

## CHAPTER 2

# RELATIONS & FUNCTIONS

# PART 1

# RELATIONS

# Definition

- If  $R$  is a relation from set  $A$  into itself, we say that  $R$  is a relation on  $A$ .

$$a \in A, b \in A \ (a, b) \in A \times A \text{ and } R \subseteq A \times A$$

- Example

Let  $A = (1, 2, 3, 4, 5)$  and  $R$  be defined by  $a, b \in A$ ,  
 $aRb \leftrightarrow b - a = 2$

$$R = \{(1, 3), (2, 4), (3, 5)\}$$

# Example

*Let  $A = \{ 1, 2, 3, 4 \}$  and  $B = \{ p, q, r \}$*

*$R = \{ (1, q), (2, r), (3, q), (4, p) \}$*

*$R \subseteq A \times B$*

*$R$  is the relation from  $A$  to  $B$*

*$1Rq$  ( 1 is related to  $q$  )*

*~~$3Rp$~~  ( 1 is not related to  $p$  )*

# Relations

- Binary relations:  $xRy$   
 On sets  $x \in X$   $y \in Y$   
 $R \subseteq X \times Y$
- Example:  
 “less than” relation  
 from  $A = \{0, 1, 2\}$  to  
 $B = \{1, 2, 3\}$

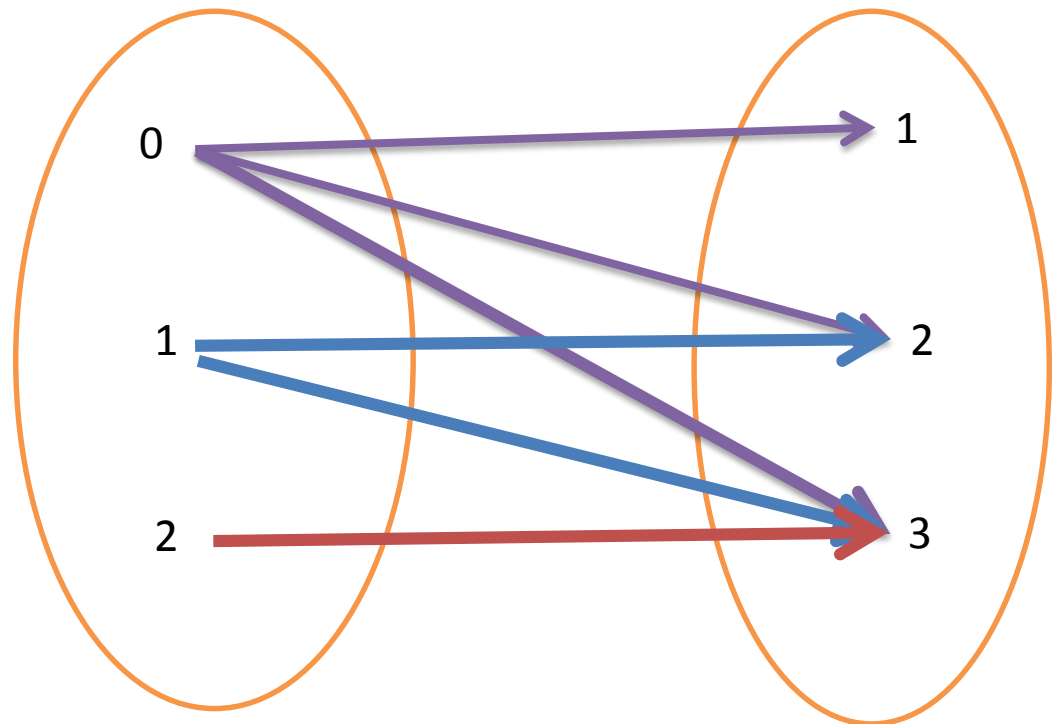
## Use traditional notation

$0 < 1, 0 < 2, 0 < 3, 1 < 2,$   
 $1 < 3, 2 < 3$

## Use set notation

$A \times B = \{(0, 1), (0, 2), (0, 3), (1,$   
 $1), (1, 2), (1, 3), (2, 1), (2, 2$   
 $), (2, 3)\}$

## Use Arrow Diagrams



$R = \{(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3)\}$

# Example

$A = \{\text{New Delhi, Ottawa, London, Paris, Washington}\}$

$B = \{\text{Canada, England, India, France, United States}\}$

Let  $x \in A, y \in B$ .

Define the relation between  $x$  and  $y$  by “ $x$  is the capital of  $y$ ”

$R = \{(\text{New Delhi, India}), (\text{Ottawa, Canada}), (\text{London, England}), (\text{Paris, France}), (\text{Washington, United States})\}$

# Domain and Range

Let  $R$ , a relation from  $A$  to  $B$ .

The set,  $\{ a \in A \mid (a,b) \in R \text{ for some } b \in B \}$   
is called the **domain** of  $R$ .

The set,  $\{ b \in B \mid (a,b) \in R \text{ for some } a \in A \}$   
is called the **range** of  $R$ .

# Example

Let  $R$  be a relation on  $X = \{1, 2, 3, 4\}$  defined by  $(x, y) \in R$  if  $x \leq y$ , and  $x, y \in X$ .

Then,

$$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$$

The **domain and range** of  $R$  are both equal to  $X$ .



# Example

Let  $X = \{2, 3, 4\}$  and  $Y = \{3, 4, 5, 6, 7\}$

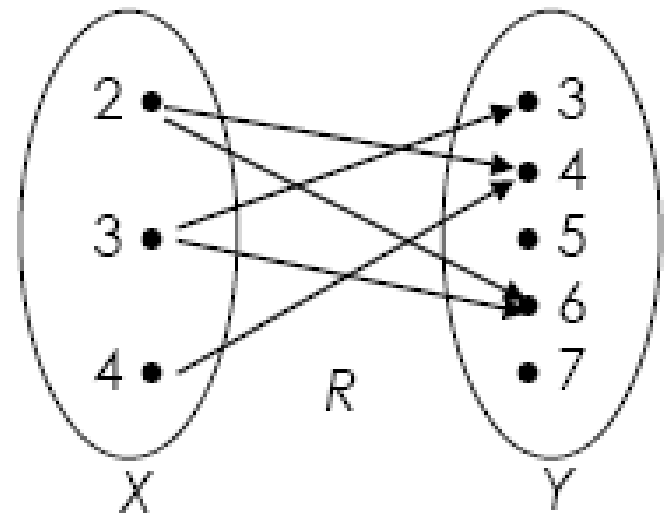
If we **define a relation  $R$  from  $X$  to  $Y$  by,**

**$(x, y) \in R$  if  $y/x$**  (with zero remainder)

We obtain,

$R = \{ (2,4), (2,6), (3,3), (3,6), (4,4) \}$

$$R = \{ (2,4), (2,6), (3,3), (3,6), (4,4) \}$$



Arrow diagram

The **domain** of  $R$  is  $\{2,3,4\}$

The **range** of  $R$  is  $\{3,4,6\}$

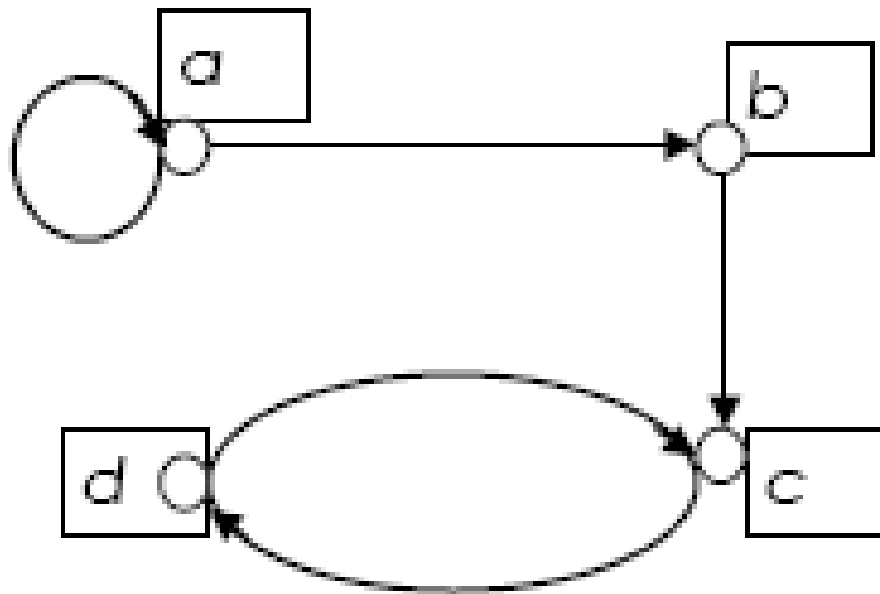
# Diagraph

An **informative way** to **picture a relation** on a set is to draw its **digraph**.

