Chapter 7 Parametric Equations and Polar Coordinates

7.1 Parametric equations

- Definition
- Sketching parametric equations

7.2 Polar Coordinates

- Polar coordinates system
- Polar coordinates and Rectangular coordinates
- Forming polar equations from Cartesian equations and vice-versa
- Sketching polar equations

Trigger Problem: Missile in Flight

Suppose that a missile is fired toward your location from 500 miles away and follows a flight path given by the parametric equations

$$x = 100t\,,\; y = 80t - 16t^2 \text{ for } 0 \le t \le 5$$

Two minutes later, you fire an interceptor missile from your location following the flight path

$$x = 500 - 200(t-2)\,,\; y = 80(t-2) - 16(t-2)^2$$

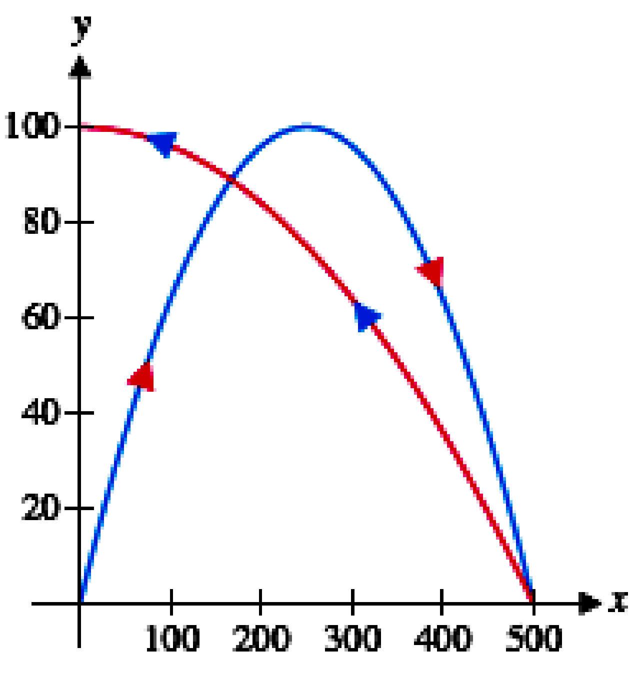
$$\text{for } 2 \le t \le 7$$

Determine whether the interceptor missile hits its target.

Solution

Missile flight paths:

The two paths intersect, but does the two missiles collide?



- Collision happens when the missiles are at the same point *at the same time*.

WANT: find values of *t* for which both paths pass through same point simultaneously

We set the two *x*-values equal:

$$100t = 500 - 200(t - 2)$$

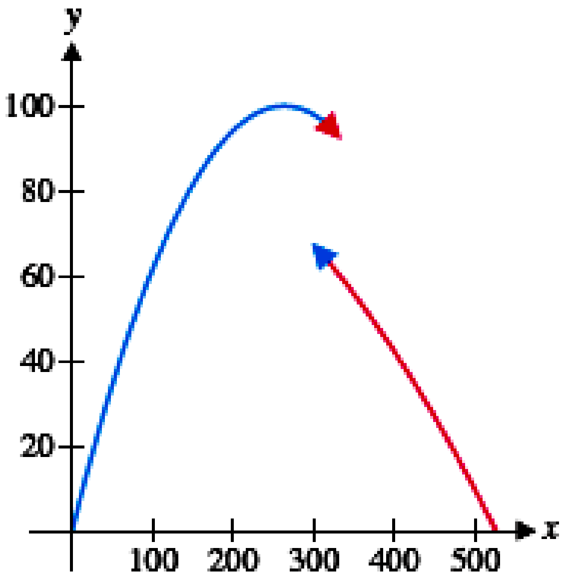
The two missiles have the same *x*-coordinate when $t = 3$.

But, the *y*-coordinates are not the same here, since when $t = 3$, we have

$$80t - 16t^2 = 96$$

But

$$80(t - 2) - 16(t - 2)^2 = 64$$



From the graph, we can see that the two missiles pass one another without colliding.

- ➢ by the time the interceptor missile intersects the flight path of the incoming missile, it is long gone!

Plane Curves

Two methods for describing curves:

- Parametric equations – use a third variable *t* called a parameter
- Polar curves – use a new coordinate system, called the polar coordinate system

7.1 Parametric Equations

- Describes where & when the particle was at a particular time
- Indicate direction of motion

Definition 7.1

Equations

$$x = f(t),\ y = g(t)$$

that express *x* and *y* in *t* is known as **parametric equations**, and *t* is called the **parameter**.

Example 7.1

Sketch and identify the curve defined by the parametric equations.

(i) $x = t^2 - 2t$, $y = t + 1$ for $0 \le t \le 4$

(ii) $x = \cos t$, $y = \sin t$ for $0 \le t \le 2\pi$

(iii) $x = \cos 2t$, $y = \sin 2t$ for $0 \le t \le \pi$

7.1.1 Parametrisation

- Not unique – different sets of parametric equations can represent the same curve.

Need to distinguish between

- Curve – set of points
- Parametric curve – points traced in a particular way

Example 7.2

Compare the curves represented by the parametric equations. How do they differ?

(a) $x = t$, $y = t^2$

(b) $x = \sin t$, $y = \sin^2 t$

(c) $x = e^t$, $y = e^{2t}$

Note

The parameter can be anything. We can use the letter *x* as the parameter instead of the letter *t*, if we prefer.

Example 7.3

Sketch the graph of the following equations

(i) $x = 2t$, $y = 4t^2 - 1$.

(ii) $x = 3t - 5$, $y = 2t + 5$.

Example 7.4

Find parametric equations to represent the line segment from (-2, 7) to (3, -1).

Example 7.5

Find the parametric equations for the path of a particle that moves along the circle $x^2 + (y-1)^2 = 4$ in the manner describe.

(a) Once around clockwise, starting at (2, 1)

(b) Three times around counterclockwise, starting at (2, 1)

(c) Halfway around counterclockwise, starting at (0, 3)

Example 7.6

Find the parametric equations for ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$

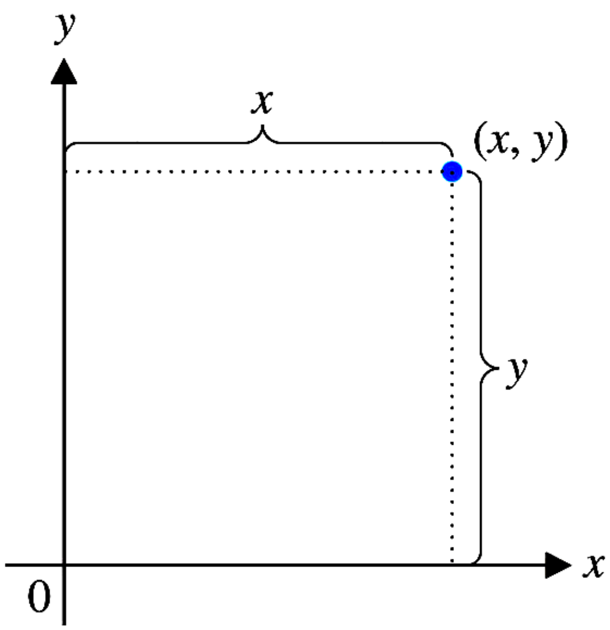
(i) Use these parametric equations to graph the ellipse when $a = 3$ and $b = 1,2$ and 4.

(ii) How does the shape of the ellipse change as *b* varies?

