Chapter 8 Complex Numbers

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8.0 Complex Numbers

- Invented as an extension of real numbers in order to have a number system in which all polynomials have roots
- Have the unique property of representing and manipulating *two* variables as a *single* quantity

Three way to express a complex number:

- the Cartesian form, $z = a + ib$
- the Polar form, $z = r(\cos\theta + i\sin\theta)$
- the exponential form, $z = re^{i\theta}$

8.1 Definition

If z is a **complex number**, then it can be expressed in the form

$$z = a + bi,$$

where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$.

The **real part** of the complex number z is a.
The **imaginary part** of the complex number z is b.

$$z = a + bi$$
$$z = \text{Re}(z) + \text{Im}(z)$$

- A complex number is represented by a single variable, even though it is composed of two parts.
- The standard symbol for the set of all complex numbers is $\mathbb{C}$.

Example 8.1

(a) If $z_1 = 2 + 3i$, then

the real part of z_1, $\text{Re}(z_1) = 2$, and

the imaginary part of z_1, $\text{Im}(z_1) = 3$.

(b) Simplify

(i) $i^{17} = i^{16+1} = i^{4(4)+1} = i^1 = i$.

(ii) i^{99}.

(iii) $4i^4 - 6$.

What is i^n for a general positive integer n?

Example 8.2

Find all the roots of the equation $x^2 + 4 = 0$, if $x \in \mathbb{C}$.

8.2 Basic Operations on Complex Numbers

Given that $z_1 = a + bi$ and $z_2 = c + di$ where $z_1, z_2 \in \mathbb{C}$.

3.2.1 Equality

Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal.

If $z_1 = z_2$, then $a = c$ and $b = d$.

Example 8.3

For what values of *x* and *y* is $3x + 4i = (2y + x) + xi$?

8.2.2 Addition and subtraction

If $z_1 = a + bi$ and $z_2 = c + di$ are two complex numbers, then

$$z_1 \pm z_2 = a + c \pm b + d \ i$$

8.2.3 Multiplications

If $z_1 = a + bi$ and $z_2 = c + di$ are two complex numbers, and *k* is a constant, then

(i) $z_1 \cdot z_2 = (a + bi) \cdot (c + di)$

$$= (ac - bd) + (ad + bc)i$$

(ii) $kz_1 = ka + kbi$

Multiplication technique:

- multiply in the usual way and use property of $i^2 = -1$

8.2.4 Division and conjugate

If $z = a + bi$ then the **conjugate** of *z* is denoted as $\overline{z} = a - bi$.

$$z \cdot \overline{z} = a + b\mathrm{i} \cdot a - b\mathrm{i} = a^2 + b^2 = |z|^2.$$

$$z^{-1} = \frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{\overline{z}}{|z|^2}$$

Division technique:

- Multiply numerator and denominator by the conjugate of the denominator

Example 8.4

Simplify $3i \div 5 - 2i$.

Solution

$$\frac{3i}{5-2i}=\frac{3i}{5-2i}\cdot\frac{5+2i}{5+2i}$$

$$=\frac{15i+6i^2}{25-4i^2}=\frac{-6+15i}{29}=-\frac{6}{29}+\frac{15}{29}i$$

Example 8.5

Given that $z_1 = 1-2i$, $z_2 = -3+4i$ and $z_3 = -2-i$ are complex numbers.

(a) Find $z_1 + 2z_2$.

(b) Find $z_2 \cdot z_3$.

(c) Find $\overline{z_1} \cdot z_3$.

(d) Write $\frac{2}{z_1}$ in $a+bi$ form.