- ❖ Let  $R$  be a relation on a finite set  $A$ .
- ❖ Draw dots (vertices) to represent the elements of  $A$ .
- ❖ If the element  $(a,b) \in R$ , draw an arrow (called a directed edge) from  $a$  to  $b$

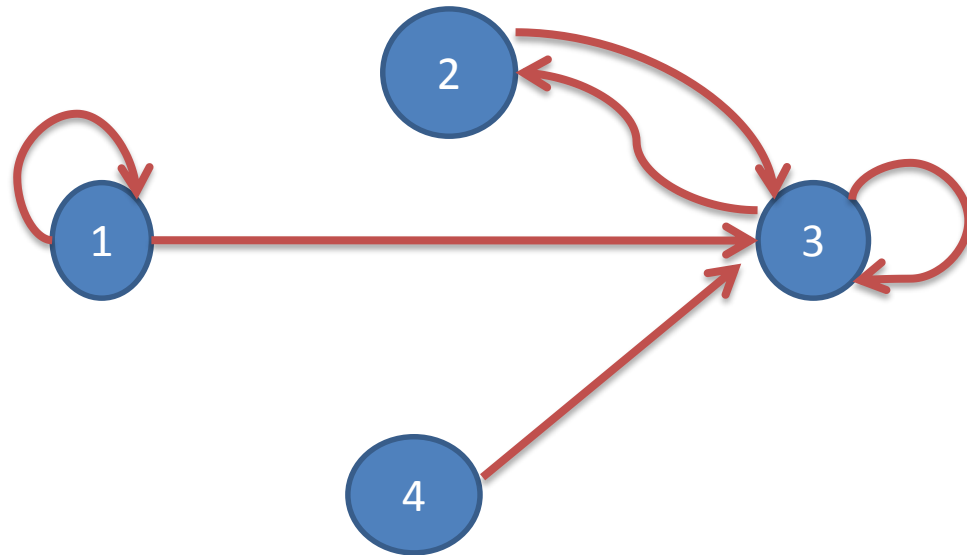
# Example

The relation  $R$  on  $A = \{a, b, c, d\}$ ,  
 $R = \{(a, a), (a, b), (c, d), (d, c), (b, c)\}$

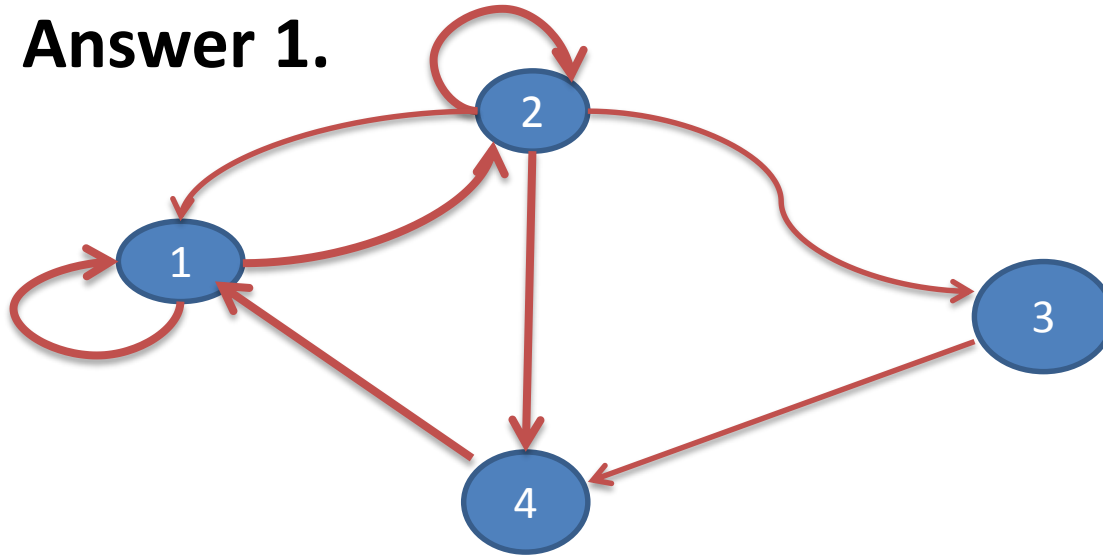


# Exercise

1. Let  $A = \{1, 2, 3, 4\}$  and  $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (2, 4), (3, 4), (4, 1)\}$ . Draw the digraph of  $R$ .
2. Find the relation determined by digraph below



**Answer 1.**



**Answer 2.**

Since  $a_i R a_j$  if and only if there is an edge from  $a_i$  to  $a_j$ ,  
 so  $R = \{ (1,1), (1,3), (2,3), (3,2), (3,3), (4,3) \}$

# Matrices of Relations

A matrix is a convenient way to represent a relation  $R$  from  $A$  to  $B$ .

- Label the rows with the elements of  $A$  (in some arbitrary order)
- Label the columns with the elements of  $B$  (in some arbitrary order)

# Matrices of Relations

- Let  $M_R = [m_{ij}]_{n \times p}$  be the Boolean  $n \times p$  matrix

$$M_R = \begin{bmatrix} m_{11} & m_{12} & \dots & \dots & m_{1p} \\ m_{21} & m_{22} & \dots & \dots & m_{2p} \\ \vdots & \vdots & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots \\ m_{n1} & m_{n2} & \dots & \dots & m_{np} \end{bmatrix}$$

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{otherwise} \end{cases}$$

# Example

- The relation,

$$R = \{ (1,b), (1,d), (2,c), (3,c), (3,b), (4,a) \}$$

from,  $X = \{ 1, 2, 3, 4 \}$  to  $Y = \{ a, b, c, d \}$

$$M_R = \begin{matrix} & a & b & c & d \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{matrix} \text{ or } M_R = \begin{matrix} & d & b & a & c \\ \begin{matrix} 2 \\ 3 \\ 4 \\ 1 \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \end{matrix}$$



# Example

The matrix of the relation  $R$  from  $\{2, 3, 4\}$  to  $\{5, 6, 7, 8\}$  defined by

$x R y$  if  $x$  divides  $y$

$$\begin{array}{c} 2 \\ 3 \\ 4 \end{array} \begin{pmatrix} 5 & 6 & 7 & 8 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

# Example

Let  $A = \{ a, b, c, d \}$

Let  $R$  be a relation on  $A$ .

$R = \{ (a,a), (b,b), (c,c), (d,d), (b,c), (c,b) \}$

$$M_R = \begin{matrix} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}$$

# Exercise

An airline services the five cities  $c_1, c_2, c_3, c_4$  and  $c_5$ . Table below gives the cost (in dollars) of going from  $c_i$  to  $c_j$ . Thus the cost of going from  $c_1$  to  $c_3$  is RM100, while the cost of going from  $c_4$  to  $c_2$  is RM200

To from	$c_1$	$c_2$	$c_3$	$c_4$	$c_5$
$c_1$		140	100	150	200
$c_2$	190		200	160	220
$c_3$	110	180		190	250
$c_4$	190	200	120		150
$c_5$	200	100	200	150	

If the relation  $R$  on the set of cities  $A = \{c_1, c_2, c_3, c_4, c_5\}$  :  $c_i R c_j$  if and only if the cost of going from  $c_i$  to  $c_j$  is defined and less than or equal to RM180.

i) Find  $R$ .

ii) Matrices of relations for  $R$

# Solution

$$R = \{(c_1, c_2), (c_1, c_3), (c_1, c_4), (c_2, c_4), (c_3, c_1), (c_3, c_2), (c_4, c_3), (c_4, c_5), (c_5, c_2), (c_5, c_4)\}$$

$$M_R = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

# In degree and out degree

If  $R$  is a relation on a set  $A$  and  $a \in A$ , then the in-degree of  $a$  (relative to relation  $R$ ) is the number of  $b \in A$  such that  $(b, a) \in R$ .

The out degree of  $a$  is the number of  $b \in A$  such that  $(a, b) \in R$ .



