Chapter 6: Vectors

6.1 Vectors in 3-dimensional space

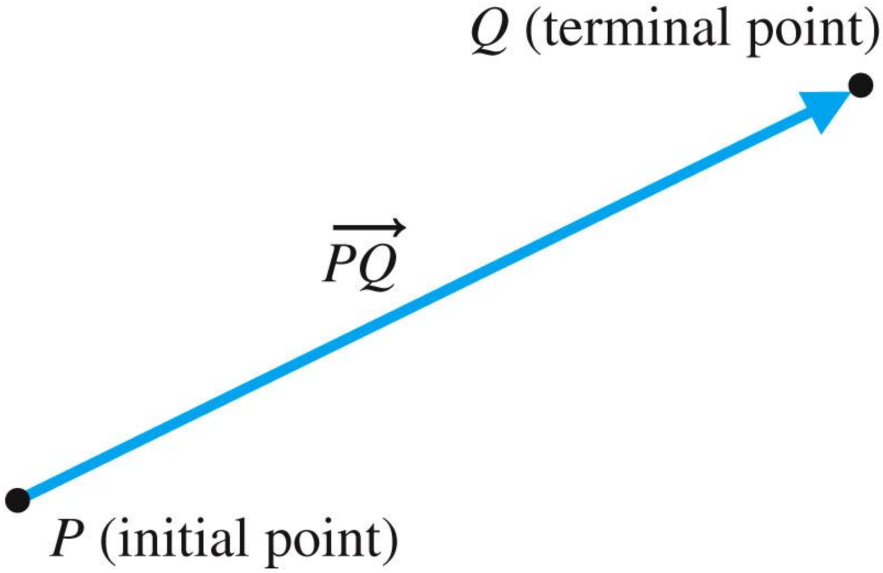
6.2 Scalar Products

6.3 Vector products

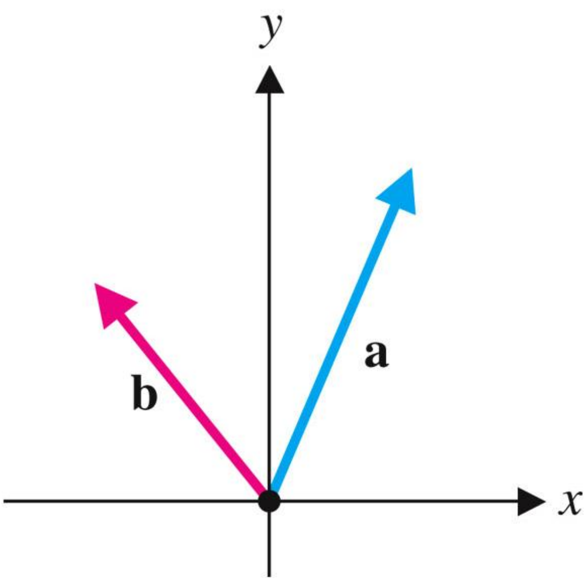
6.4 Lines and Planes in space

6.1 Vector in 3-Dimensional Space

A **vector** is a quantity that has both **magnitude** and **direction**.