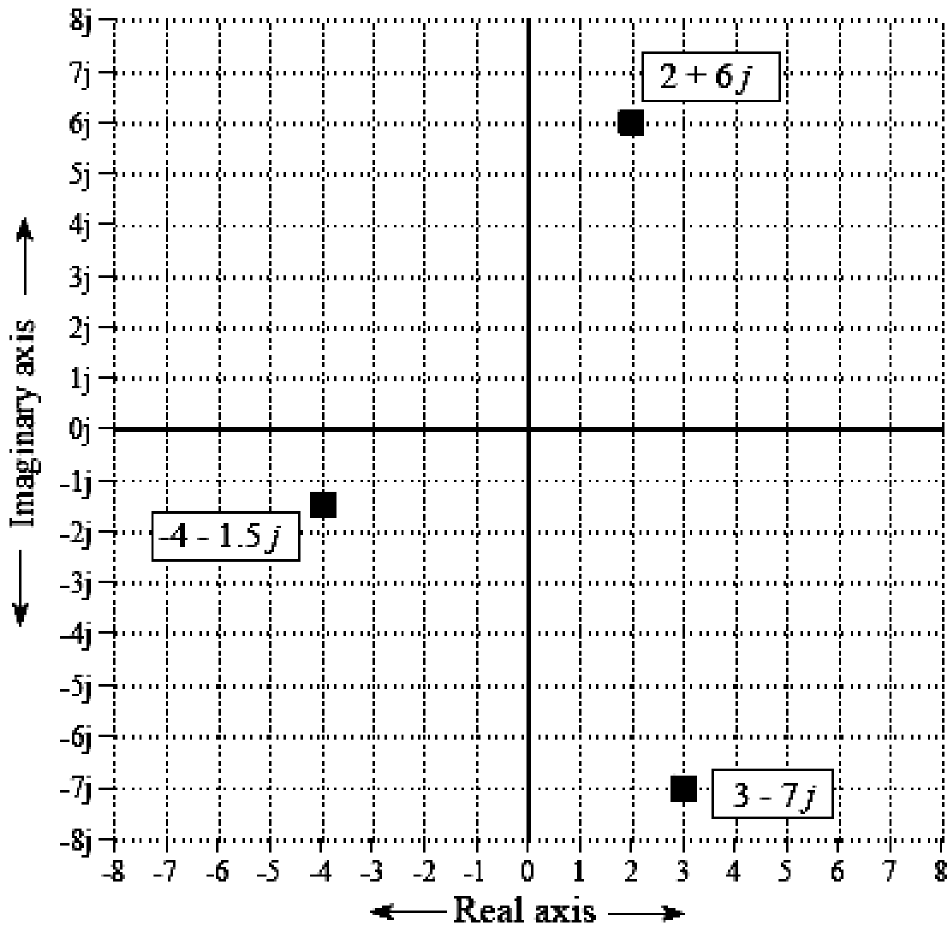
(e) Find $\frac{z_2}{z_3}$, and express it in $a+bi$ form.

8.3 Graphing Complex Numbers

8.3.1 The Complex Plane/Argand diagram

Complex numbers are represented by locations in a two-dimensional display called the **complex plane**.

- Every complex number has a unique location in the complex plane, as illustrated by the three examples shown here.
- The horizontal axis represents the real part, while the vertical axis represents the imaginary part.



Note

The *real number line* is the same as the *x-axis* of the complex plane.

Example 8.6

Sketch the following complex numbers on the same diagram.

(a) $z_1 = 3 + 2i$ (b) $z_2 = 3 - 2i$

(c) $z_3 = -3 - 2i$ (d) $z_4 = -3 + 2i$

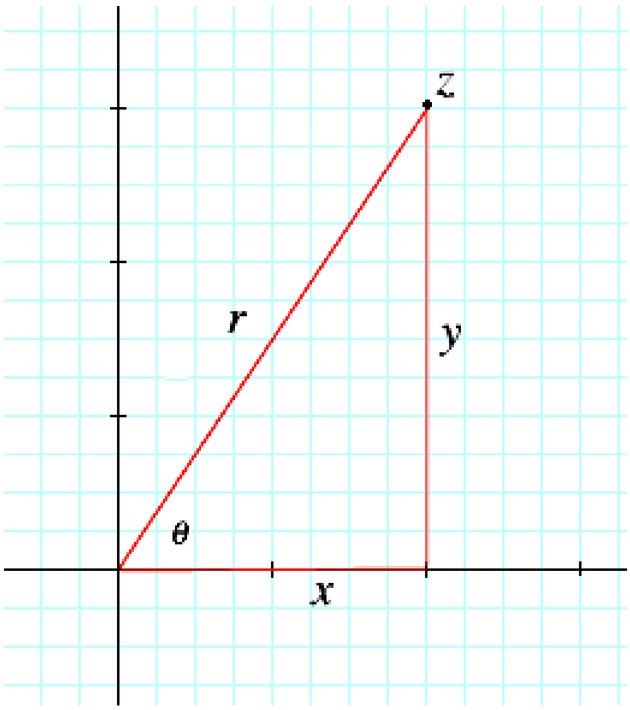
Remark

Real numbers are special cases of complex numbers. For instance, the real number 2 is $2 + 0i$.