Meaning that, in terms of the digraph of  $R$ , is that the in-degree of a vertex is

**“the number of edges terminating at the vertex”**

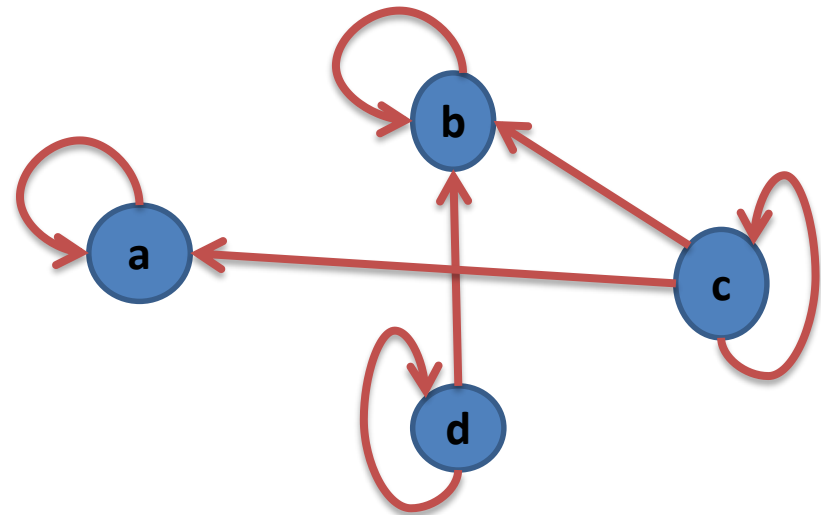
The out-degree of a vertex is

**“ the number of edges leaving the vertex”**

# Example

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

	a	b	c	d
In-degree	2	3	1	1
Out-degree	1	1	3	2



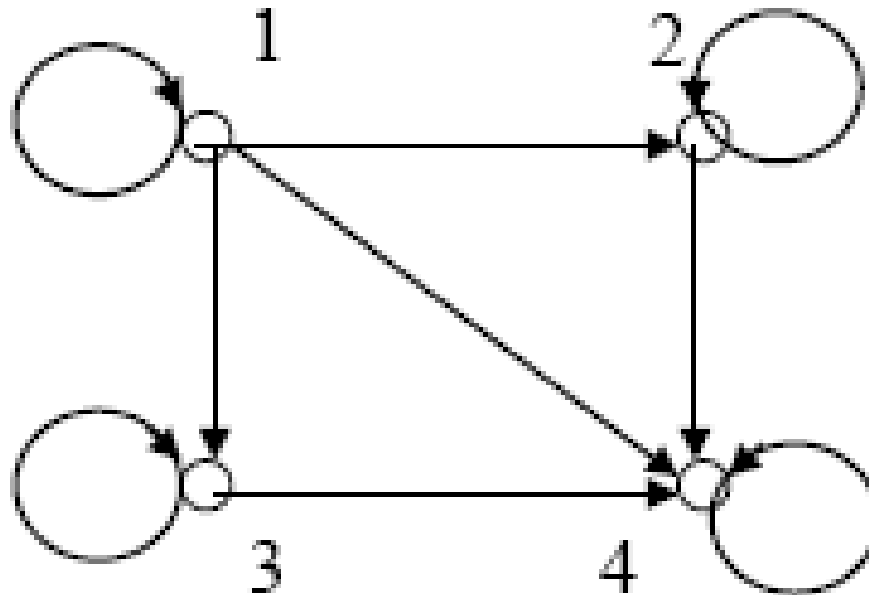
# Reflexive Relations

- Reflexive
  - A Relation  $R$  on set  $A$  is called **reflexive** if every  $a \in A$  is related to itself OR
  - A relation  $R$  on a set  $X$  is called **reflexive** if all pair  $(x,x) \in R; \forall x: x \in X$
- Irreflexive
  - A relation  $R$  on a set  $A$  is **irreflexive** if  $x \not R x$  or  $(x,x) \notin R; \forall x: x \in X$
- Not Reflexive
  - A Relation  $R$  is **not reflexive** if at least one pair of  $(x,x) \notin R, \forall x: x \in X$

# Reflexive Relations

The digraph of a reflexive relation has a loop at every vertex.

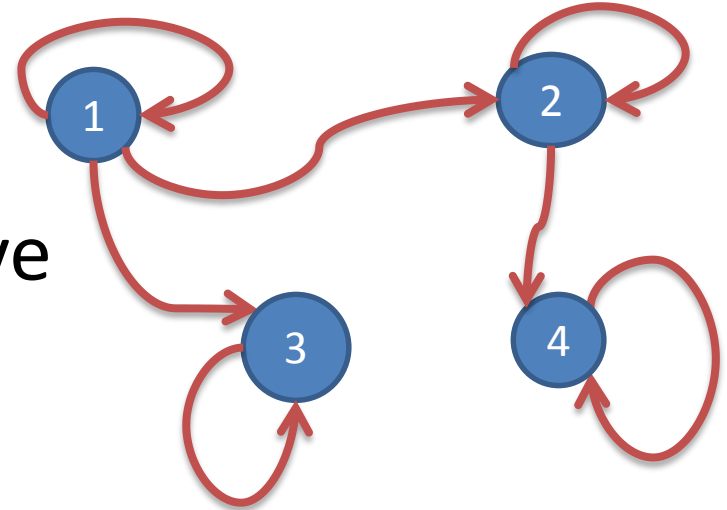
example



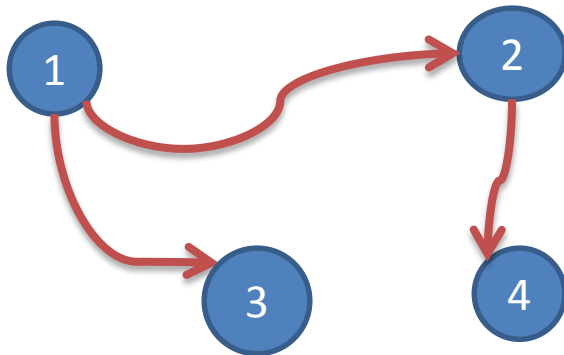


# Reflexive Relations

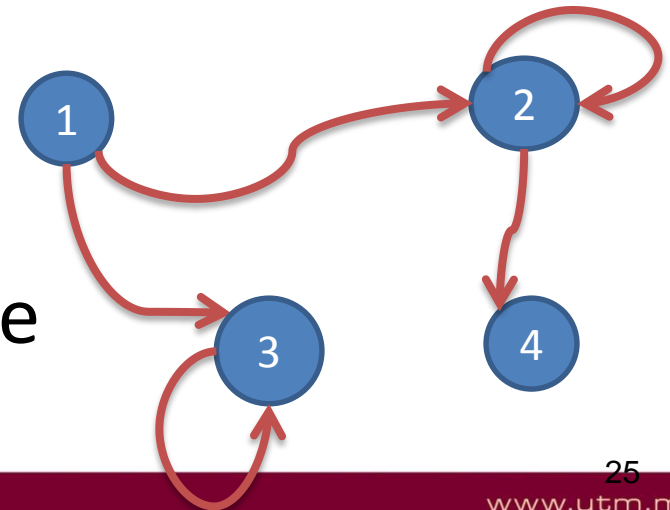
- Reflexive



- Irreflexive



- Not Reflexive



# Reflexive Relations

The relation  $R$  is reflexive if and only if the matrix of relation has 1's on the main diagonal.

example

	$a$	$b$	$c$	$d$
$a$	1	0	0	0
$b$	0	1	1	0
$c$	0	1	1	0
$d$	0	0	0	1

# Reflexive Relations

The relation  $R$  is *irreflexive* if and only if the matrix relation have all 0's on its main diagonal

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

# Reflexive Relations

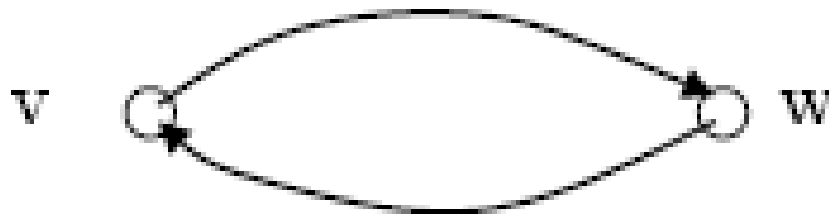
The relation  $R$  is not reflexive.

	$a$	$b$	$c$	$d$
$a$	1	0	0	0
$b$	0	0	1	0
$c$	0	1	1	0
$d$	0	0	0	1

$b \in X$   
 $(b, b) \notin R$

# Symmetric Relations

The digraph of a symmetric relation has the property that whenever there is a directed edge from  $v$  to  $w$ , there is also a directed edge from  $w$  to  $v$ .