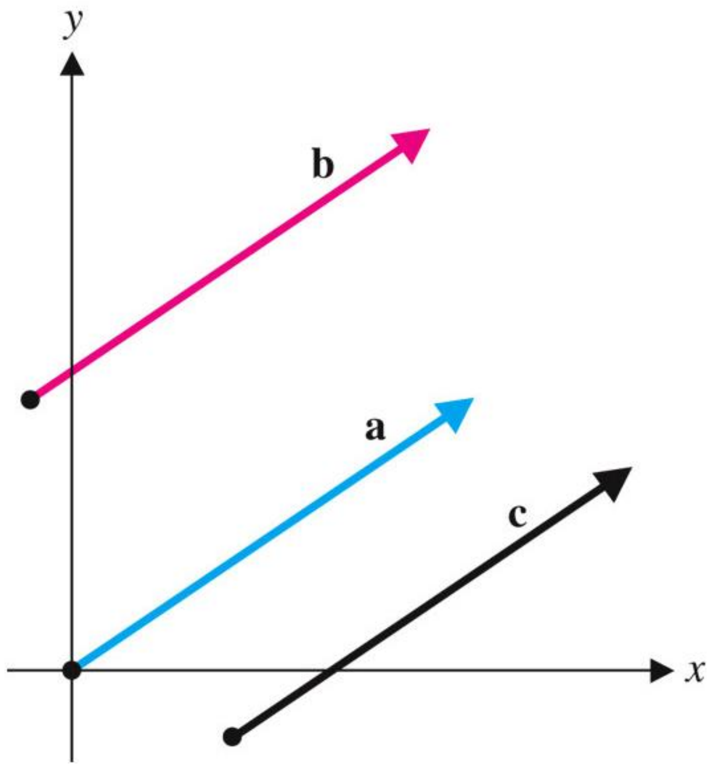


Any point in 2-dimensional plane or 3-dimensional space can represent a **position vector.**

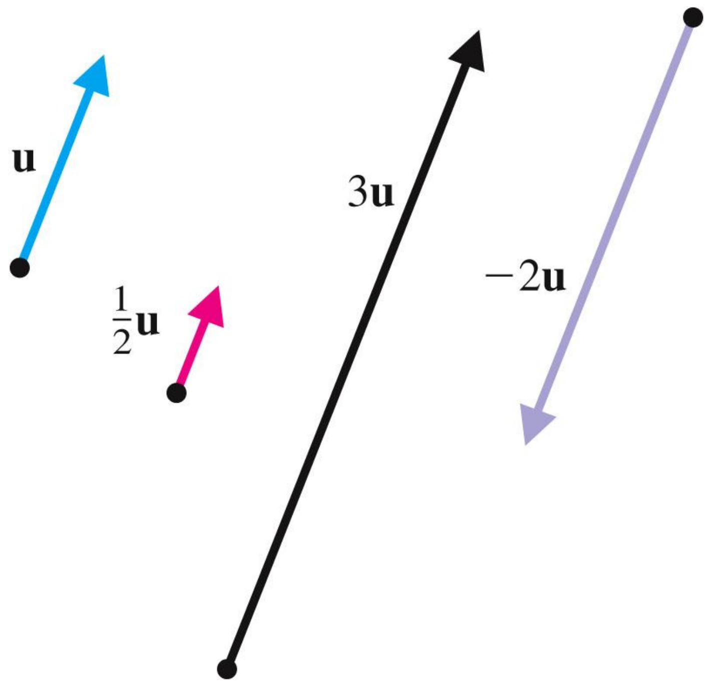


A vector can be written as $\overrightarrow{PQ}$, or **a**, or $\hat{a}$.

Two vectors are equal if both vectors have the **same magnitude** and **direction**.

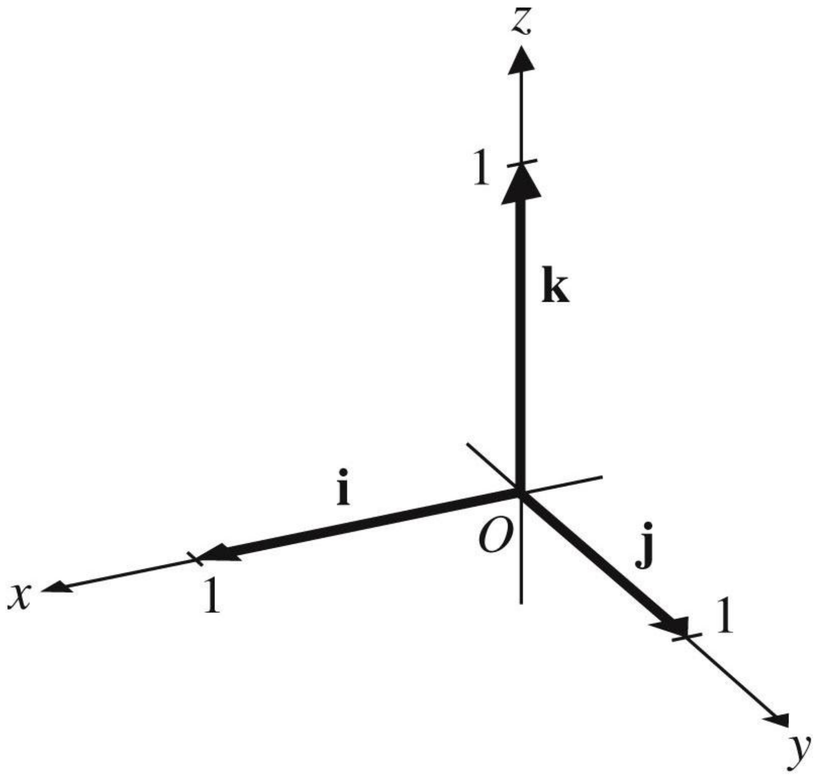


If $\mathbf{v} = k\mathbf{w}$ with k as a constant, then **v** and **w** are parallel vectors.