8.3.2 Polar form of Complex Numbers

In rectangular coordinates, the *x* and *y* specify a complex number $z = x + yi$ by giving the distance *x* right and the distance *y* up.

Polar coordinates specify the same point *z* by stating how far *r* away from the origin 0, and the angle θ for the line from the origin to the point.



The distance *r*, is known as the **modulus** of *z*, and is denoted as $|z|$.

$$r = |z| = \sqrt{x^2 + y^2}\,.$$

Angle θ is called the **argument** of *z*, denoted Arg (*z*).

θ is known as the **principle argument** if $-\pi \leq \theta \leq \pi$.

From the diagram above, we find the following three relations:

$$\tan\theta = \frac{y}{x},\ x = r\cos\theta, \text{ and } y = r\sin\theta.$$

If we apply these relations to our complex number $z = x + yi$, then we get an alternate description for *z*,

$$\boxed{z = r(\cos\theta + i\sin\theta)}.$$

Example 8.7

Express the following complex numbers in polar form.

(a) $z_1 = 1 + i$ (b) $z_2 = -1 + i$

(c) $z_3 = -2 + 2i$ (d) $z_4 = -2 - 2i$

Products and quotients in polar form

Consider two complex numbers:

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$
$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

Multiplying:

$$z_1 z_2 = r_1 r_2\left[\cos\left(\theta_1 + \theta_2\right) + i\sin\left(\theta_1 + \theta_2\right)\right]$$

so $|z_1 z_2| = r_1 r_2$ and $\mathrm{Arg}(z_1 z_2) = \theta_1 + \theta_2$.

Similarly for quotients of complex numbers in polar form

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$= \frac{r_1}{r_2}\left[\cos\left(\theta_1 - \theta_2\right) + i\sin\left(\theta_1 - \theta_2\right)\right].$$

so $\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2}$ and $\mathrm{Arg}\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2$.

Thus,

$|z_1 z_2| = |z_1||z_2|$ and
$\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$

$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ and $\mathrm{Arg}\left(\frac{z_1}{z_2}\right) = \mathrm{Arg}(z_1) - \mathrm{Arg}(z_2)$

8.4 De Moivre's Theorem and its Applications

De Moivre's Theorem:

If $z = r(\cos\theta + i\sin\theta)$ and $n \in \mathbb{R}$, then

$$z^n = r^n \ \cos n\theta + i\sin n\theta \ .$$

Example 8.8

(a) Write $1-i$ in the polar form, then find the value of $(1-i)^{12}$ by using De Moivre's Theorem.

(b) Find $\left[\sqrt{2}\ \cos 10^0 + i\sin 10^0\ \right]^{10}$.

Solution

(a) Let $z = 1 - i$.

$$|z| = \sqrt{a^2 + b^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}.$$

As z is in the fourth quadrant,

$$\text{Arg}(z) = \theta = \tan^{-1}(1) = -\frac{\pi}{4}$$

So $z = \sqrt{2}\ \cos\ -\frac{\pi}{4}\ + i\sin\ -\frac{\pi}{4}\ $.

Using De Moivre's Theorem,

$$1-i\ ^{12}=\sqrt{2}\ ^{12}\ \cos\ -\frac{12\pi}{4}\ +i\sin\ -\frac{12\pi}{4}$$

$$=64\ \cos\ -3\pi\ +i\sin\ -3\pi\ \ =-64$$

(b) *Complete the solution…*

Example 8.9

Simplify $\dfrac{1+i\ ^{6}}{1-i\sqrt{3}\ ^{4}}$.

Example 8.10

Simplify (i) $\dfrac{3-3i\ ^{4}}{\sqrt{3}+i\ ^{3}}$ (ii) $\dfrac{\sqrt{3}+i\ ^{4}}{1-i\ ^{3}}$

Ans.: (i) $\dfrac{81i}{2}$

(ii) $4\sqrt{2}\ \cos\ -\dfrac{7\pi}{12}\ +i\sin\ -\dfrac{7\pi}{12}$

De Moivre 's Theorem can be used to find all of the ***n*th roots** of any number.

