value **0 and 1** on its diagonal.

- Digraph R will have some loop at some vertices.

Not Reflexive Relations

Digraph



Matrix

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Example

The relation R on **X** = {1, 2, 3, 4} defined by $(x,y) \in R$ if $x \leq y$, $x,y \in$ **X.** Determine whether R is a reflexive relation.

Solution:

For each element $x \in$ **X**, $(x,x) \in R$

=> (1,1), (2,2), (3,3), (4,4) are each in R.

Example

The relation $R = \{(a,a), (b,c), (c,b), (d,d)\}$ on **X** = $\{a, b, c, d\}$ is not reflexive. Why?

Solution:

This is because $b \in$ **X**, but $(b,b) \notin R$. Also $c \in$ **X**, but $(c,c) \notin R$.

Exercise 1

- Consider the following relations on the set {1,2,3}:

R_1={(1,1), (1,2), (2,1), (2,2), (3,1), (3,3)}

R_2={(1,1), (1,3), (2,2), (3,1)}

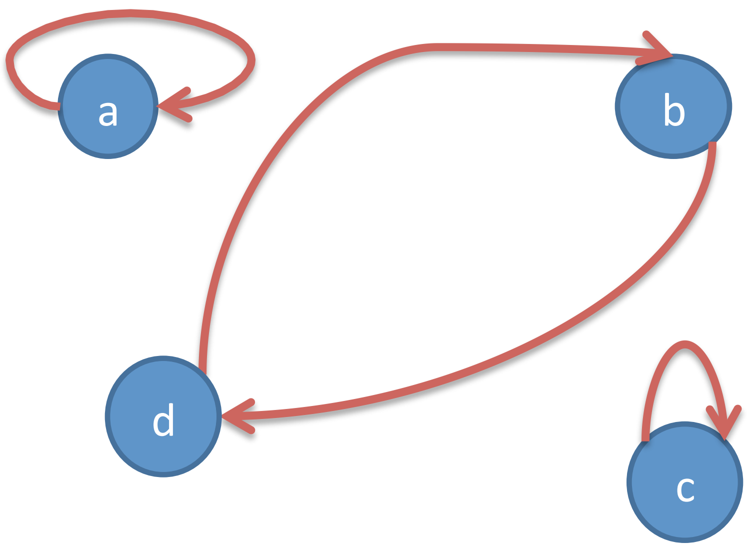
R_3={(2,3)}

R_4={(1,1)}

Which of them are reflexive?

Exercise 2

The relation *R* on ***X***={a, b, c, d} given by the below digraph. Is *R* a reflexive relation? Justify your answer.



Exercise 3

Let **A** = { 1,2,3,4). Construct the matrix of relation of R. Then, determine whether the relation is reflexive, not reflexive or irreflexive.

(i) R_1 = { (1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,4))

(ii) R_2 = { (1,1), (1,2), (1,3), (3,1), (3,3), (4,4)}

(iii) R_3 ={ (1,1), (1,2), (1,3), (1,4), (3,1), (3,2), (3,3), (3,4), (4,2)}

(iv) R_4 ={ (1,2), (1,3), (1,4), (3,2), (3,4), (4,2)}

Symmetric Relations

A relation R on a set **X** is called **symmetric** if for all $x,y \in$ **X**, if $(x,y) \in R$, then $(y, x) \in R$.

$$\forall x,y \in \mathbf{X}, (x,y) \in R \rightarrow (y,x) \in R$$

Let M be the matrix of relation R.

The relation R is symmetric if and only if
for all i and j, the ij-th entry of M is equal to the ji-th entry of M.

Symmetric Relations (cont.)

The matrix of relation M_R is symmetric if $M_R = M_R^T$

example

$$M_R = \begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} = M_R^T$$

Symmetric Relations (cont.)

The digraph of a symmetric relation has the property that whenever there is a directed edge from v to w, there is also a directed edge from w to v.

Example

- The relation $R = \{ (a,a), (b,c), (c,b), (d,d) \}$ on $X = \{ a, b, c, d \}$

$$\boxed{\begin{array}{l}(b,c) \in R \\ (c,b) \in R\end{array}}$$

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

symmetric

Antisymmetric

- A relation R on set **A** is antisymmetric if $a \neq b$, whenever aRb, then bRa.
- In other word if whenever $a\mathbf{R}b$, then $b\mathbf{R}a$ then it implies that $a=b$

$$\forall a,b \in \mathbf{A},\ (a,b) \in R \land a \neq b \rightarrow (b,a) \notin \mathbf{R}$$