Any vector that has a magnitude of one unit is called a **unit vector**.

Three **standard unit vectors** are: **i**, **j** and **k**.



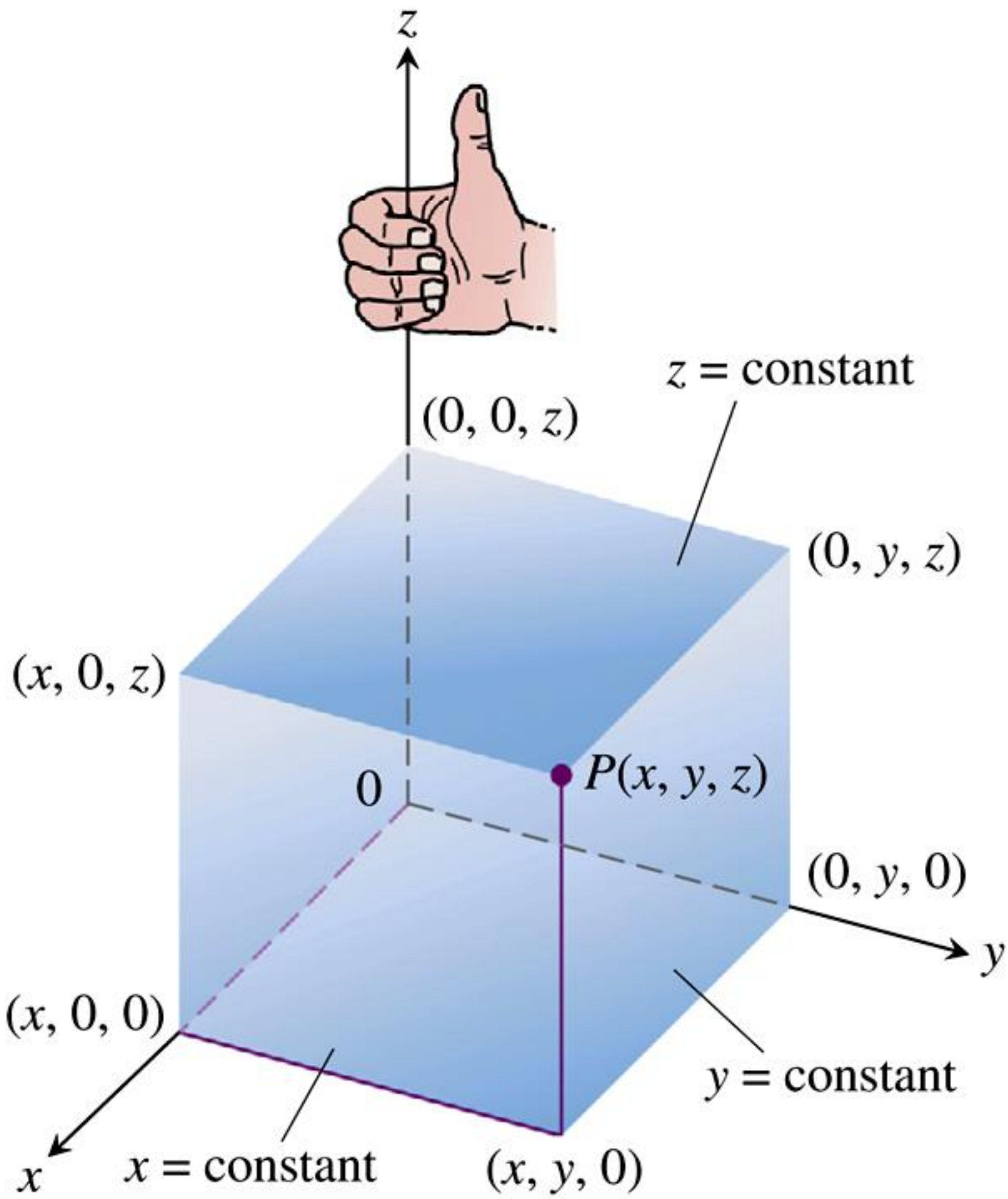
Vectors **i**, **j** and **k** can be written in components form:

$$\mathbf{i} = \langle 1,0,0 \rangle,\ \mathbf{j} = \langle 0,1,0 \rangle \text{ and } \mathbf{k} = \langle 0,0,1 \rangle$$