# Example

- The relation  $R = \{ (a,a), (b,c), (c,b), (d,d) \}$  on  $X = \{ a, b, c, d \}$

$$\begin{aligned} (b,c) &\in R \\ (c,b) &\in R \end{aligned}$$

	$a$	$b$	$c$	$d$
$a$	1	0	0	0
$b$	0	0	1	0
$c$	0	1	0	0
$d$	0	0	0	1

symmetric

# Antisymmetric Relations

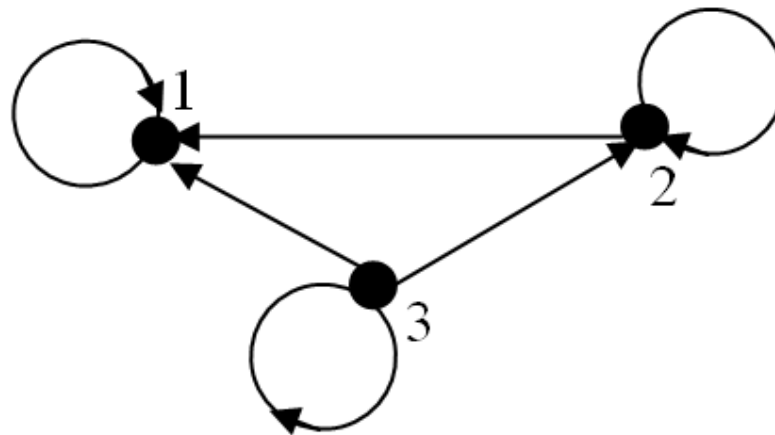
- Matrix  $M_R = [M_{ij}]$  of an antisymmetric relation  $R$  satisfies the property that if  $i \neq j$ , then  $m_{ij}=0$  or  $m_{ji}=0$
- If  $R$  is antisymmetric relation, then for different vertices  $i$  and  $j$  there cannot be an edge from vertex  $i$  to vertex  $j$  and an edge from vertex  $j$  to vertex  $i$
- At least one directed relation and one way

# Example

- Let  $R$  be a relation on  $A = \{1, 2, 3\}$  defined as  $(a, b) \in R$  if  $a \geq b$ ,  $a, b \in A$  is an antisymmetric relation because for all  $a, b \in A$ ,  $(a, b) \in R$  and  $a \neq b$ , then  $(b, a) \notin R$ , for example

$(3, 2) \in R$  but  $(2, 3) \notin R$

$(3, 3) \in R$  and  $(3, 3) \in R$  implies  $a = b$





# Example

- The relation  $R$  on  $X = \{1, 2, 3, 4\}$  defined by,

$$(x, y) \in R \quad \text{if } x \leq y, x, y \in X$$

$$\begin{aligned} (1, 2) &\in R \\ (2, 1) &\notin R \end{aligned}$$

	1	2	3	4
1	1	1	1	1
2	0	1	1	1
3	0	0	1	1
4	0	0	0	1

antisymmetric

# Example

The relation

$$R = \{ (a,a), (b,b), (c,c) \}$$

on  $X = \{ a, b, c \}$

$R$  has no members of the form  $(x,y)$  with  $x \neq y$ , then  $R$  is antisymmetric

	$a$	$b$	$c$
$a$	1	0	0
$b$	0	1	0
$c$	0	0	1

# Asymmetric

- A relation is asymmetric if and only if it is both antisymmetric and irreflexive.
- The matrix  $M_R = [m_{ij}]$  of an asymmetric relation  $R$  satisfies the property that
  - If  $m_{ij} = 1$  then  $m_{ji} = 0$
  - $m_{ii} = 0$  for all  $i$  (the main diagonal of matrix  $M_R$  consists entirely of 0's or otherwise)
- If  $R$  is asymmetric relation, then the digraph of  $R$  cannot simultaneously have an edge from vertex  $i$  to vertex  $j$  and an edge from vertex  $j$  to vertex  $i$
- All edges are “one way street” and no loop at every vertex

# Example

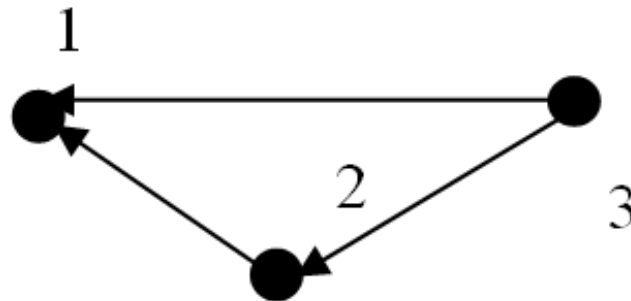
- Let  $R$  be the relation on  $A = \{1, 2, 3\}$  defined by  $(a, b) \in R$  if  $a > b$ ,  $a, b \in A$  is an asymmetric relation because,

$$(2, 1) \in R \text{ but } (1, 2) \notin R$$

$$(3, 1) \in R \text{ but } (1, 3) \notin R$$

$$(3, 2) \in R \text{ but } (2, 3) \notin R$$

$$(1, 1) \notin R, (2, 2) \notin R, (3, 3) \notin R$$



# Not Symmetric

- Let  $R$  be a relation on a set  $A$ . Then  $R$  is called ***not symmetric***, if for all  $a, b \in A$ , if  $(a, b) \in R$ , there exist  $(b, a) \notin R$ .

$$\exists a, b \in A, (a, b) \in R \rightarrow (b, a) \notin R$$

# Not Symmetric AND not antisymmetric

- Let  $R$  be a relation on a set  $A$ . Then  $R$  is called ***not symmetric*** and ***not antisymmetric***, if for all  $a, b \in A$ , if  $(a, b) \in R$ , there exist  $(b, a) \notin R$  and if  $(a, b) \in R$ , there exist  $(b, a) \in R$ .

$$\exists a, b \in A, (a, b) \in R \rightarrow (b, a) \in R$$

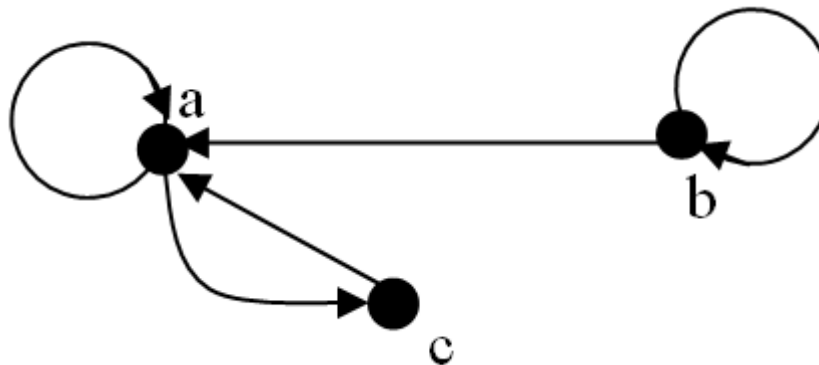
AND

$$\exists a, b \in A, (a, b) \in R \rightarrow (b, a) \notin R$$

# Example

- Relation  $R = \{(a, c), (b, b), (c, a), (b, a), (a, a)\}$  on  $A = \{a, b, c\}$  is not symmetric and not antisymmetric relation because there is,

$(a, c), (c, a) \in R$  and also  $(b, a) \in R$  but  $(a, b) \notin R$



# Example

- The relation  $R = \{ (a,a), (b,c), (c,b), (d,d) \}$  on  $X = \{ a, b, c, d \}$

$$\begin{aligned} (b,c) &\in R \\ (c,b) &\in R \end{aligned}$$

$$\begin{array}{c} \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} \end{array} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Symmetric  
and  
not  
antisymmetric



# Exercise

1. Let  $A = \mathbb{Z}$ , the set of integers and let  $R = \{(a, b) \in A \times A \mid a < b\}$ . So that  $R$  is the relation “less than”.  
Is  $R$  symmetric, asymmetric or antisymmetric?
2. Let  $A = \{1, 2, 3, 4\}$  and let  $R = \{(1, 2), (2, 2), (3, 4), (4, 1)\}$   
Determine whether  $R$  symmetric, asymmetric or antisymmetric.

# Solution

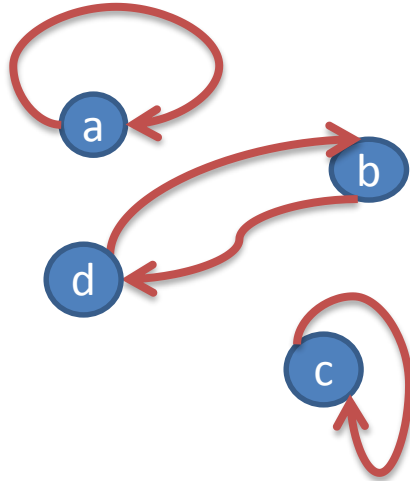
## Answer 1

- Symmetric : If  $a < b$ , then it is not true that  $b < a$ , so  $R$  is not symmetric
- Assymetric : If  $a < b$  then  $b > a$  ( $b$  is greater than  $a$ ), so  $R$  is assymetric
- Antisymmetric : If  $a \neq b$ , then either  $a > b$  or  $b > a$ , so  $R$  is antisymmetric

## Answer 2

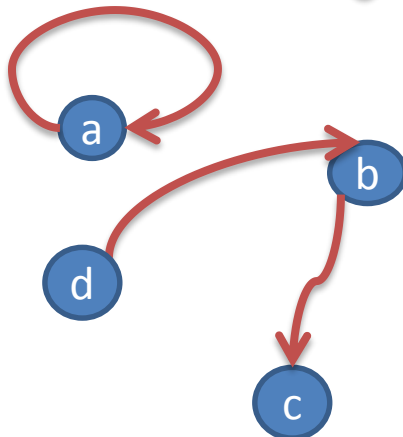
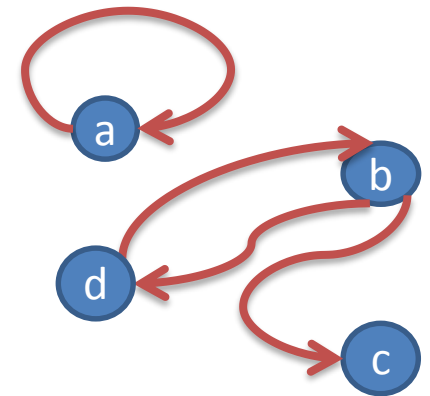
- $R$  is not symmetric since  $(1,2) \in R$ , but  $(2,1) \notin R$
- $R$  is not asymmetric, since  $(2,2) \in R$
- $R$  is antisymmetric, since  $a \neq b$ , either  $(a,b) \notin R$  or  $(b,a) \notin R$