Or

$$\forall a,b \in \mathbf{A},\ (a,b) \in R \land (b,a) \in R \rightarrow a = b$$

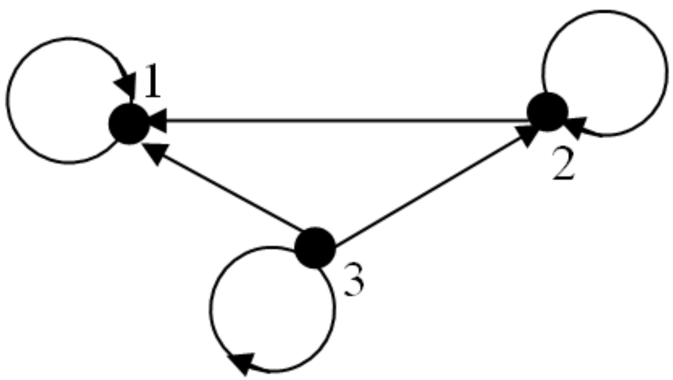
Antisymmetric (cont.)

- Matrix $M_R = [M_{ij}]$ of an antisymmetric relation R satisfies the property that if $i \neq j$, then $m_{ij}=0$ or $m_{ji}=0$.
- If R is antisymmetric relation, then for different vertices i and j there cannot be an edge from vertex i to vertex j and an edge from vertex j to vertex i.
- At least one directed relation and one way.

Example

- Let R be a relation on **A** = {1, 2, 3} defined as $(a, b) \in R$ if $a \geq b$; $a, b \in$ **A** is an antisymmetric relation because for all $a, b \in$ ***A***, $(a, b) \in$ ***R*** and $a \neq b$, then $(b, a) \notin R$.
- For example: $(3, 2) \in R$ but $(2, 3) \notin R$

  $(3, 3) \in R$ and $(3, 3) \in R$ implies $a = b$



Example

- The relation R on $X = \{1, 2, 3, 4\}$ defined by,

$$(x,y) \in R \quad \text{if } x \le y,\ x,y \in X$$

$$\boxed{\begin{array}{l}(1,2) \in R \\ (2,1) \notin R\end{array}}$$

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

antisymmetric

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Example

The relation

$R = \{ (a,a), (b,b), (c,c) \}$

on $X = \{ a, b, c \}$

$$\begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

R has no members of the form (x,y) with $x \neq y$, then R is antisymmetric

Asymmetric

- Let R be a relationon on a set **A.** Then R is called **asymmetric** if $\forall a,b \in \mathbf{A}$, if $(a,b) \in R$, then $(b, a) \notin R$.

$$\boxed{\forall a,b \in \mathbf{A}, (a,b) \in R \rightarrow (b,a) \notin R}$$

- In this sense, a relation is asymmetric if and only if it is both antisymmetric and irreflexive.
- The matrix $M_R = [m_{ij}]$ of an asymmetric relation R satisfies the property that,

  If $m_{ij} = 1$ then $m_{ji} = 0$

  $m_{ii} = 0$ for all i (**the main diagonal of matrix M_R consists entirely of 0's or otherwise**)

Asymmetric (cont.)

- If *R* is asymmetric relation, then the digraph of *R* cannot simultaneously have an edge from vertex *i* to vertex *j* and an edge from vertex *j* to vertex *i*
- All edges are “one way street”.
- $M_R \neq M_R^T$

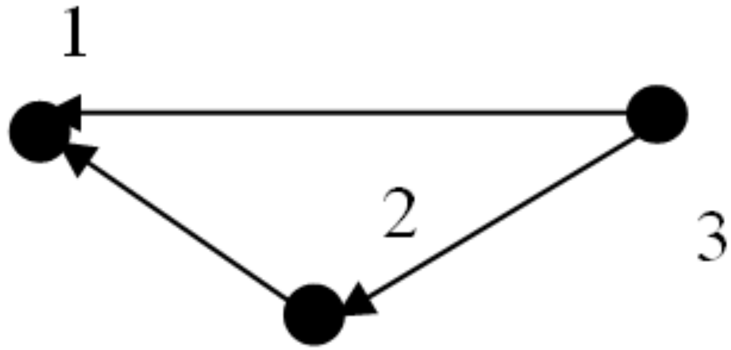
Example

Let R be the relation on **A** = {1, 2, 3} defined by $(a, b) \in R$ if $a > b$; $a,b \in$ **A** is an asymmetric relation because,

$$(2, 1) \in R \text{ but } (1, 2) \notin R$$

$$(3, 1) \in R \text{ but } (1, 3) \notin R$$

$$(3, 2) \in R \text{ but } (2, 3) \notin R$$



Not Symmetric

Let R be a relation on a set **A**. Then R is called ***not symmetric***, if for all $a, b \in$ **A,** if $(a, b) \in R$, there exist $(b, a) \notin R$.