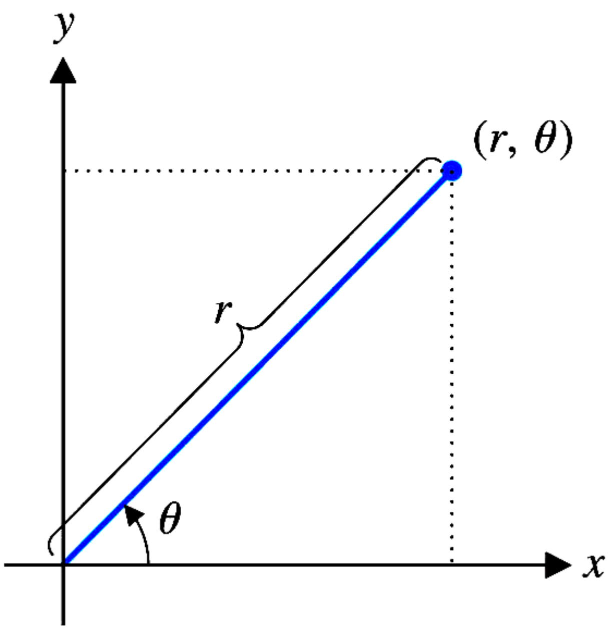
7.2 Polar Coordinates

Polar coordinates of a point in a coordinate plane give its distance and direction from the origin.

Rectangular & polar coordinates systems



Rectangular coordinates

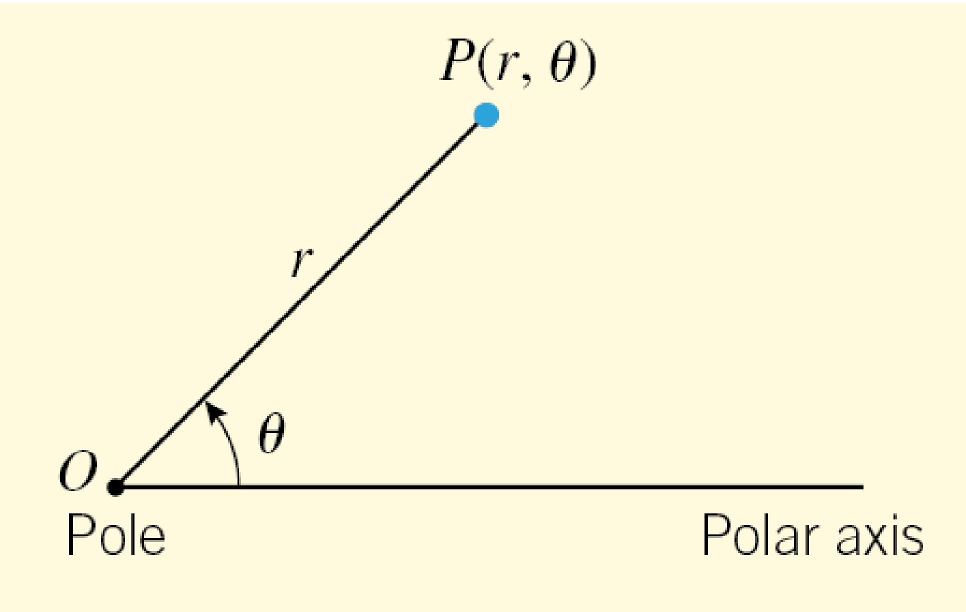


Polar coordinates

- **rectangular coordinates –** describe a point in terms of the horizontal and vertical distances from the origin
- **polar coordinates –** describe a point by specifying the distance r from the point to the origin and an angle θ (in radians) measured from the positive x-axis counterclockwise to the ray connecting the point and the origin

Definition 7.2

The polar coordinates of a point *P* other than the origin in an *xy*-plane are r,θ where *r* is the distance form *P* to the origin and θ is an angle from the positive *x*-axis to the line between the origin and *P*.



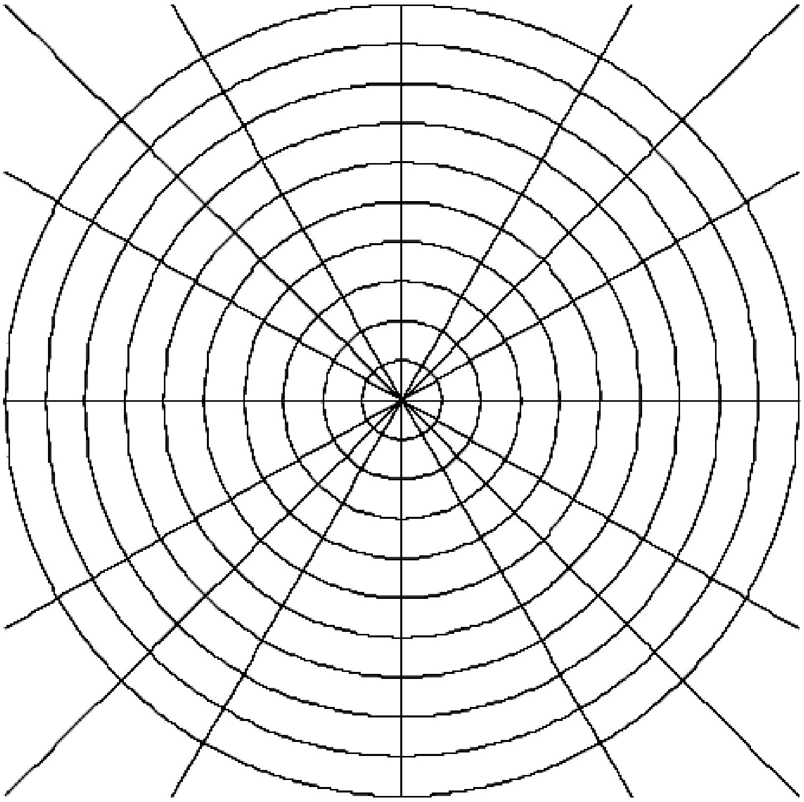
Note:

(i) Polar coordinate of a point is not unique.

(ii) θ is positive in an anticlockwise direction, and it is negative if it is taken in clockwise direction.

(iii) A point $-r,\theta$ is in the opposite direction of point r,θ .

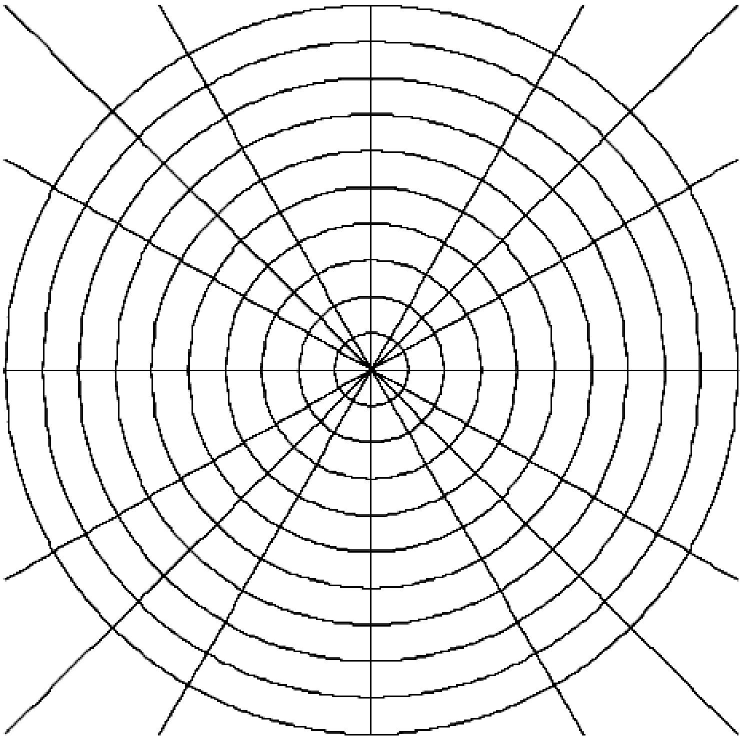
Polar Grid



Example 7.7 Plot the following points:

(a) $3, 225^\circ$ (b) $1, 225^\circ$ (c) $-3, 225^\circ$

(d) $2, \frac{\pi}{3}$ (e) $2, -\frac{\pi}{3}$ (f) $-2, \frac{\pi}{3}$



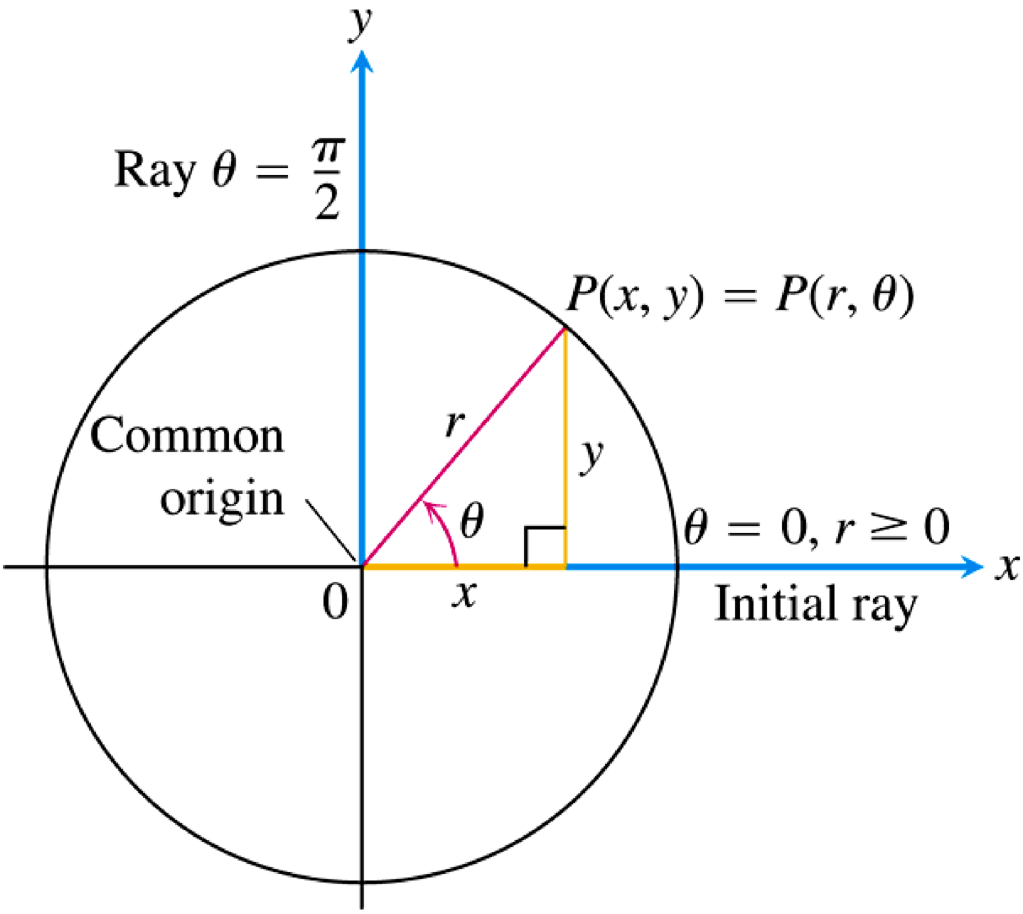
Example 7.8

Find all possible polar coordinates of the points whose polar coordinates are given as the following:

(a) $P\ 1, 45^\circ$ (b) $Q\ 2, -60^\circ$

(c) $R\ -1, 225^\circ$

7.2.1 Polar Coordinates and Rectangular Coordinates



Relationship between polar and Cartesian Coordinates:

$x = r\cos\theta$	$y = r\sin\theta$
$r^2 = x^2 + y^2$	$\tan\theta = \dfrac{y}{x}$

Example 7.9 Find the rectangular coordinates of the points whose polar coordinates are given as

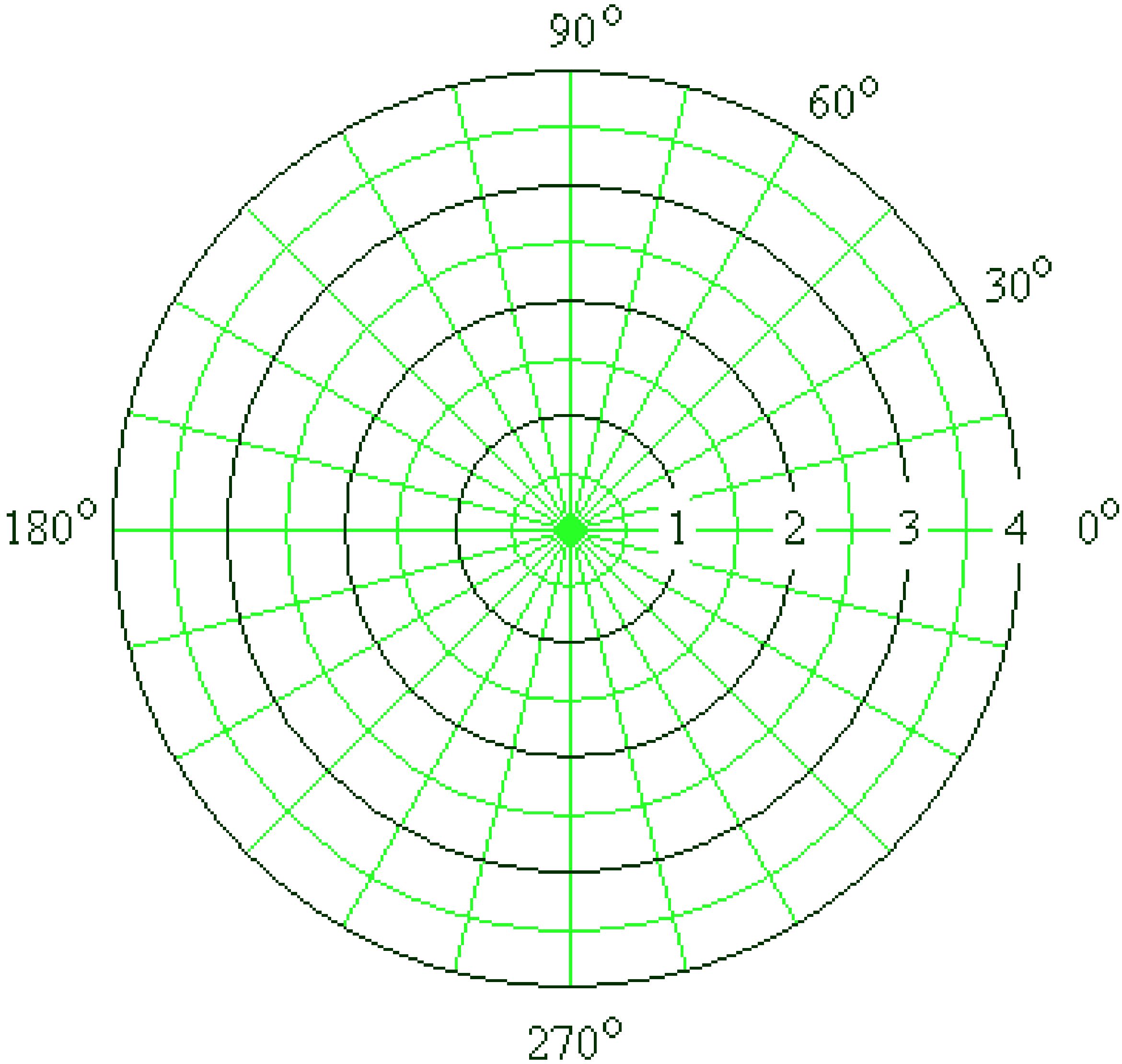
(a) $1, \frac{7\pi}{4}$ (b) $-4, \frac{\pi}{3}$ (c) $-2, -30^\circ$

Example 7.10 Find all polar coordinates of the points whose rectangular coordinates are given as

(a) $11, 5$ (b) $0, 2$ (c) $-4, -4$

Plot the points on the following polar grid:

a) (2, 60°) b) (4, 165°) c) (3, 315°)



7.2.2 Forming polar equation from rectangular equation and vice-versa

Example 7.11 Express the following rectangular equations in polar equations.

(a) $y = x^2$ (b) $x^2 + y^2 = 16$

(c) $xy = 1$

Example 7.12 Express the following polar equations in rectangular equations.