# Summary on Symmetric



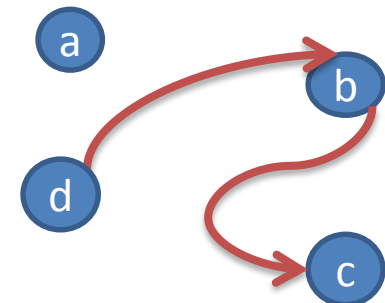
Symmetric

Not Symmetric



Antisymmetric

Asymmetric



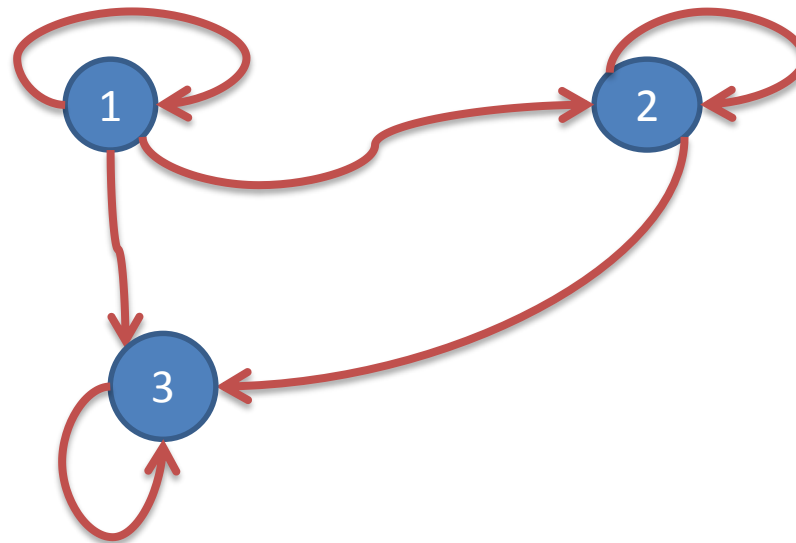
# Transitive Relations

- A relation  $R$  on set  $A$  is transitive if for all  $a, b \in A$ ,  $(a, b) \in R$  and  $(b, c) \in R$  implies that  $(a, c) \in R$
- In the diagraph of  $R$ ,  $R$  is a transitive relation if and only if there is a directed edge from one vertex  $a$  to another vertex  $b$ , and if there exists a directed edge from vertex  $b$  to vertex  $c$ , then there must exist a directed edge from  $a$  to  $c$

# Example

$$R=\{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$$

The diagram:



# Transitive Relations

The matrix of the relation  $M_R$  is transitive if

$$M_R \otimes M_R = M_R$$

$\otimes$  is the product of boolean

# Matrix multiplication

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

A
B
C

A, B and C are square matrices of size  $N \times N$

a, b, c and d are submatrices of A, of size  $N/2 \times N/2$

e, f, g and h are submatrices of B, of size  $N/2 \times N/2$

# Matrix multiplication

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \times \begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix} = \begin{bmatrix} aj + bm + cp & ak + bn + cq & al + bo + cr \\ dj + em + fp & dk + en + fq & dl + eo + fr \\ gj + hm + ip & gk + hn + iq & gl + ho + ir \end{bmatrix}$$



# Matrix multiplication

$$\begin{aligned}
 \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix} &= \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} \otimes (a \ d) + \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} \otimes (b \ e) + \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} \otimes (c \ f) \\
 &= \begin{pmatrix} 1a & 1d \\ 4a & 4d \\ 7a & 7d \end{pmatrix} + \begin{pmatrix} 2b & 2e \\ 5b & 5e \\ 8b & 8e \end{pmatrix} + \begin{pmatrix} 3c & 3f \\ 6c & 6f \\ 9c & 9f \end{pmatrix} \\
 &= \begin{pmatrix} 1a + 2b + 3c & 1d + 2e + 3f \\ 4a + 5b + 6c & 4d + 5e + 6f \\ 7a + 8b + 9c & 7d + 8e + 9f \end{pmatrix}.
 \end{aligned}$$

# Example

The relation  $R$  on  $A=\{1,2,3\}$  defined by  $(a,b) \in R$  if  $a \leq b$ ,  $a,b \in A$ , is a transitive. The matrix of relation  $M_R$ ,

$$M_R = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

The product of boolean,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that,  $(1,2)$  and  $(2,3) \in R$ ,  $(1,3) \in R$

# Example

The relation  $R$  on  $A=\{a,b,c,d\}$  is  $r=\{(a,a), (b,b), (c,c), (d,d), (a,c), (c,b)\}$  is not transitive. The matrix of relation  $M_R$ ,

$$M_R = \begin{matrix} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 1 & \textcircled{0} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

$M_R \otimes M_R \neq M_R$

The product of boolean,

$$\begin{matrix} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix} \otimes \begin{matrix} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix} = \begin{matrix} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 1 & \textcircled{1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

Note that,  $(a,c)$  and  $(c,b) \in R$ ,  $(a,b) \notin R$

# Example

Let  $R$  be a relation on  $A=\{1,2,3\}$  is defined by  $(a,b) \in R$  if  $a \leq b$ ,  $a,b \in A$ . Find  $R$ . Is  $R$  a transitive relation?

Solution:

$$R = \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$$

$$M_R = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

$R$  is a transitive relation because

$(1,2)$  and  $(2,2) \in R, (1,2) \in R$

**$(1,2)$  and  $(2,3) \in R, (1,3) \in R$**

**$(1,3)$  and  $(3,3) \in R, (1,3) \in R$**

$(2,2)$  and  $(2,3) \in R, (2,3) \in R$

$(2,3)$  and  $(3,3) \in R, (2,3) \in R$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

# Equivalence Relations

Relation  $R$  on set  $A$  is called an equivalence relation if it is a **reflexive, symmetric and transitive**.

## Example

Let  $R=\{(1,1), (1,3), (2,2), (3,1), (3,3)\}$  on  $\{1,2,3\}$ , the matrix relation  $M_R$ ,

$$M_R = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{matrix}$$

All the main diagonal matrix elements are 1 and the matrix is **reflexive**.

# Example - cont

The transpose matrix  $M_R, M_R^T$  is equal to  $M_R$ , so  $R$  is **symmetric**

$$M_R = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{matrix} \quad M_R^T = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{matrix}$$

The product of boolean show that the matrix is **transitive**

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

So  $R$  is an **equivalence relation**.

# Partial Order Relations

Relation  $R$  on set  $A$  is called a partial order relation if it is a **reflexive, antisymmetric and transitive**.

## Example:

Let  $R$  be a relation on a set  $A=\{1,2,3\}$  defined by  $(a,b) \in R$  if  $a \leq b, a,b \in R$ .

$$R=\{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$$

$R$  is **reflexive, antisymmetric and transitive**.

So  $R$  is a **partial order relation**.

# PART 2

# FUNCTIONS

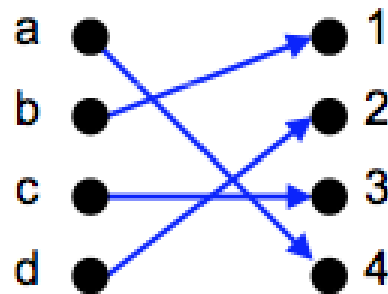


# FUNCTION

- Let **X** and **Y** are nonempty sets.
- A **function** ( $f$ ) from **X** to **Y** is a relation from **X** to **Y** having a properties:
  - The domain of  $f$  is **X**
  - If  $(x, y), (x, y') \in f$ , then  $y = y'$

# Relations vs. Functions

- Not all relations are functions
- But consider the following function:



- All functions are relations!

# Relations vs Functions

## When to use which?

- A function is used when you need to obtain a SINGLE result for any element in the domain
  - Example: sin, cos, tan
- A relation is when there are multiple mappings between the domain and the co-domain
  - Example: students enrolled in multiple courses

# Domain, Co-domain, Range

- A function from **X** to **Y** is denoted,  $f: \mathbf{X} \rightarrow \mathbf{Y}$
- The **domain** of  $f$  is the set **X**.
- The set **Y** is called the **co-domain** or target of  $f$ .
- The set  $\{ y \mid (x,y) \in f \}$  is called the **range**.