$\sin\theta = \sin\ \theta + k2\pi$ and

$\cos\theta = \cos\ \theta + k2\pi$ for all $k \in \mathbb{Z}$.

$$r\ \cos\theta + i\sin\theta\ ^{\frac{1}{n}}$$

$$= \ r\ \cos\ \theta + k2\pi\ + i\sin\ \theta + k2\pi\ ^{1/n}$$

$$= r^{1/n}\left(\cos\frac{\theta + k2\pi}{n} + i\sin\frac{\theta + k2\pi}{n}\right)$$

for $k = 0,1,2,\cdots,n-1$

Substituting $k = 0, 1, 2,\cdots,n-1$ yields the *n*th roots of the given complex number.

Example 8.11

Let $z = -\frac{1}{2}$, find the sixth root of *z*.

Example 8.12

Find the three cube roots of $z = -\frac{1}{2} + \frac{1}{2}i$.

Complex Roots of Unity

- Solutions of $z^n = 1$ with $n \in \mathbb{N}$ are called roots of unity
- Zeros of polynomial $z^n - 1$, so expect *n* solutions

Example: Cube root of unity

- $n = 3$: $z^3 = 1$

  $|z| = 1\,, \theta = 0$

  $$z^3 = 1\ \cos\ 2n\pi\ + i\sin\ 2n\pi$$

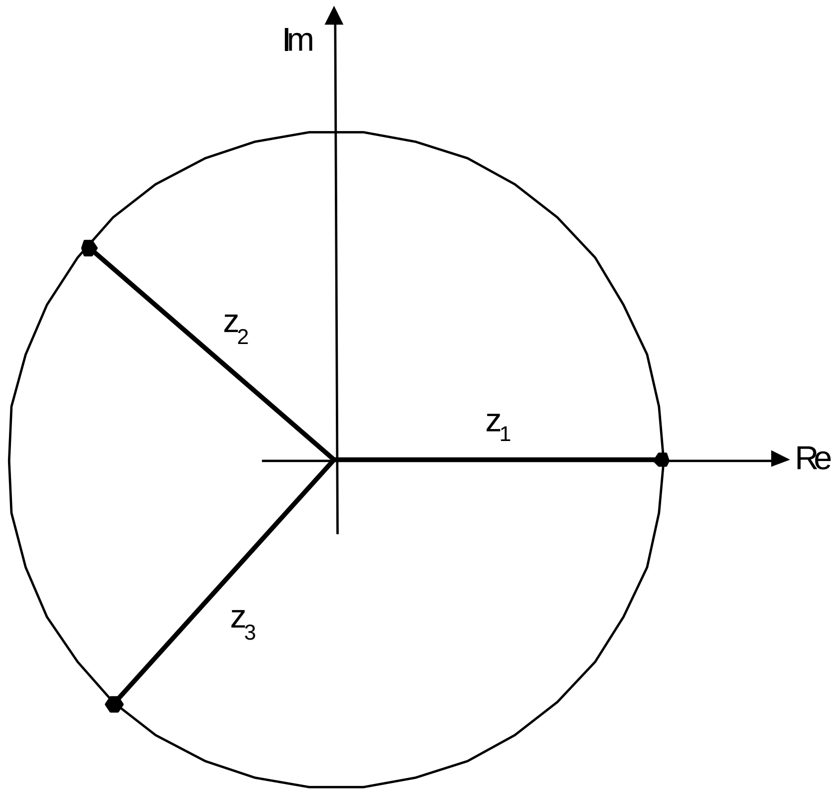
  $$\therefore z = 1^{1/3}\ \cos\ \frac{2n\pi}{3}\ + i\sin\ \frac{2n\pi}{3}$$

  Hence,

  $$z_1 = 1$$

  $$z_2 = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

  $$z_3 = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$



Now look at what happens when we square z_2 :

$$z_2^{\ 2} = \left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^2 = -\frac{1}{2}-\frac{\sqrt{3}}{2}i = z_3$$

Similarly $z_3^{\ 2} = z_2$.