(a) $r = 2\sin\theta$ (b) $r = \dfrac{3}{4\cos\theta + 5\sin\theta}$

(c) $r = 4\cos\theta + 4\sin\theta$

Some equivalent equations expressed in terms of polar coordinates and Cartesian coordinates

Polar equation	Cartesian equivalent
$r\cos\theta = 2$	$x = 2$
$r^2\cos\theta\sin\theta = 4$	$xy = 4$
$r^2\cos^2\theta - r^2\sin^2\theta = 1$	$x^2 - y^2 = 1$
$r = 1 + 2r\cos\theta$	$y^2 - 3x^2 - 4x - 1 = 0$
$r = 1 - \cos\theta$	$x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$

- **Some curves are more simply expressed with polar coordinates, others are not**

7.2.3 Sketching graphs of polar equations

The **graph** of a polar equation $r = f\ \theta$ is the set of all points (x, y) for which

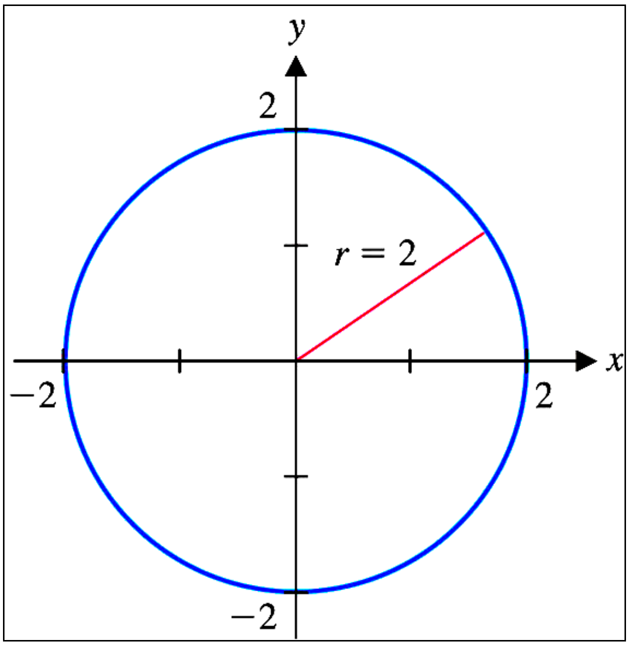
$$x = r\cos\theta,\ y = r\sin\theta \text{ and } r = f\ \theta\ .$$

Example 7.13

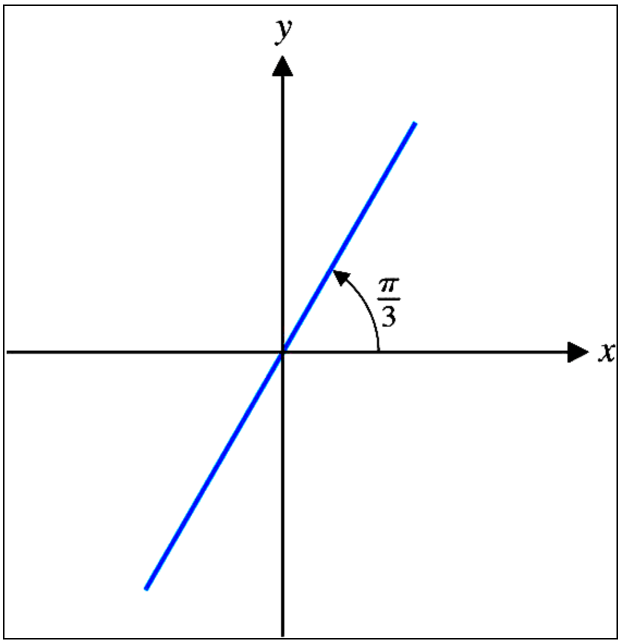
Sketch the graphs of (a) $r = 2$ and (b) $\theta = \dfrac{\pi}{3}$.

Solution

(a)



(b)



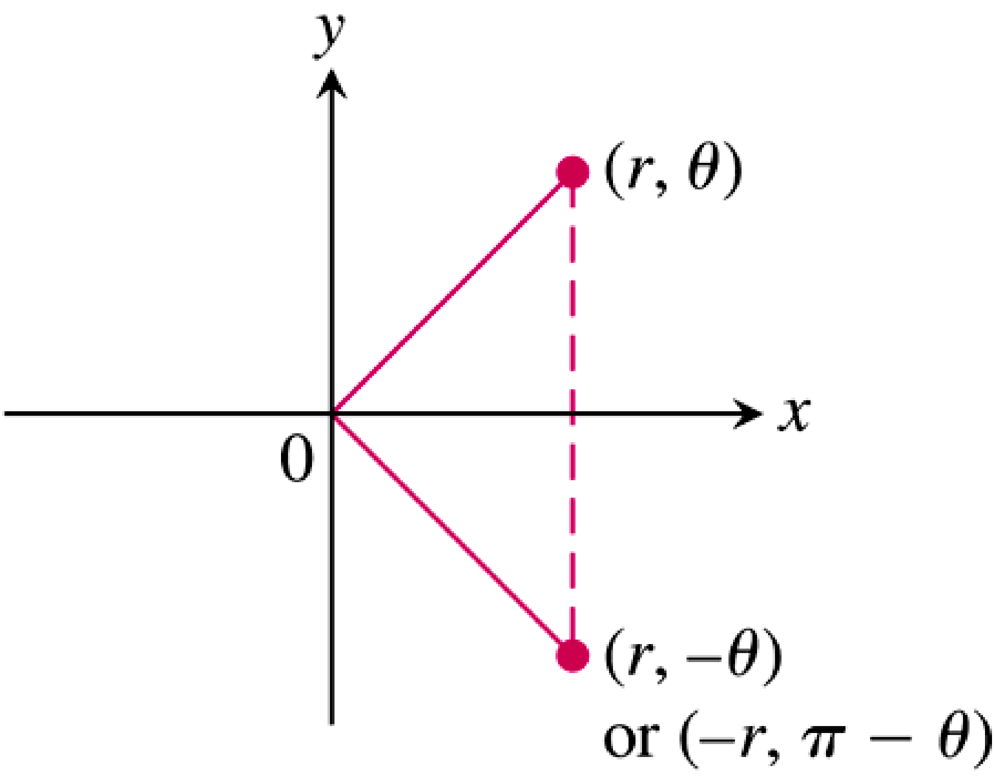
There are two methods to sketch a graph of $r = f(\theta)$:

(i) Form a table for r and θ. $(0 \le \theta \le 2\pi)$.
From the table, plot the (r, θ) points.

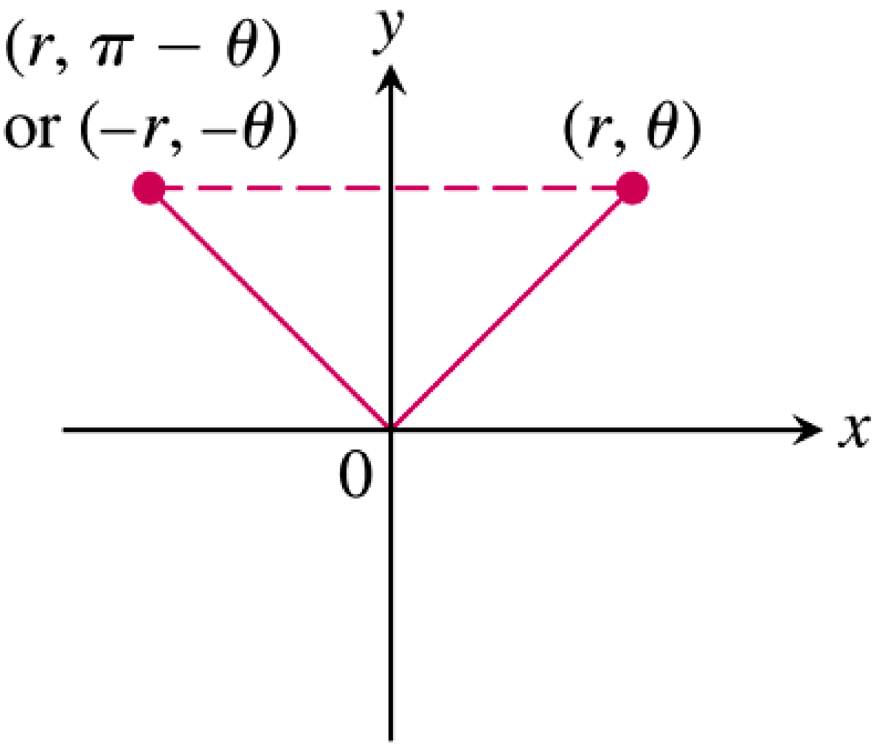
(ii) Symmetry test of the polar equation.
The polar equations is symmetrical
(a) about the x-axis if $f(\theta) = f(-\theta)$
(b) about the y-axis if $f(\theta) = f(\pi - \theta)$
(c) at the origin if $f(\theta) = f(\pi + \theta)$