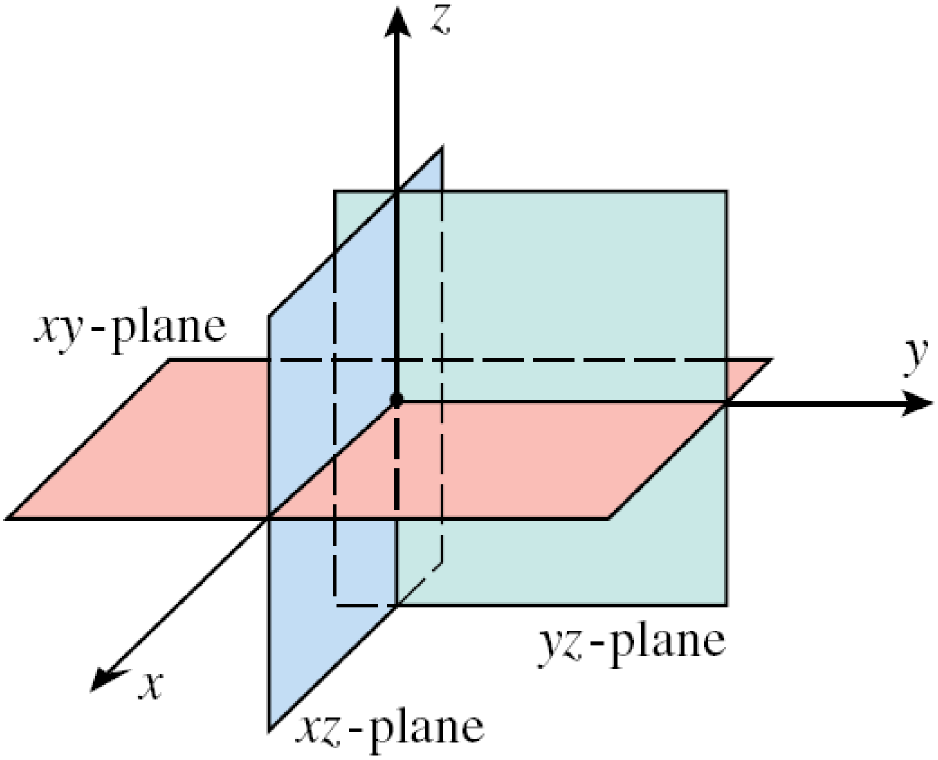
Any vector can be written as a linear combination of the vectors **i**, **j** and **k**