Hence the roots of unity can be denoted as

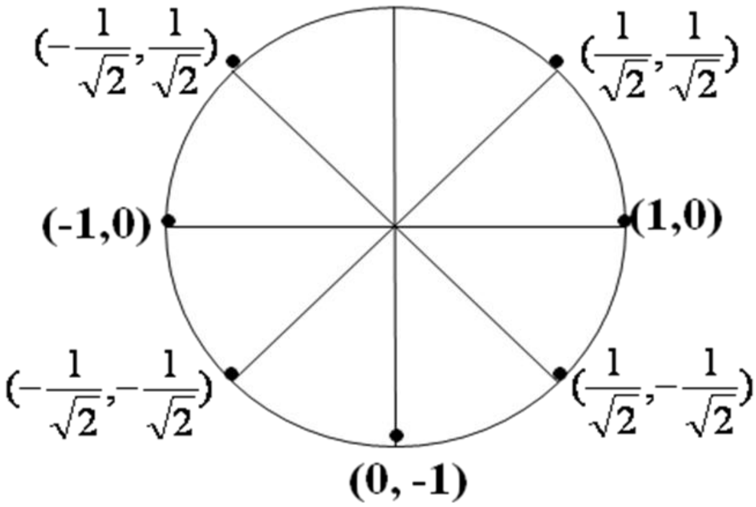
$$1, \omega, \omega^2$$

where ω is a complex root of unity.

It can also be shown that

$$1+\omega+\omega^2=0.$$

- $n=8$:

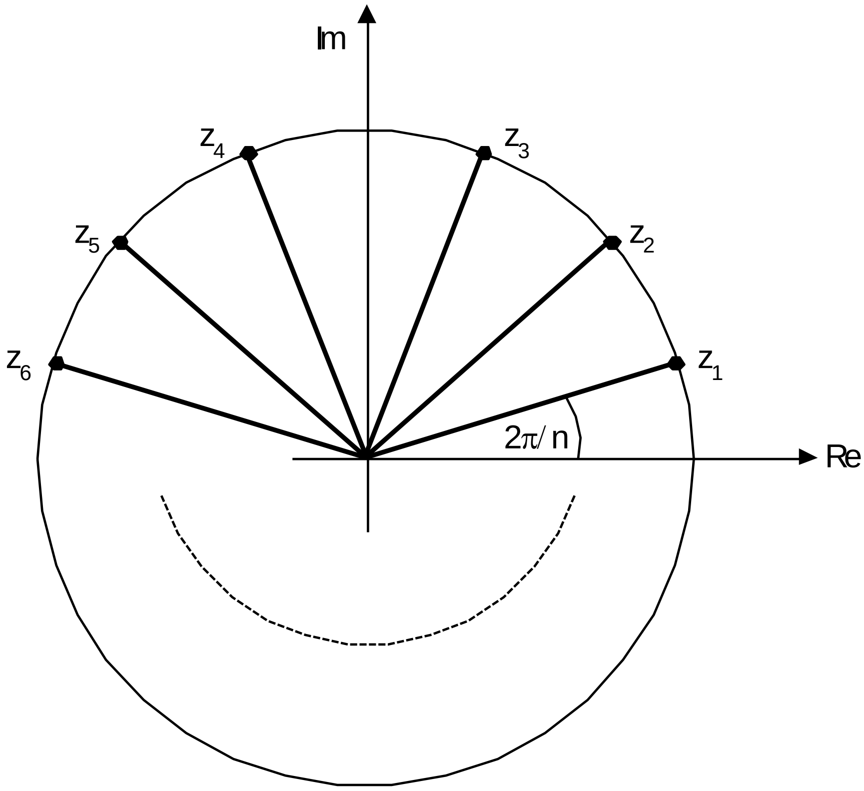


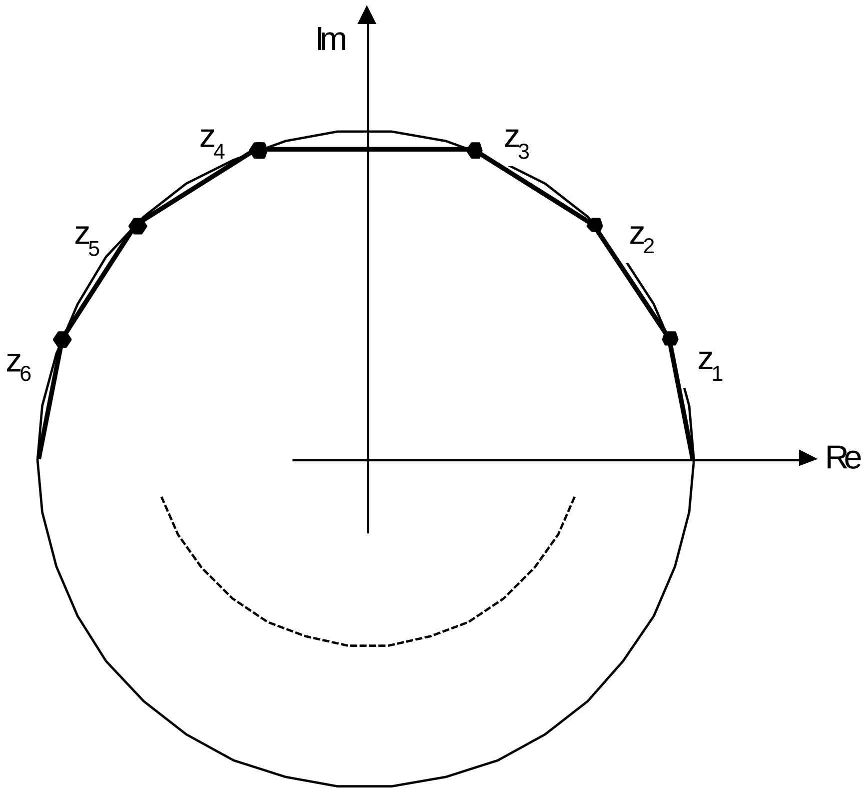
- What relationship would exists between the roots?
- Can you deduce a pattern for the relationship between the n^{th} roots of unity?

In general the n^{th} roots of unity $z=1^{1/n}$

- will produce *n* equally spaced roots, separated by angles of $\frac{2\pi}{n}$,
- all roots will lie on the unit circle

- The complex numbers form vertices of an *n*-sided polygon





8.5 EULER'S FORMULA

- Polar form of complex numbers

Definition: Euler's Formula

Euler's formula states that

$$e^{i\theta} = \cos\theta + i\sin\theta$$

It follows that

$$e^{in\theta} = \cos n\theta + i\sin n\theta$$

Form the definition, if *z* is any complex number with modulus *r* and Arg(z) θ, then

$$\begin{aligned} z &= r\cos\theta + ir\sin\theta \\ &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

Example 8.13

Express the following complex numbers in the form of $re^{i\theta}$.

(a) $2 + 2\sqrt{3}i$ (b) $2 - 4i$

(c) $-5i$ (d) -6

Solution

(a) $2+2\sqrt{3}i \Rightarrow r = \sqrt{(2)^2 + \left(2\sqrt{3}\right)^2} = 4$

$$\theta = \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

Thus, $2+2\sqrt{3}i = 4e^{\frac{\pi}{3}i}$

(b) *Complete the solution…*

Example 8.14

Find complex number expressions, in Cartesian form, for

(a) $e^{\frac{\pi}{4}i} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i.$

(b) e^{-i}

(c) $e^{i\pi}$

Conjugation, Multiplication, Division in Polar Form

- If $z = re^{i\theta}$, then $\bar{z} = r\overline{e^{i\theta}} = re^{\overline{i\theta}} = re^{-i\theta}$
- So conjugation corresponds to $\theta \to -\theta$
- For two numbers in polar form, $z = re^{i\theta}$ and $w = \rho e^{i\phi}$, we have

$$zw = r\rho e^{i(\theta+\phi)}$$

and

$$\frac{z}{w} = \frac{r}{\rho} e^{i(\theta-\phi)}$$

- So to multiply/divide: multiply/divide moduli, and add/subtract angles

8.5.1 Euler's Formula and the *n*th Power of a Complex Number

We know that a complex number can be express as $z = re^{i\theta}$, then

$$z^2 = r^2 e^{i2\theta}$$
$$z^3 = r^3 e^{i3\theta}$$
$$z^4 = r^4 e^{i4\theta}$$
$$\vdots$$
$$z^n = r^n e^{in\theta}$$