Symmetry Tests for Polar Graphs

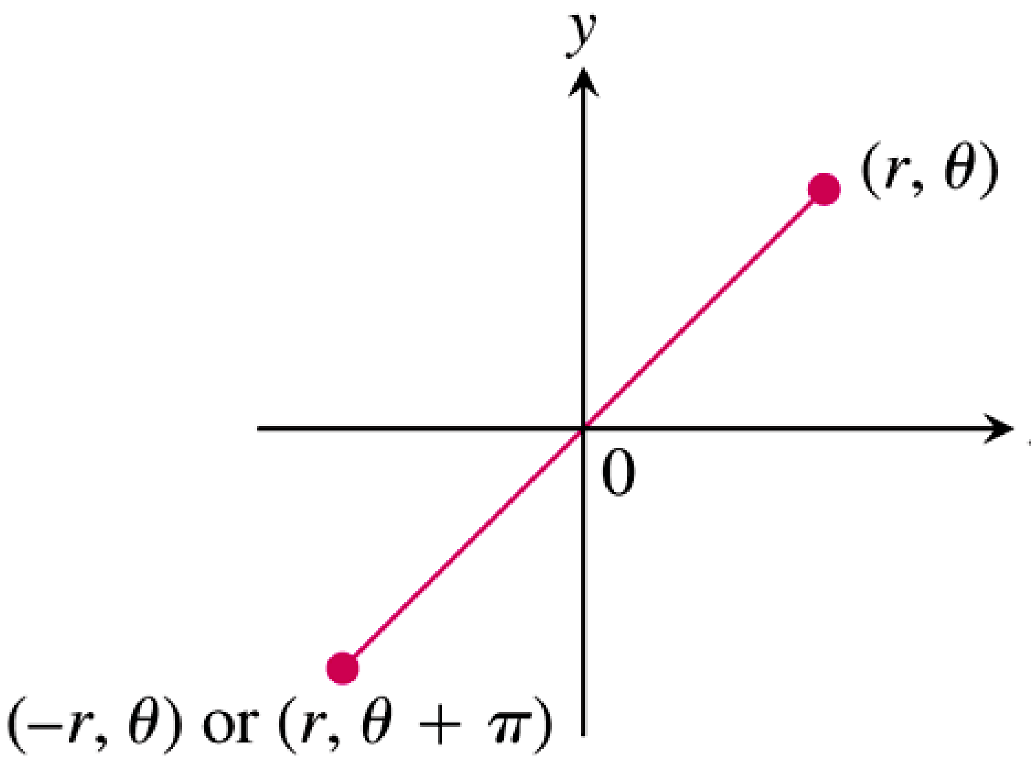
(a) About the x-axis



(b) About the *y*-axis



(c) About the origin/pole



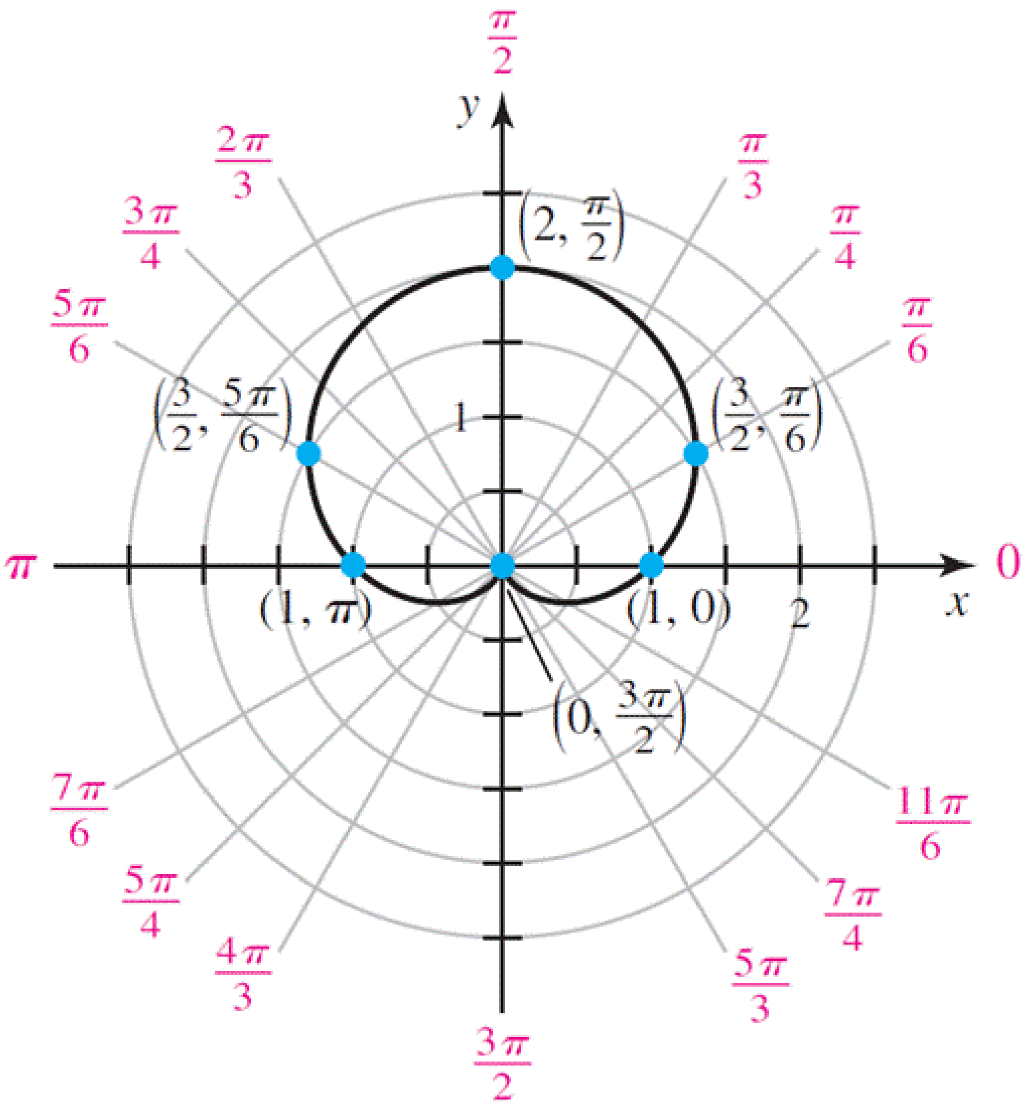
Example 7.14 Sketch the graphs of

(i) $r = 1 + \sin\theta$

(ii) $r = 1 - \cos\theta$

Solution

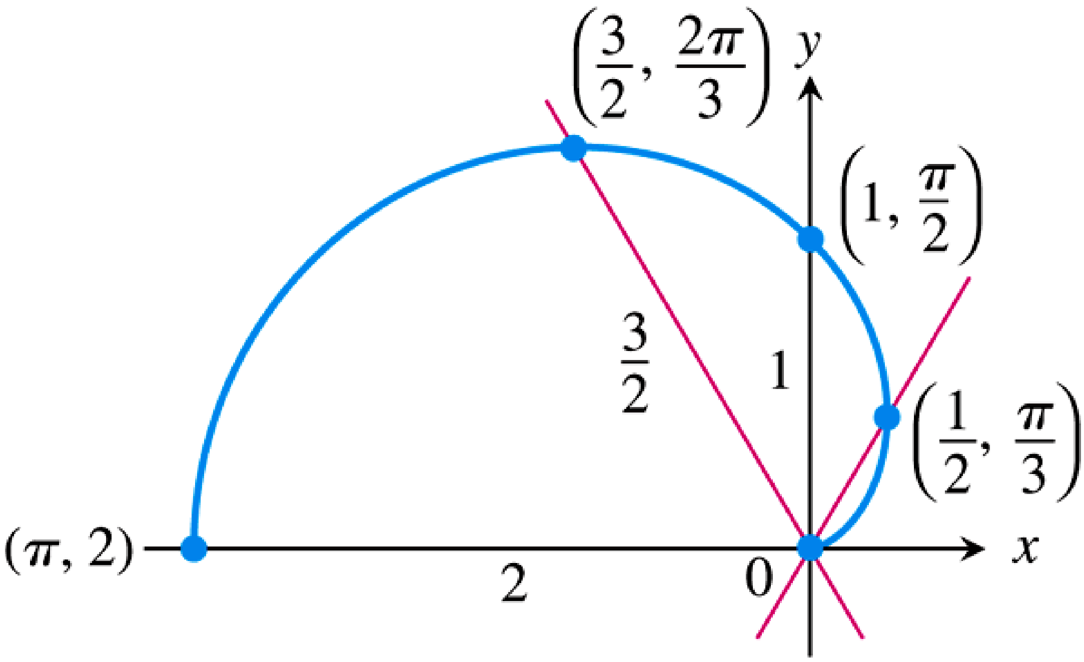
(i) $r = 1 + \sin\theta$, cardiod

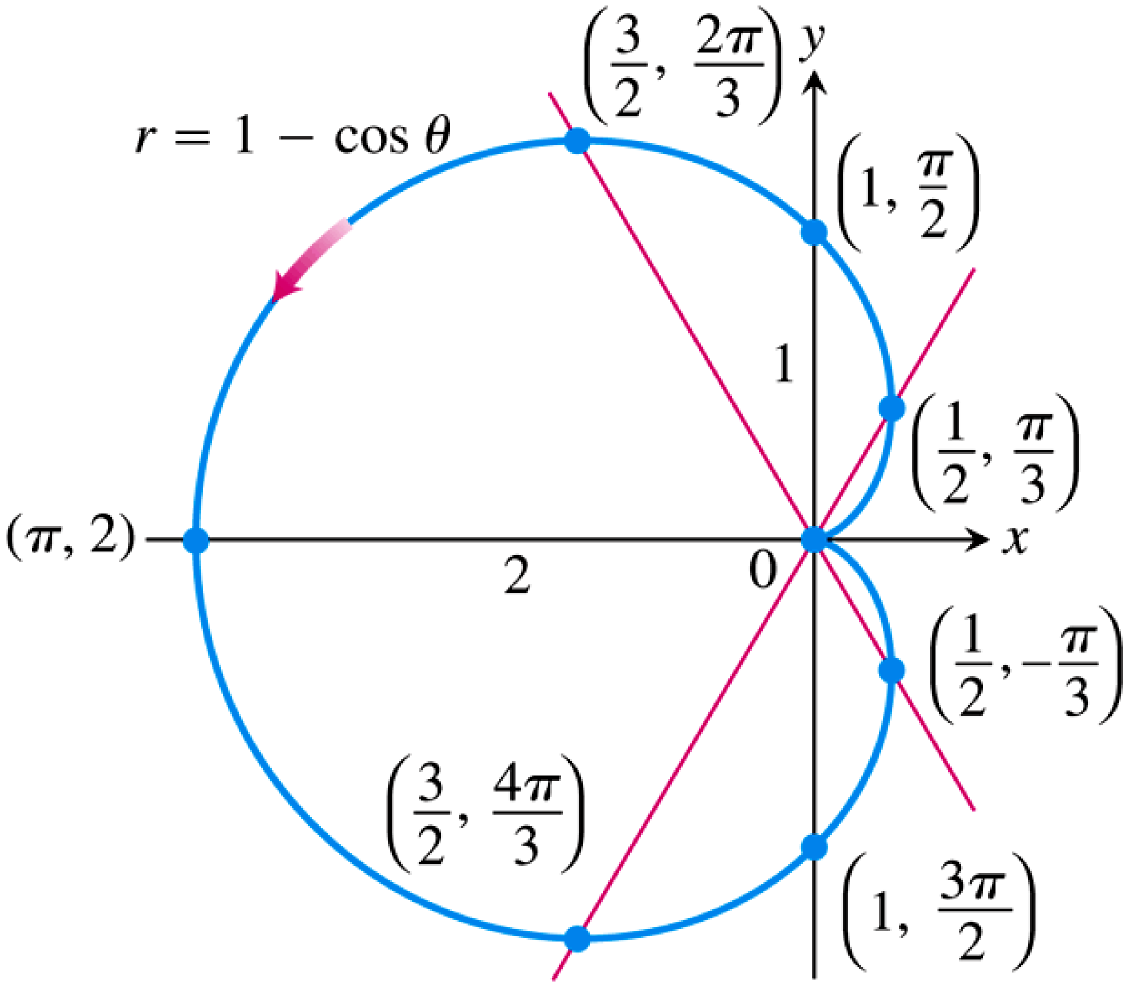


Cardioid $r = 1 + \sin\theta$

(ii)

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
$r = 1 - \cos\theta$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2





$\left(\frac{3}{2}, \frac{2\pi}{3}\right)$
y
$r = 1 - \cos\theta$
$\left(1, \frac{\pi}{2}\right)$
1
$\left(\frac{1}{2}, \frac{\pi}{3}\right)$
$(\pi, 2)$
x
2
0
$\left(\frac{1}{2}, -\frac{\pi}{3}\right)$
$\left(\frac{3}{2}, \frac{4\pi}{3}\right)$
$\left(1, \frac{3\pi}{2}\right)$