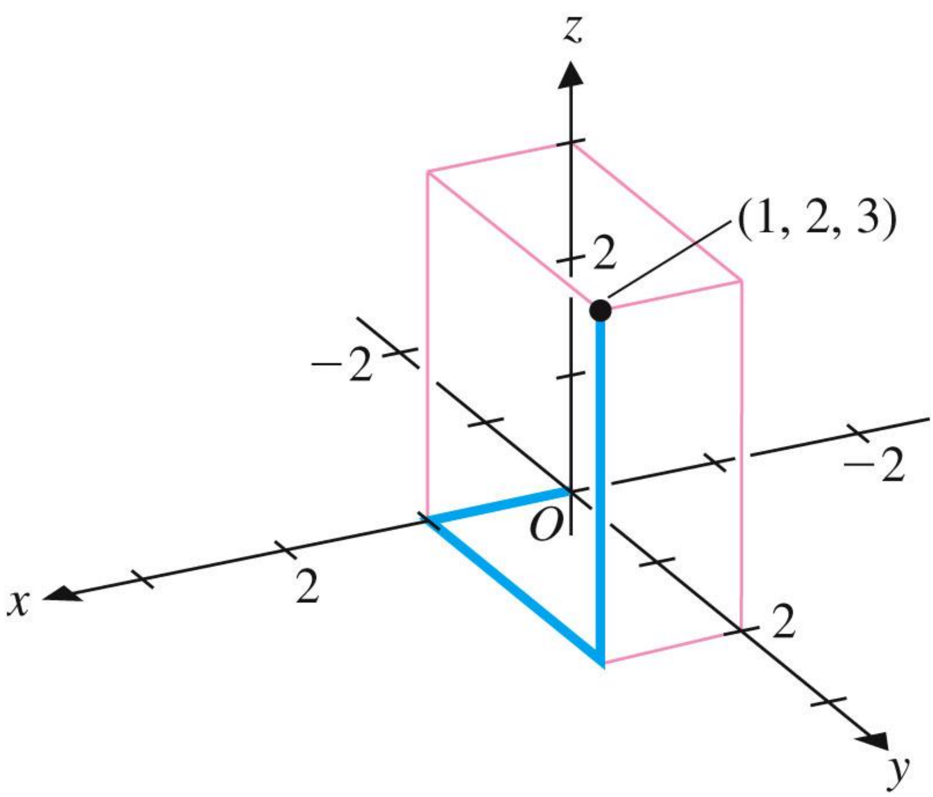
3-D Coordinate System

The Cartesian coordinate system is right-handed.



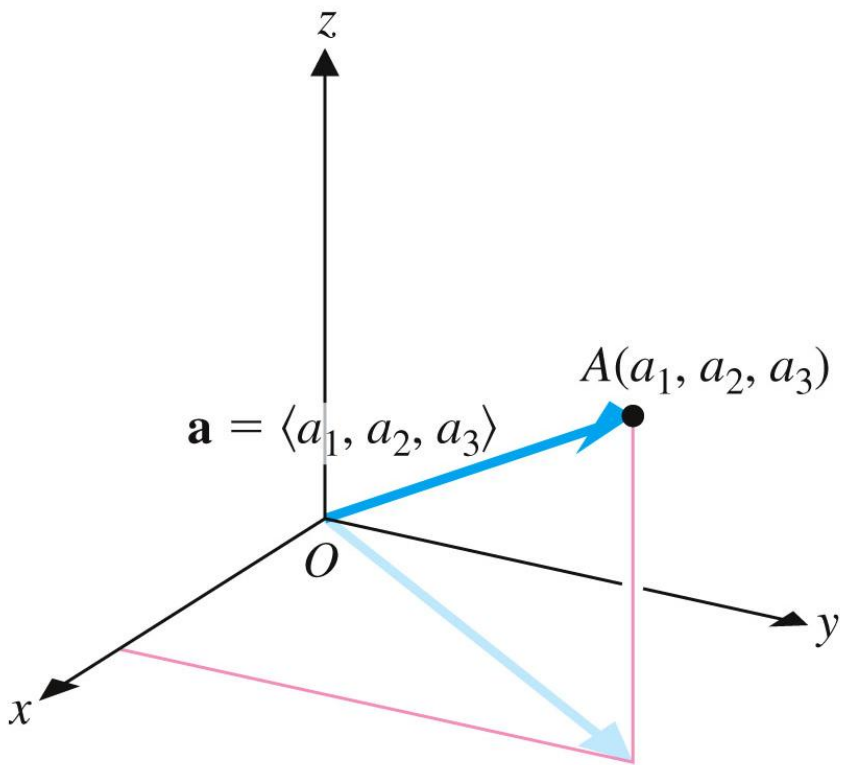
The coordinate planes divide space into 8 octants:

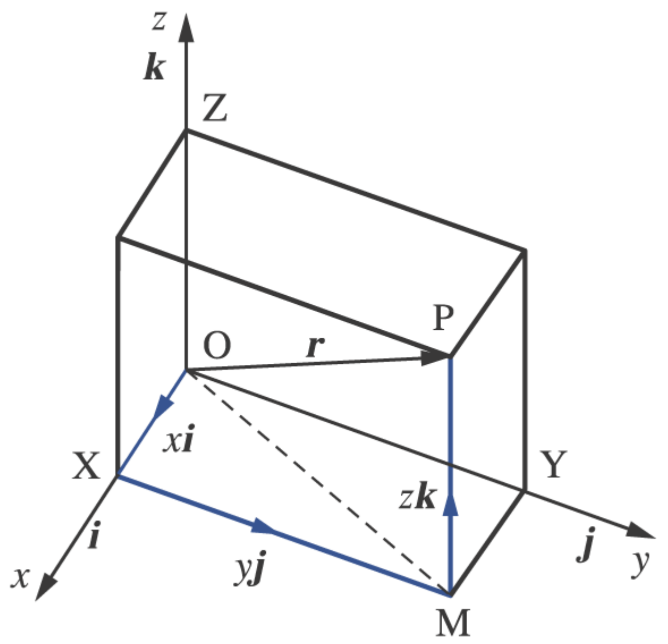

z
xy-plane
y
x
yz-plane
xz-plane


z
(1, 2, 3)
2
−2
−2
O
x
2
2
y

The **position vector** $\overrightarrow{OP}$ of a point x, y, z in space is the vector from the origin to the point. It can be expressed in the form

$\overrightarrow{OP} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \langle x, y, z \rangle$.