$$\exists\, a,b \in \mathbf{A}, (a, b) \in R \wedge (b, a) \notin R$$

Not Symmetric AND Not Antisymmetric

Let R be a relation on a set **A**. Then R is called ***not symmetric*** and ***not antisymmetric***, if and only if

$$\exists\, a,b \in \mathbf{A},\ (a, b) \in R \wedge (b, a) \notin R$$

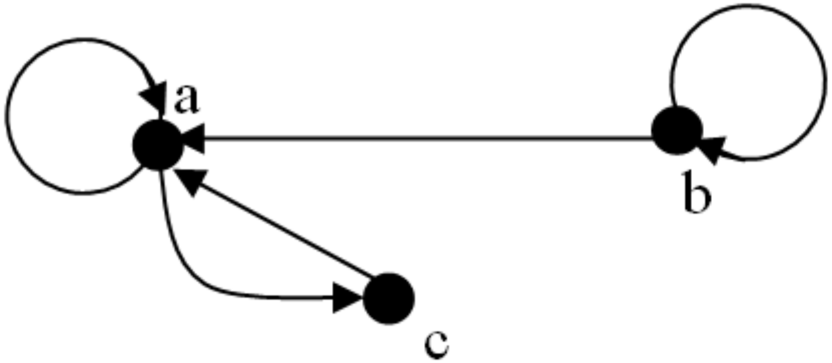
AND

$$\exists\, a,b \in \mathbf{A},\ (a, b) \in R \wedge a \neq b \wedge (b, a) \in R$$

Example

Relation $R = \{(a, c), (b, b), (c, a), (b, a), (a, a)\}$ on **A** $= \{a, b, c\}$ is not symmetric and not antisymmetric relation because there is,

$$(a,c), (c,a) \in R \text{ and also } (b,a) \in R \text{ but } (a,b) \notin R$$



Example

- The relation $R = \{ (a,a), (b,c), (c,b), (d,d) \}$ on $X = \{ a, b, c, d \}$

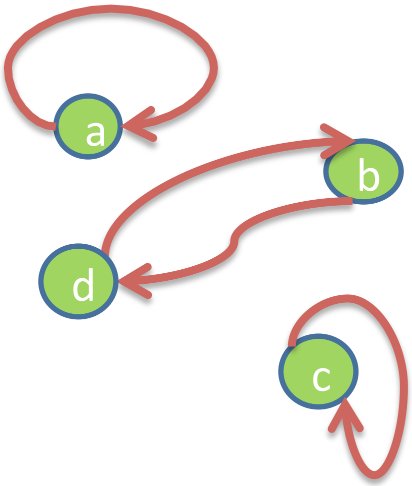
$$\boxed{\begin{array}{l}(b,c) \in R \\ (c,b) \in R\end{array}}$$

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Not Symmetric and

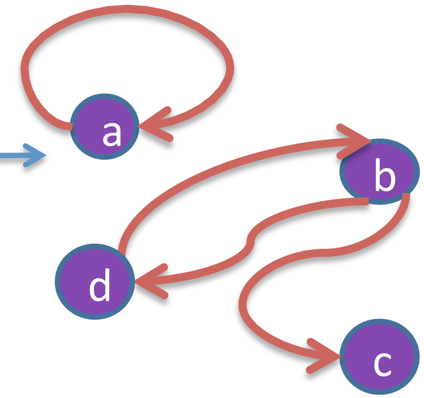
not antisymmetric

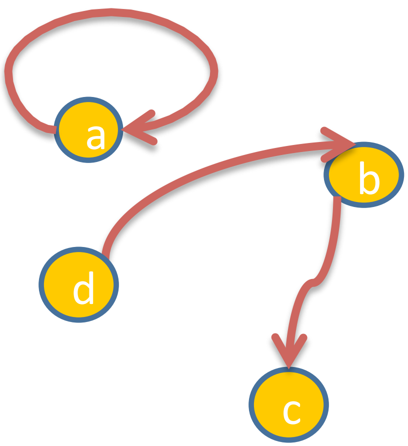
Summary



Symmetric

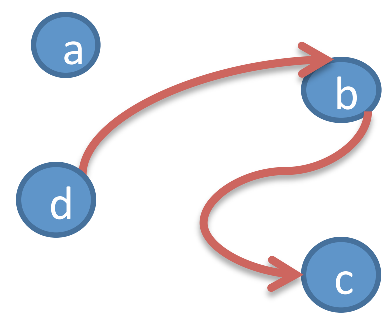
Not Symmetric





Antisymmetric

Asymmetric



Exercise 4

Let **A**=$\{1,2,3,4\}$ and let $R = \{(1,2), (2,2), (3,4), (4,1)\}$

Determine whether R is symmetric, asymmetric or antisymmetric.

Exercise 5

Let **A**=**Z**, the set of integers and let,

$$R = \{(a,b) \in \mathbf{A}\times\mathbf{A} \mid a<b\}.$$

So that *R* is the relation "less than". Is *R* symmetric, asymmetric or antisymmetric?