Example 8.15

Express the complex number $z = -1 + \sqrt{3}i$ in the form of $re^{i\theta}$. Then find

(a) z^2 (b) z^3 (c) z^7

8.5.2 Euler's Formula and the *n*th Roots of a Complex Number

The *n*-th roots of a complex number can be found using the Euler's formula. Note that if $z^n = re^{i\theta}$, then,

$$z = \left[re^{i\theta}\right]^{\frac{1}{n}} = r^{\frac{1}{n}} e^{i\,\frac{\theta+2k\pi}{n}}$$

for $k = 0, 1, 2, \cdots, n-1$

Substituting $k = 0, 1, 2, \cdots, n-1$ yields the *n*th roots of the given complex number.

Example 8.16

Solve $z^2 = 1 + i$.

Example 8.17

Find all the roots of $z^3 = 3 + 4i$.

Ans.: cube roots of 3+4*i* ≈ 1.63+0.52*i*, −1.26+1.15*i*, −0.36−1.67*i*

Example 8.18

Find all the roots of $z^3 = \sqrt{3} - i$.

Example 8.19

(a) Solve the equation $z + 2i \;^3 = 216i$

(b) Prove that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$, hence find the roots of the equation $16p^4 - 16p^2 + 3 = 0$.

Ans.: (b) $p_1 = \cos\ 30^\circ \ = 0.866$ $\quad p_2 = \cos\ 60^\circ \ = 0.5$

$p_3 = \cos\ 120^\circ \ = -0.5$ $\quad p_4 = \cos\ 150^\circ \ = -0.866$

Cos and sin as complex exponentials

Definition

$$\cos\,\theta\ = \frac{1}{2}\ e^{i\theta} + e^{-i\theta}$$

$$\sin\,\theta\ = \frac{1}{2}\ e^{i\theta} - e^{-i\theta}$$

- For real θ, these give the real and imaginary part of $e^{i\theta}$ respectively
- For complex θ, they provide the “natural” extensions of cos and sin to the whole complex plane

Application to higher-degree equations

Certain higher-degree equations can be brought into quadratic form and solved this way. For example, the 6th-degree equation in *x*:

$$x^6 - 4x^3 + 8 = 0$$

can be rewritten as:

$$(x^3)^2 - 4(x^3) + 8 = 0$$

or, equivalently, as a quadratic equation in a new variable *u*:

$$u^2 - 4u + 8 = 0$$

where $u = x^3$.

Solving the quadratic equation for *u* results in the two solutions:

$$u = 2 \pm 2i$$

Thus $$x^3 = 2 \pm 2i$$

Finding the three cube roots of $2 + 2i$ – the other three solutions for x will be their complex conjugates

Rewriting the right-hand side using Euler's formula:

$$x^3 = 2^{\frac{3}{2}} e^{\frac{1}{4}\pi i} = 2^{\frac{3}{2}} e^{\frac{8k+1}{4}\pi i}$$

(since $e^{2k\pi i} = 1$), gives the three solutions:

$$x = 2^{\frac{1}{2}} e^{\frac{8k+1}{12}\pi i} \qquad , k = 0, 1, 2$$

Using Eulers' formula again together with trigonometric identities such as $\cos(\pi/12) = (\sqrt{2} + \sqrt{6}) / 4$, and adding the complex conjugates, gives the complete collection of solutions as:

$$x_{1,2} = -1 \pm i$$

$$x_{3,4} = \frac{1 + \sqrt{3}}{2} \pm \frac{1 - \sqrt{3}}{2} i$$

and

$$x_{5,6} = \frac{1-\sqrt{3}}{2} \pm \frac{1+\sqrt{3}}{2}\, i$$

Example 8.20

Find the fifth-roots of $\sqrt{3}+i$ expressed in trigonometric form.

Ans.: The five fifth-roots are

$$z_1 = 2^{1/5}\ \cos 6^\circ + i \sin 6^\circ$$

$$z_2 = 2^{1/5}\ \cos 78^\circ + i \sin 78^\circ$$

$$z_3 = 2^{1/5}\ \cos 150^\circ + i \sin 150^\circ$$

$$z_4 = 2^{1/5}\ \cos 222^\circ + i \sin 222^\circ$$

$$z_5 = 2^{1/5}\ \cos 294^\circ + i \sin 294^\circ$$