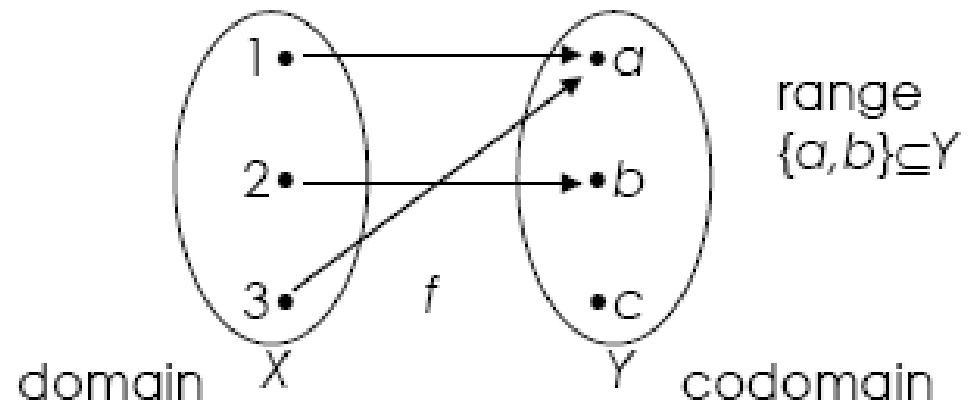
# Example

Given the relation,  $f = \{ (1,a), (2,b), (3,a) \}$  from  $X = \{ 1, 2, 3 \}$  to  $Y = \{ a, b, c \}$  is a function from  $X$  to  $Y$ . State the domain, co-domain and range.

## Solution:

- ✓ The domain of  $f$  is  $X$
- ✓ Co-domain of  $f$  is  $Y$
- ✓ The range of  $f$  is  $\{a, b\}$

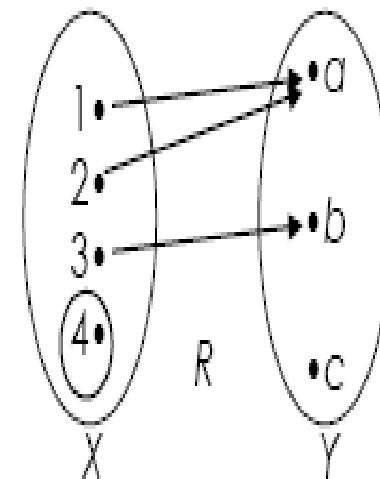
$$f = \{ (1,a), (2,b), (3,a) \}$$



# Example

- The relation,  $R = \{(1,a), (2,a), (3,b)\}$  from  $X = \{1, 2, 3, 4\}$  to  $Y = \{a, b, c\}$  is NOT a function from  $X$  to  $Y$ .
- The domain of  $R$ ,  $\{1, 2, 3\}$  is not equal to  $X$ .

$$R = \{(1,a), (2,a), (3,b)\}$$



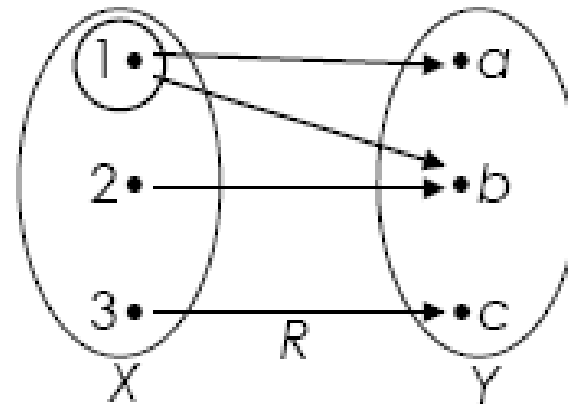
There is no arrow from 4

# Example

- The relation,  $R = \{(1,a), (2,b), (3,c), (1,b)\}$  from  $X = \{1, 2, 3\}$  to  $Y = \{a, b, c\}$  is NOT a function from  $X$  to  $Y$
- $(1,a)$  and  $(1,b)$  in  $R$  but  $a \neq b$ .

$$R = \{(1,a), (2,b), (3,c), (1,b)\}$$

There are 2  
arrows from 1



# Notation of function: $f(x)$

- For the function,  $f = \{(1,a), (2,b), (3,a)\}$
- We may write,  $f(1)=a, f(2)=b, f(3)=a$
- Notation  $f(x)$  is used to define a function.

## Example :

- Defined:  $f(x) = x^2$
- $f(2) = 4, f(-3.5) = 12.25, f(0) = 0$
- Notation :  $f = \{(x, x^2) | x \text{ is a real number}\}$



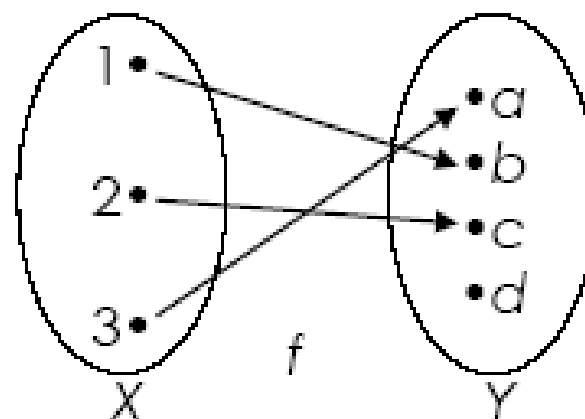
# One-to-One Function

- A function  $f$  from  $X$  to  $Y$ , is said one-to-one (or injective) if for each  $y \in Y$ , there is at most one  $x \in X$ , with  $f(x)=y$ .
- For all  $x_1, x_2$ , if  $f(x_1) = f(x_2)$ , then  $x_1=x_2$ .  
$$\forall x_1 \forall x_2 ((f(x_1) = f(x_2)) \rightarrow (x_1=x_2))$$

# Example

- The function,  $f = \{ (1,b), (3,a), (2,c) \}$  from  $X = \{ 1, 2, 3 \}$  to  $Y = \{ a, b, c, d \}$  is **one-to-one**.

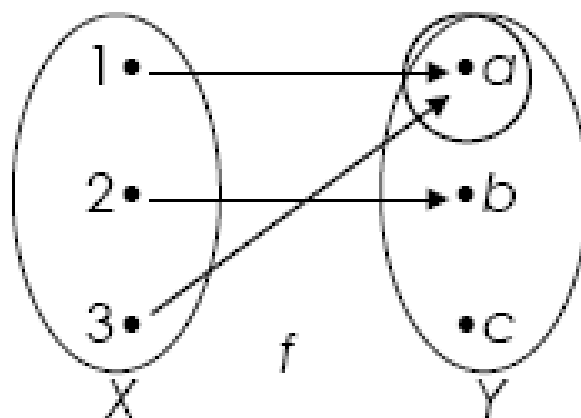
Each element in  $Y$   
has at most one  
arrow pointing to it



# Example

- The function,  $f = \{ (1,a), (2,b), (3,a) \}$  from  $X = \{ 1, 2, 3 \}$  to  $Y = \{ a, b, c \}$  is **NOT** one-to-one.
- $f(1) = a = f(3)$

$$f = \{ (1,a), (2,b), (3,a) \}$$



$a$  has 2 arrows pointing to it

# Onto Function

- If  $f$  is a function from  $\mathbf{X}$  to  $\mathbf{Y}$  and the range of  $f$  is  $\mathbf{Y}$ ,  $f$  is said to be **onto**  $\mathbf{Y}$  (or an onto function or a surjective function)
- For every  $y \in \mathbf{Y}$ , there exists at least one  $x \in \mathbf{X}$  such that  $f(x) = y$

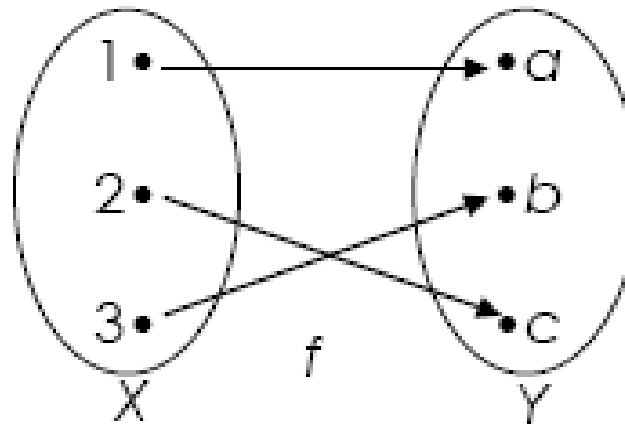
$$\forall y \in \mathbf{Y} \exists x \in \mathbf{X} (f(x) = y)$$

# Example

- The function,  $f = \{ (1,a), (2,c), (3,b) \}$  from  $X = \{ 1, 2, 3 \}$  to  $Y = \{ a, b, c \}$  is **one-to-one** and **onto**  $Y$ .