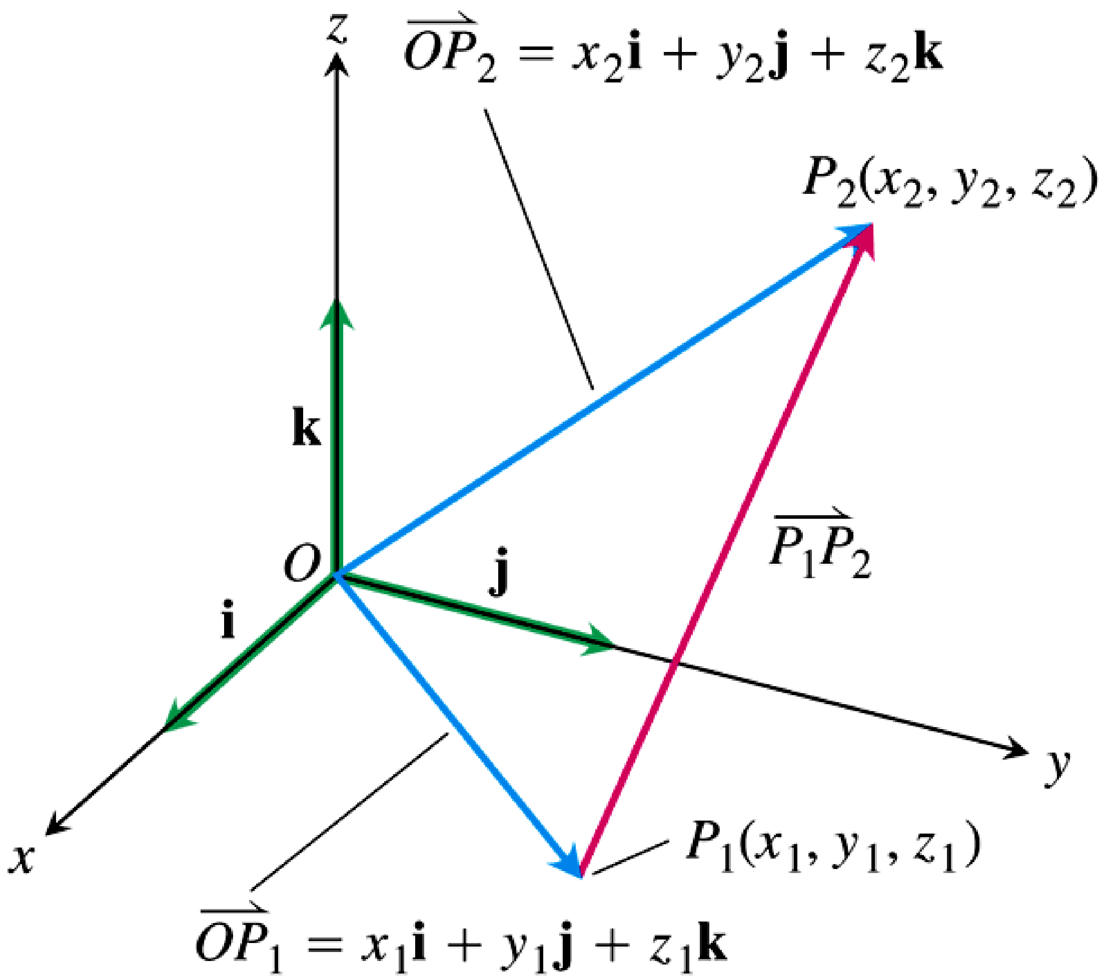


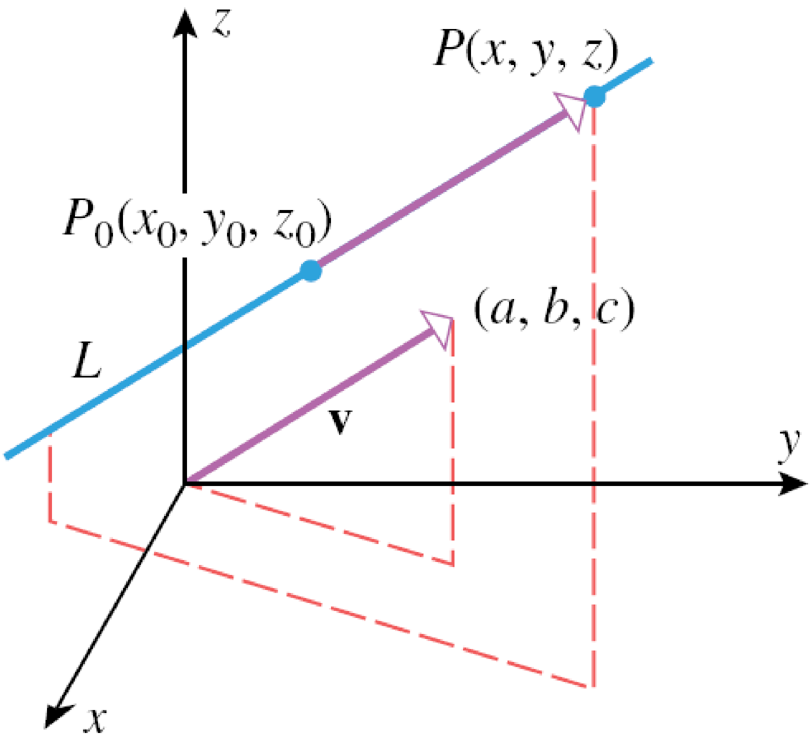
The displacement vector $\overrightarrow{P_1P_2}$ from the point $P_1 = \ x_1, y_1, z_1$ to the point $P_2 = \ x_2, y_2, z_2$ can