Exercise 6

- Consider the following relations on the set {1,2,3}:

$R_1=\{(1,1), (1,2), (2,1), (2,2), (3,1), (3,3)\}$

$R_2=\{(1,1), (1,3), (2,2), (3,1)\}$

$R_3=\{(2,3)\}$

$R_4=\{(1,1)\}$

Which of them are symmetric?

Which of them are antisymmetric?

Exercise 7

Let **A** = { 1,2,3,4). For each question below (i – iii), construct the matrix of relation of *R*. Then, determine whether the relation is symmetric, asymmetric, not symmetric or antisymmetric.

(i) $R = \{ (1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,4))$

(ii) $R = \{(1,1), (1,2), (1,3), (3,1), (3,3), (4,4)\}$

(iii) $R =\{ (1,1), (1,2), (1,3), (1,4), (3,2), (3,3), (3,4), (4,2)\}$

Transitive Relations

A relation *R* on set **A** is **transitive** if for all $a,b,c \in$ **A**, if (a,b) and $(b,c) \in R$, then $(a,c) \in R$

$$\boxed{\forall (a,b) \in A, (a,b) \in R \wedge (b,c) \in R \rightarrow (a,c) \in R}$$

Transitive Relations (cont.)

Normally, the matrix of the relation M_R is transitive if

$$M_R \otimes M_R = M_R$$

$\otimes$: product of Boolean

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How to do Product of Boolean

$$(1 \wedge 1) \vee (0 \wedge 0) \vee (1 \wedge 0) \vee (0 \wedge 0) = 1$$

$$\begin{array}{c|c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \otimes \begin{array}{c|c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} = \begin{array}{c|c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$(1 \wedge 0) \vee (0 \wedge 1) \vee (1 \wedge 1) \vee (0 \wedge 0) = 1$$

Example

- Consider the following relations on the set {1,2,3}:

R_1={(1,1),(1,2),(2,3)}

R_2={(1,2),(2,3),(1,3) }

Which of them is transitive?

Solution:

R_1 is not transitive because $(1, 2) \in R \wedge (2,3) \in R$, but $(1,3) \notin R$

R_2 is transitive because $(1, 2) \in R \wedge (2, 3) \in R \rightarrow (1,3) \in R$

Example

The relation R on **A**={a,b,c,d} is R = {(a,a), (b,b), (c,c), (d,d), (a,c), (c,b)} is not transitive. The matrix of relation $M_{R,}$

$$M_R = \begin{array}{c|cccc} & a & b & c & d \\ a & 1 & 0 & 1 & 0 \\ b & 0 & 1 & 0 & 0 \\ c & 0 & 1 & 1 & 0 \\ d & 0 & 0 & 0 & 1 \end{array}$$

$M_R \otimes M_R \neq M_R$

The product of boolean,

$$\begin{array}{c}\\ a\\ b\\ c\\ d\end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\end{array} \otimes \begin{array}{c}\\ a\\ b\\ c\\ d\end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\end{array} = \begin{array}{c}\\ a\\ b\\ c\\ d\end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\end{array}$$

Note that,(a,c) and (c,b) $\in R$, (a,b)$\notin R$

Exercise 8

Let R be a relation on **A**={1,2,3} is defined by $(a,b) \in \boldsymbol{R}$ if $a \leq b$, $a,b \in \boldsymbol{A}$.

i) Find R.

ii) Is R a transitive relation?

Equivalence Relations

Relation R on set **A** is called an **equivalence relation** if it is a **reflexive, symmetric and transitive**.

Example: Let R={(1,1), (1,3), (2,2), (3,1), (3,3)} on {1,2,3}, the matrix of the relation M_R,

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

All the main diagonal matrix elements are 1, so it is **reflexive**.

Equivalence Relations (cont.)

The transpose matrix M_R, M_R^T is equal to M_R, so R is **symmetric**.

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array} \qquad M_R^T = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

The product of Boolean show that the matrix is **transitive**.

$$\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{matrix} \otimes \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Partial Order Relations

Relation R on set **A** is called a **partial order relation** if it is a reflexive, antisymmetric and transitive.

Example: Let R be a relation on a set **A**={1,2,3} defined by $(a,b) \in R$ if $a \leq b$, $a,b \in R$.

R ={(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)}

R is reflexive, antisymmetric and transitive.

So R is a partial order relation.

Exercise 9

i) The relation R on the set {1,2,3,4,5} defined by the rule $(x,y) \in R$ if $x+y \leq 6$.

(i) List the elements of R.

(ii) Find the domain of R.

(iii) Find the range of R.

(iv) Is the relation of R reflexive, symmetric, asymmetric, antisymmetric, transitive, and/or equivalence relation or partial order?