$$f = \{ (1,a), (2,c), (3,b) \}$$

**One-to-one**  
 Each  
 element in  $Y$   
 has at most  
 one arrow

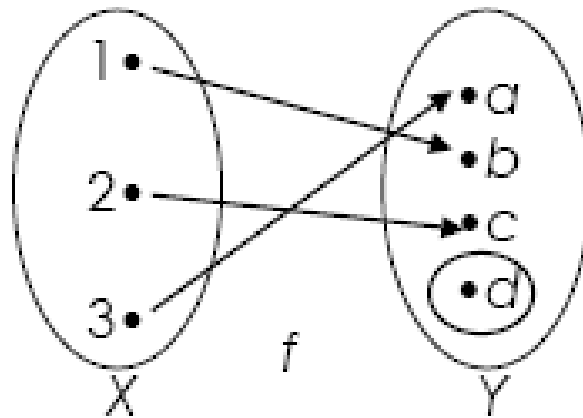


**Onto**  
 Each  
 element in  $Y$   
 has at least  
 one arrow  
 pointing to it

# Example

- The function,  $f = \{ (1,b), (3,a), (2,c) \}$  is **not onto**  $Y = \{a, b, c, d\}$

$$f = \{ (1,b), (3,a), (2,c) \}$$

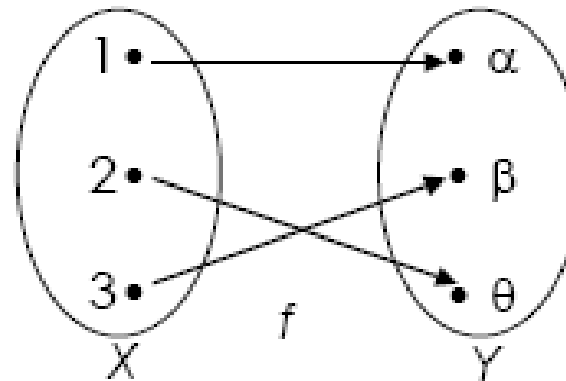


**not onto**  
 no arrow  
 pointing to  $d$

# Bijection Function

- A function,  $f$  is called **one-to-one correspondence** (or bijective/bijection) if  $f$  is both one-to-one and onto.
- Example

$$\blacklozenge f = \{ (1, \alpha), (2, \theta), (3, \beta) \}$$



One-to-one  
and onto Y  
**-bijection**

# Exercise

Determine which of the relations  $f$  are functions from the set  $\mathbf{X}$  to the set  $\mathbf{Y}$ . In case any of these relations are functions, determine if they are one-to-one, onto  $\mathbf{Y}$ , and/or bijection.

a)  $\mathbf{X} = \{ -2, -1, 0, 1, 2 \}$ ,  $\mathbf{Y} = \{ -3, 4, 5 \}$  and  
 $f = \{ (-2,-3), (-1,-3), (0,4), (1,5), (2,-3) \}$

b)  $\mathbf{X} = \{ -2, -1, 0, 1, 2 \}$ ,  $\mathbf{Y} = \{ -3, 4, 5 \}$  and  
 $f = \{ (-2,-3), (1,4), (2,5) \}$

c)  $\mathbf{X} = \mathbf{Y} = \{ -3, -1, 0, 2 \}$  and  
 $f = \{ (-3,-1), (-3,0), (-1,2), (0,2), (2,-1) \}$



# Solution

Determine which of the relations  $f$  are functions from the set  $\mathbf{X}$  to the set  $\mathbf{Y}$ . In case any of these relations are functions, determine if they are one-to-one, onto  $\mathbf{Y}$ , and/or bijection.

a)  $\mathbf{X} = \{ -2, -1, 0, 1, 2 \}$ ,  $\mathbf{Y} = \{ -3, 4, 5 \}$  and  
 $f = \{ (-2,-3), (-1,-3), (0,4), (1,5), (2,-3) \}$

=> Function, ONTO

b)  $\mathbf{X} = \{ -2, -1, 0, 1, 2 \}$ ,  $\mathbf{Y} = \{ -3, 4, 5 \}$  and  
 $f = \{ (-2,-3), (1,4), (2,5) \}$

=> Not Function

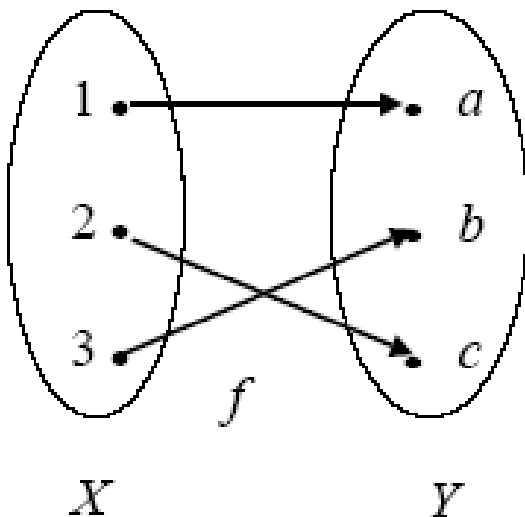
c)  $\mathbf{X} = \mathbf{Y} = \{ -3, -1, 0, 2 \}$  and  
 $f = \{ (-3,-1), (-3,0), (-1,2), (0,2), (2,-1) \}$

=> Not Function

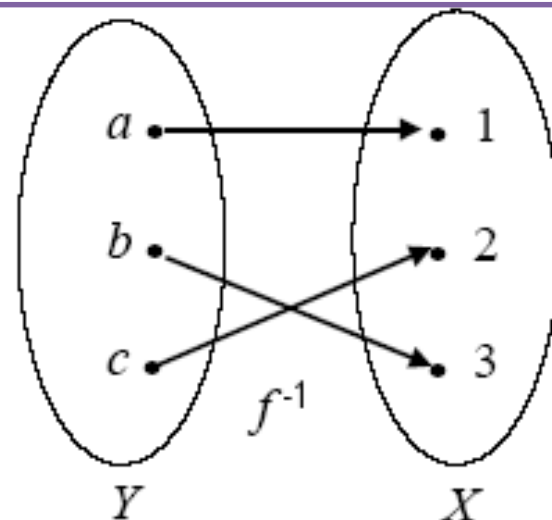
# Inverse Function

- Let  $f: \mathbf{X} \rightarrow \mathbf{Y}$  be a function.
- The **inverse** relation  $f^{-1} \subseteq \mathbf{Y} \times \mathbf{X}$  is a function from  $\mathbf{Y}$  to  $\mathbf{X}$ , if and only if  $f$  is both one-to-one and onto  $\mathbf{Y}$ .
- Example:**

$$f = \{(1,a), (2,c), (3,b)\}$$



$$f^{-1} = \{(a,1), (c,2), (b,3)\}$$



# Composition

- Suppose that  $g$  is a function from  $\mathbf{X}$  to  $\mathbf{Y}$  and  $f$  is a function from  $\mathbf{Y}$  to  $\mathbf{Z}$ .

- The **composition** of  $f$  with  $g$ ,  
 $f \circ g$

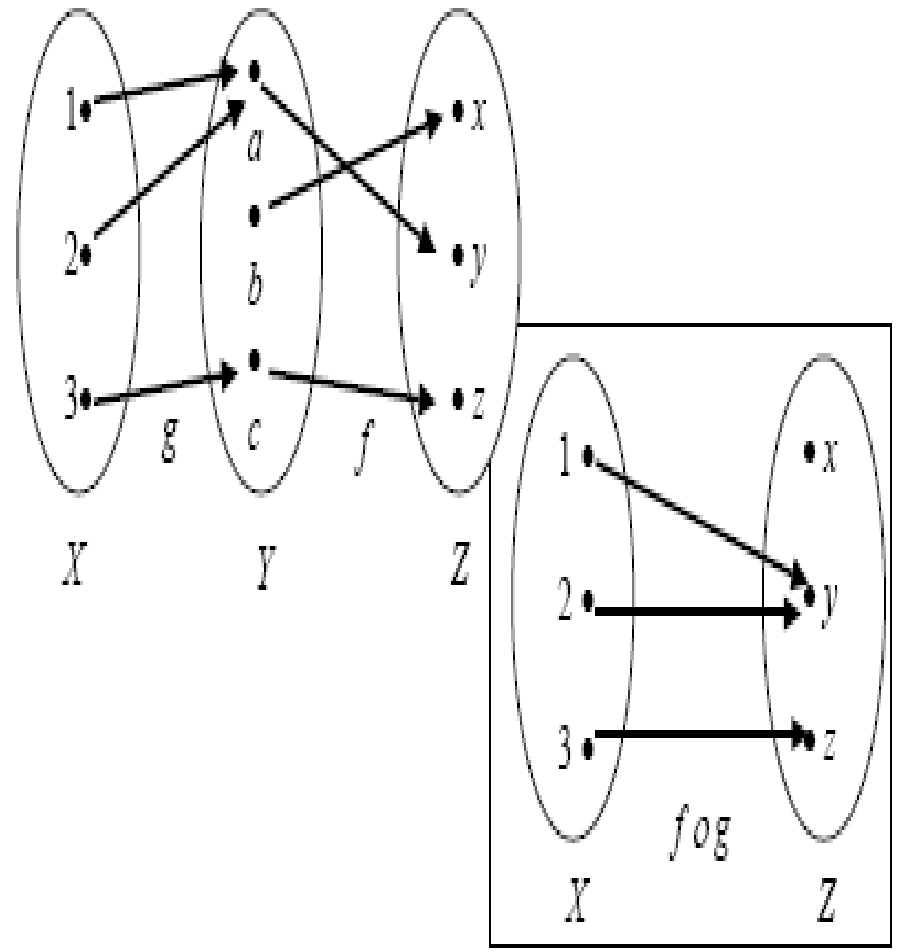
is a function

$$(f \circ g)(x) = f(g(x))$$

from  $\mathbf{X}$  to  $\mathbf{Z}$ .