Example 7.15 Sketch the graphs of

(i) $r = 1 - 2\sin\theta$

(ii) $r = 2 - 2\sin\theta$

Example 7.16 Sketch the graphs of

(a) $r = 2\sin\theta$

(b) $r = \sin 2\theta$

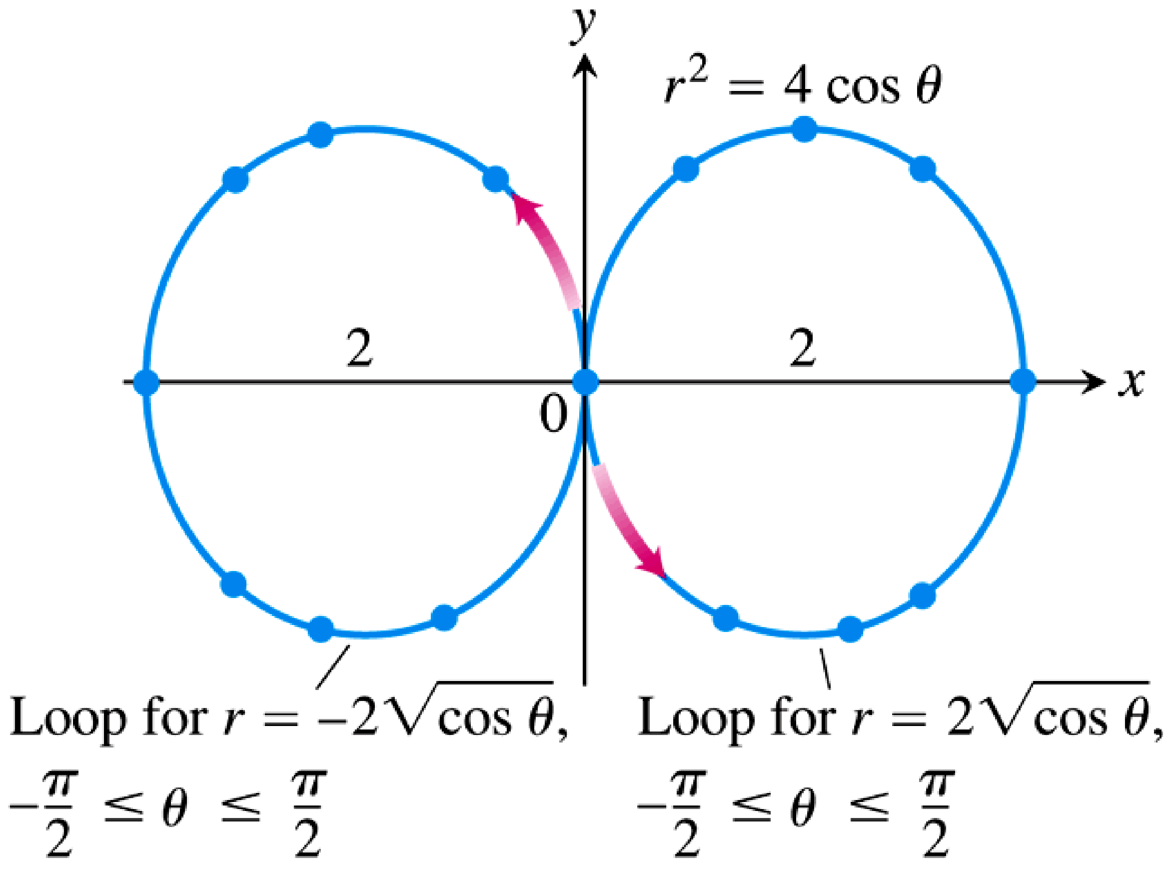
(c) $r = 2\sin^2\theta$

Example 7.17 Sketch the graph of the polar equation $r^2 = 4\cos\theta$, for $\theta \geq 0$.

Solution

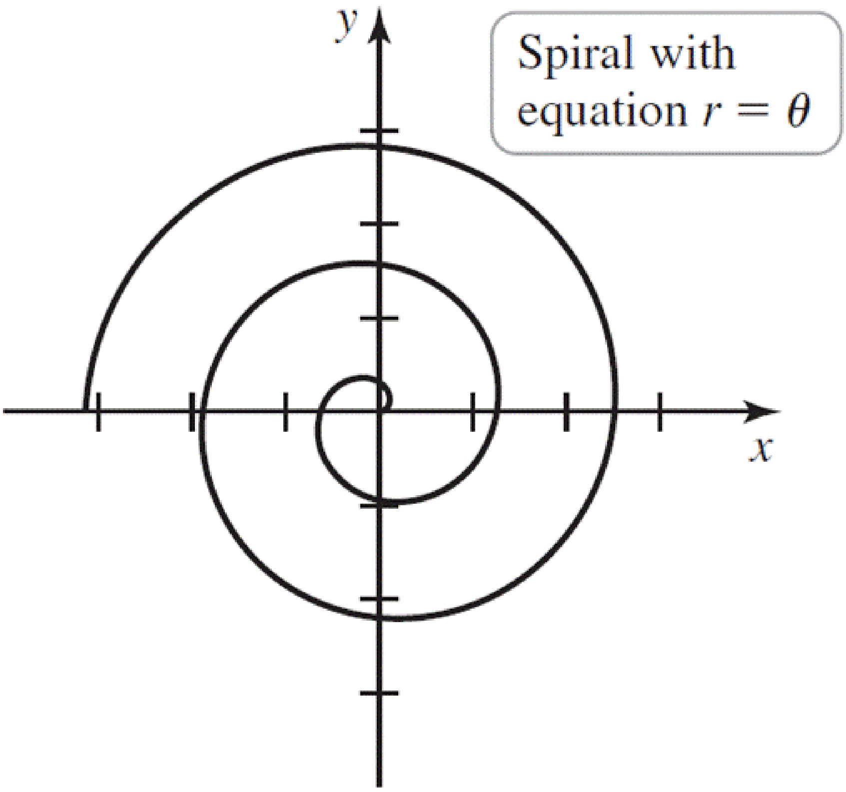
$r^2 = 4\cos\theta$, lemniscate

θ	$\cos\theta$	$r = \pm 2\sqrt{\cos\theta}$
0	1	± 2
$\pm\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\approx \pm 1.9$
$\pm\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\approx \pm 1.7$
$\pm\frac{\pi}{3}$	$\frac{1}{2}$	$\approx \pm 1.4$
$\pm\frac{\pi}{2}$	0	0



Example 7.18 Sketch the graph of the polar equation $r = \theta$, for $\theta \geq 0$.

Solution



7.2.4 Intersection of Curves in Polar Coordinates

- In Cartesian form, there is 1 – 1 correspondence between ordered pairs satisfying an equations and points on its graph.
- In Polar form, the 1 – 1 property is lost. Therefore, you must draw the graphs of the curves to locate all points of intersection.
- Method for finding points of intersection of Polar form curves must include sketching the curves.

Example 7.19

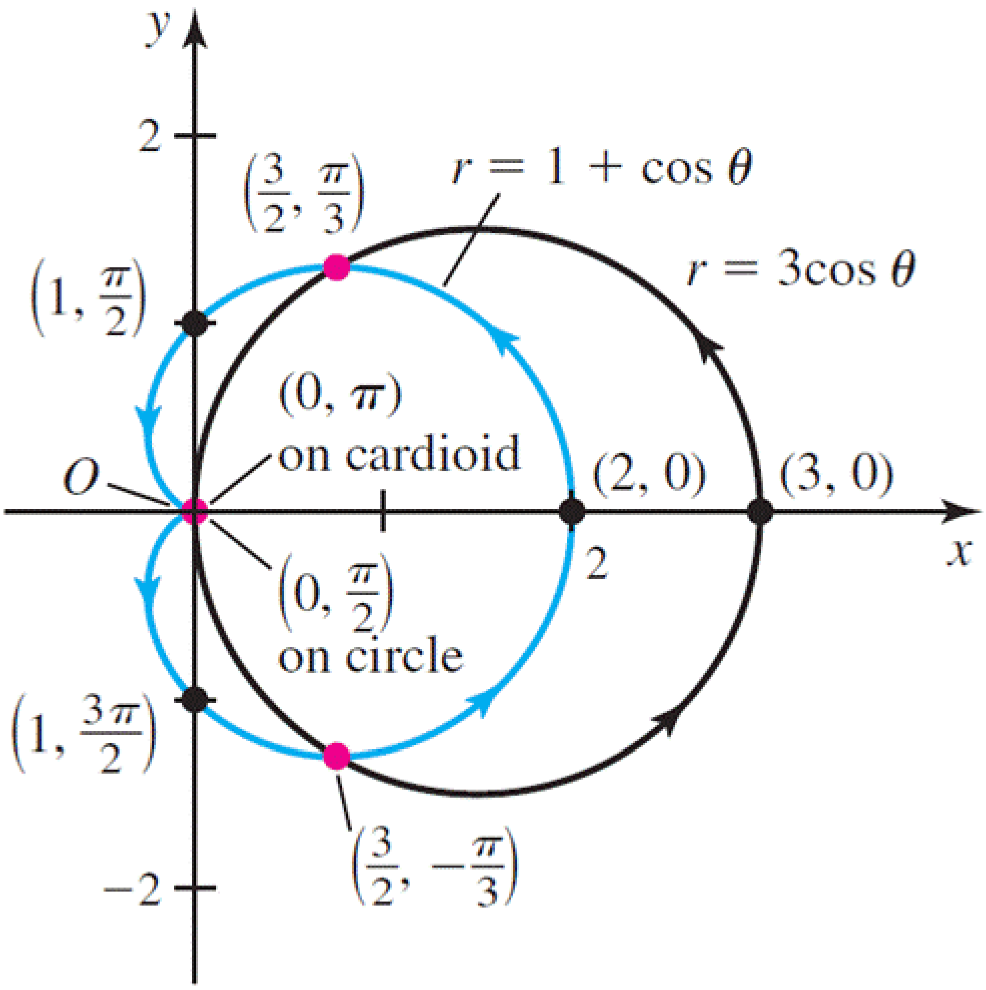
Find the points of intersection of the cardiod $r = 1 + \cos\theta$ and circle $r = 3\cos\theta$ for $0 \leq \theta \leq 2\pi$.