be computed from the coordinates of the points by the formula

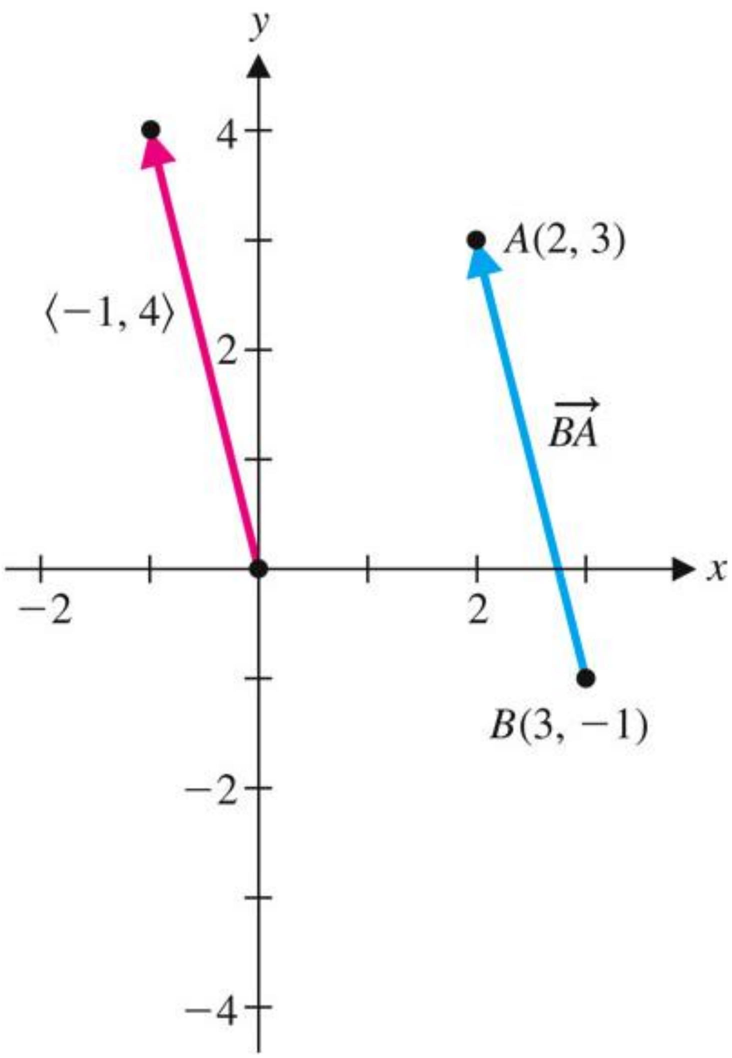
$$\overrightarrow{P_1P_2} = \ x_2 - x_1 \ \mathbf{i} + \ y_2 - y_1 \ \mathbf{j} + \ z_2 - z_1 \ \mathbf{k}$$