# Example

- Given,  $g = \{ (1,a), (2,a), (3,c) \}$  a function from  $\mathbf{X} = \{1, 2, 3\}$  to  $\mathbf{Y} = \{a, b, c\}$  and,  $f = \{ (a,y), (b,x), (c,z) \}$  a function from  $\mathbf{Y}$  to  $\mathbf{Z} = \{x, y, z\}$ .
- The **composition** function from  $\mathbf{X}$  to  $\mathbf{Z}$  is the function  $f \circ g = \{ (1,y), (2,y), (3,z) \}$



# Example

$$f(x) = \log_3 x \text{ and } g(x) = x^4$$

➤  $f(g(x)) = \log_3 (x^4)$

➤  $g(f(x)) = (\log_3 x)^4$

➤ Note:  $f \circ g \neq g \circ f$

# Example

$$f(x) = \frac{1}{5}x \qquad g(x) = x^2 + 1$$

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) = g\left(\frac{x}{5}\right) \\ &= \left(\frac{x}{5}\right)^2 + 1 = \frac{x^2}{25} + 1\end{aligned}$$

# PART 3

# RECURRENCE RELATION

# Recursion

- “A description of something that refers to itself is called a *recursive definition*.”
- In mathematics, certain recursive definitions are used all the time.
- The classic recursive example in mathematics is the definition of **factorial**.



# Recursive

- ❑ A *recursive procedure* is a procedure that invokes itself
  - ❑ Example: given a positive integer  $n$ , **factorial of  $n$**  is defined as the product of  $n$  by all numbers less than  $n$  and greater than 0. Notation:  $n! = n(n-1)(n-2)\dots 3.2.1$
- ❑ Observe that  $n! = n(n-1)! = n(n-1)(n-2)!$ , etc.
- ❑ A *recursive algorithm* is an algorithm that contains a recursive procedure

# Recursive vs. Iteration

- Iteration can be used in place of recursion
  - An iterative algorithm uses a *looping construct*
  - A recursive algorithm uses a *branching structure*
- Recursive solutions are often less efficient, in terms of both *time* and *space*, than iterative solutions
- Recursion can simplify the solution of a problem, often resulting in *shorter*, more easily understood source code

# Recursive vs. Iteration

**Q:** Does using recursion usually make your code **faster**?

**A:** No, it's usually slower (due to the overhead of maintaining the stack)

**Q:** Does using recursion usually use **less memory**?

**A:** No, it usually uses **more** memory (for the stack)

**Q:** Then **why** use recursion?

**A:** It sometimes makes your code much **simpler**!

# Recursive vs. Iteration

## Recursive version

```
int factorial (int n)
{
    if (n == 0)
        return 1;
    else
        return n * factorial (n-1);
}
```



Recursive Call

## Iterative version

```
int factorial (int n)
{
    int i, product=1;
    for (i=n; i>1; --i)
        product=product * i;

    return product;
}
```

# Factorial

## Factorial Notation

If  $n$  is a positive integer, the notation  $n!$  is the product of all positive integers from  $n$  down through 1.

$$n! = n(n-1)(n-2)\dots(3)(2)(1)$$

$0!$  , by definition is 1.



# Factorial

## Factorial Notation (n!)

- ▶ For  $n$  as a positive integer,  $n$  factorial is defined by

$$n! = n \times (n - 1) \times (n - 2) \times \dots \times 2 \times 1$$

- ▶  $0! = 1$

- ▶ Note that  $n!$  can also be written as

$$n! = n \times (n - 1)! = n \times (n - 1) \times (n - 2)! \text{ and so on}$$

# Recursive Factorial

## A simple problem - Factorial

- We have to calculate the factorial of a number 'N'.
- We know how to calculate the factorial.
- Logic 1:  
$$N! = N * N-1 * N-2 * \dots * N-(N-1) \quad \{N-(N-1)=1\}$$
- Logic 2:  
$$N! = 1 * 2 * 3 * \dots * N$$
- Logic 3:  
$$N! = N * (N-1)! \quad (\text{Recursion})$$

# Recursive Factorial algorithm

## Example:

Give a recursive algorithm for computing  $n!$ , where  $n$  is a nonnegative integer.

## Solution:

Use the recursive definition of the factorial function.

```
procedure factorial( $n$ : nonnegative integer)
  if  $n = 0$ 
    then return 1
  else
    return  $n \cdot \text{factorial}(n - 1)$ 
  {output is  $n!$ }
```



# Fibonacci

What is the Fibonacci Sequence of Numbers?



The Fibonacci numbers are a unique sequence of integers, starting with 1, where each element is the sum of the two previous numbers. For example: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, etc.

# Fibonacci

*Fibonacci relationship*

$$F_1 = 1$$

$$F_2 = 1$$

$$F_3 = 1 + 1 = 2$$

$$F_4 = 2 + 1 = 3$$

$$F_5 = 3 + 2 = 5$$

*In general :*

$$F_n = F_{n-1} + F_{n-2}$$

*or*

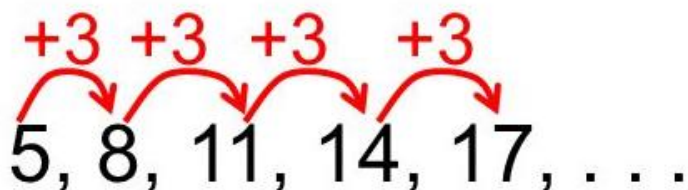
$$F_{n+1} = F_n + F_{n-1}$$

# Recursive Fibonacci algorithm

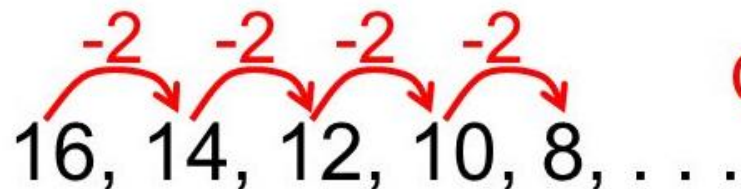
```
procedure fib(n: nonnegative integer)
if n = 0 then fib(0) := 0
else if n = 1 then fib(1) := 1
else fib(n) := fib(n - 1) + fib(n - 2)
```

# Arithmetic sequence

- Sequence in which each term after the first is obtained by adding a fixed number, called the difference, to the previous term.


$$5, 8, 11, 14, 17, \dots$$

Common difference is 3.  
( $d = 3$ )


$$16, 14, 12, 10, 8, \dots$$

Common difference is -2.  
( $d = -2$ )

# Arithmetic sequence

An arithmetic sequence has a common difference.

The formula for the  $n^{\text{th}}$  term is

$$a_n = a + (n - 1)d$$

where  $a_n$  =  $n^{\text{th}}$  term of the sequence

$a$  = first term of the sequence

$d$  = common difference

# Recursive Arithmetic sequence

The  $n$ th term is of an arithmetic sequence can be found by:

Explicit Formula:

$$a_n = dn + c$$

$d$  = common difference

$n$  = # of the term

$$c = a_1 - d$$

Recursive Formula:

$$a_{n+1} = a_n + d$$

$a_{n+1}$  = next term

$a_n$  = current term

$d$  = common difference



# Recursive Arithmetic sequence

Explicit Formula:

$$a_n = 3n - 2$$

$$a_n = 3(1) - 2, 3(2) - 2, 3(3) - 2, 3(4) - 2, \dots$$

$$a_n = 1, 4, 7, 10, \dots$$

Recursive Formula:

$$a_1 = 1, a_{n+1} = a_n + 3$$

$$a_n = 1, 1+3, (1+3)+3, [(1+3)+3]+3 \dots$$

$$a_n = 1, 4, 7, 10, \dots$$

# Recursive Arithmetic algorithm

## Recursion\*

- ▶ Recursive formula has a correspondence in programming language: recursive function calls:

$$\left\{ \begin{array}{l} a_1 = 0 \\ a_2 = 1 \\ a_n = a_{n-1} + a_{n-2} \end{array} \right.$$

- ▶ Pseudo-code for function a(n)

- ▶ int a(n)
- ▶ {
  - If n==1, return 0;
  - If n==2, return 1
  - Return (a(n-1)+a(n-2));
- ▶ }