Solution

Sketch the graphs:
From the sketch that there are three intersections of the two curves.

The third intersection is at the origin. Notice that this does not arise from any solution of the equation $1 + \cos\theta = 3\cos\theta$.

- while both curves pass through the origin, they each do so for *different* values of θ.



Remark

To find points of intersection of two polar curves $r = f(\theta)$ and $r = g(\theta)$, keep in mind that points have more than one representation in polar coordinates. In particular, this says that points of intersection need not correspond to solutions of $f(\theta) = g(\theta)$.

Example 7.20

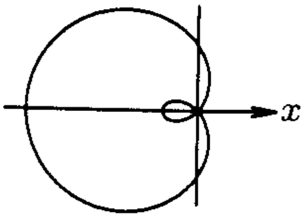
Find the points of intersection of the circle $r = 2\cos\theta$ and $r = 2\sin\theta$ for $0 \le \theta \le \pi$.

Example 7.21

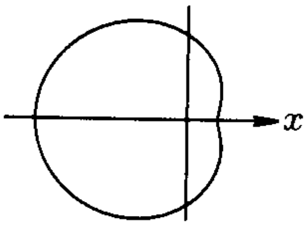
Find the points of intersection of the curves $r = 3/2 - \cos\theta$ and $\theta = 2\pi/3$.

Shapes of Curves In Polar Equation

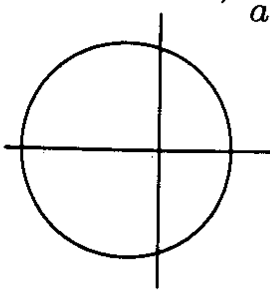
(a) LIMAÇON



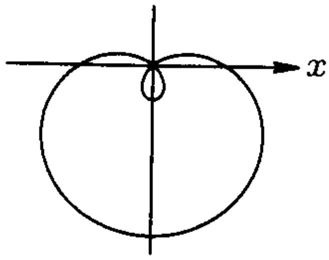
$r = b - a\cos\theta,\ \frac{b}{a} < 1$



$r = b - a\cos\theta,\ 1 < \frac{b}{a} < 2$



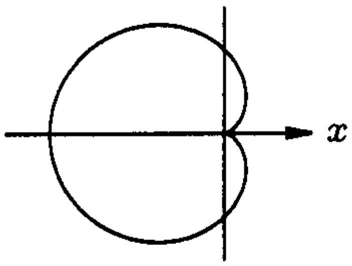
$r = b - a\cos\theta,\ \frac{b}{a} \geq 2$



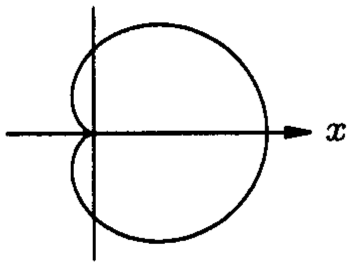
$r = b - a\sin\theta,\ \frac{b}{a} < 1$

Figure 4.29

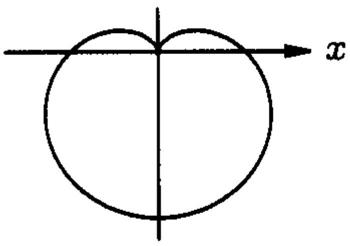
(b) CARDIOID



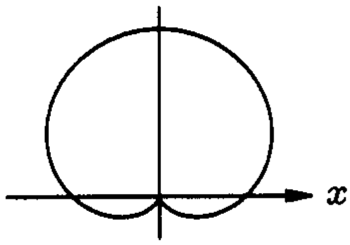
$r = a - a\cos\theta$



$r = a + a\cos\theta$

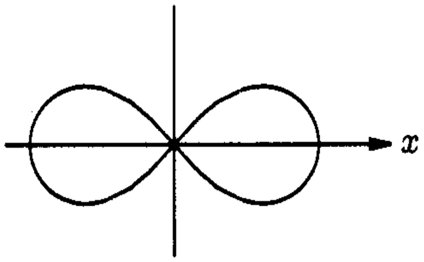


$r = a - a\sin\theta$

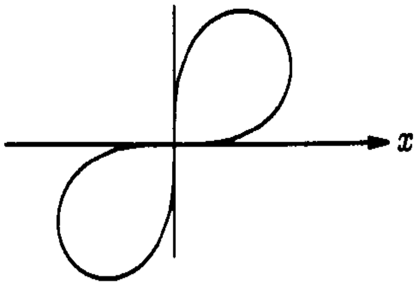


$r = a + a\sin\theta$

(c) LEMINISCATE

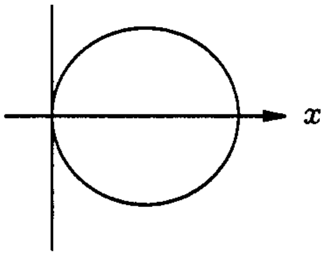


$r^2 = a^2 \cos 2\theta$

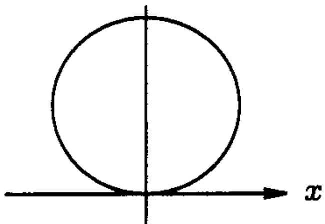


$r^2 = a^2 \sin 2\theta$

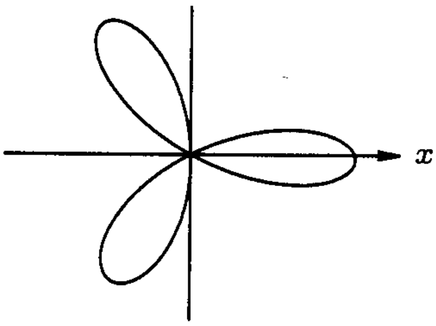
(d) ROSE PETALS



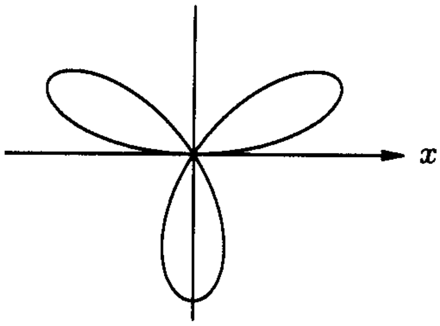
$r = a \cos \theta$



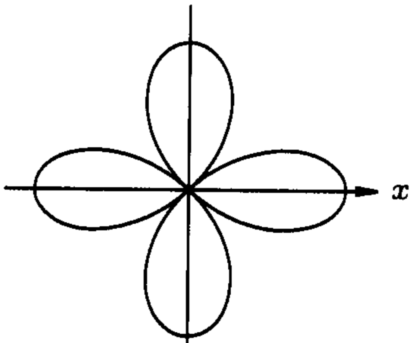
$r = a \sin \theta$



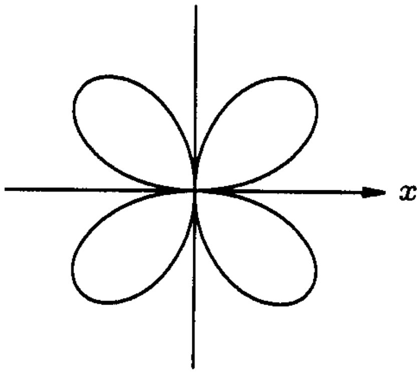
$r = a \cos 3\theta$



$r = a \sin 3\theta$



$r = a \cos 2\theta$



$r = a \sin 2\theta$