or

$$\overrightarrow{P_1P_2} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$




z
P(x, y, z)
P0(x0, y0, z0)
(a, b, c)
L
v
y
x


y
4
A(2, 3)
⟨−1, 4⟩
2
BA
x
−2
2
B(3, −1)
−2
−4

Properties of vectors in space

Let $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ be vectors in 3-dimensional space, and k is a constant.

1. $\mathbf{v} = \mathbf{w}$ if and only if

$$v_1 = w_1, v_2 = w_2, v_3 = w_3.$$

2. The magnitude of $\mathbf{v}$ is $|\mathbf{v}| = \sqrt{v_1^{\ 2} + v_2^{\ 2} + v_3^{\ 2}}$.

3. The unit vector in the direction of $\mathbf{v}$ is

$$\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\langle v_1, v_2, v_3 \rangle}{|\mathbf{v}|}.$$

4. $\mathbf{v} + \mathbf{w} = \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle$

5. $k\mathbf{v} = \langle kv_1, kv_2, kv_3 \rangle$

6. Zero vector is denoted as $\mathbf{0} = \langle 0, 0, 0 \rangle$.

Example: Express the vector $\overrightarrow{PQ}$ if it starts at point $P = (6,5,8)$ and stops at point $Q = (7,3,9)$ in components form.

Example: Given that $\mathbf{a} = \langle 3,1,-2 \rangle$, $\mathbf{b} = \langle -1,6,4 \rangle$. Find

(a) $\mathbf{a} + 3\mathbf{b}$ (b) $|\mathbf{b}|$

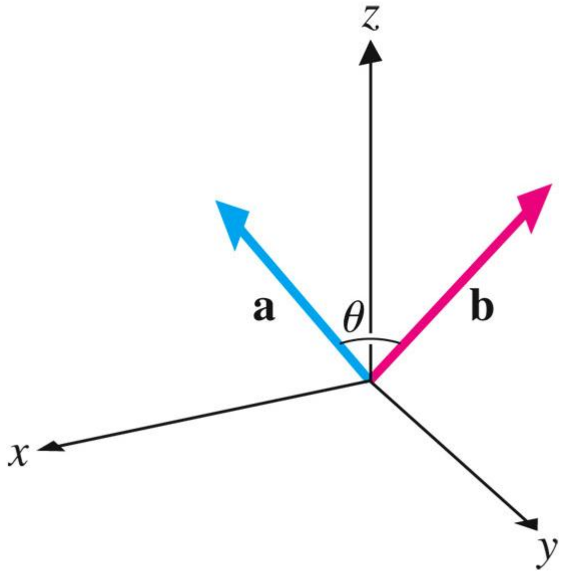
(c) a unit vector which is in the direction of **b**.

6.2 The Scalar Product (The dot product)

The **scalar product** between two vectors $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ is

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}| \cos\theta$$

where θ is the angle between **v** and **w**.



Example:
If $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{w} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and the angle between **v** and **w** is 60°, find $\mathbf{v} \cdot \mathbf{w}$.

Theorem 6.1:
If $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$, then the scalar product $\mathbf{v} \cdot \mathbf{w}$ is

$$\begin{aligned} \mathbf{v} \cdot \mathbf{w} &= \langle v_1, v_2, v_3 \rangle \cdot \langle w_1, w_2, w_3 \rangle \\ &= v_1 w_1 + v_2 w_2 + v_3 w_3 \end{aligned}$$

Example:

Given that $\mathbf{u} = \langle 2,-2,3 \rangle$, $\mathbf{v} = \langle 5,8,1 \rangle$ and $\mathbf{w} = \langle -4,3,-2 \rangle$, find

(a) $\mathbf{u} \cdot \mathbf{v}$ (b) $\mathbf{u} \cdot \mathbf{v}\ \mathbf{w}$ (c) $\mathbf{u} \cdot\ \mathbf{v} + \mathbf{w}$

(d) the angle between **u** and **v**

(e) the angle between **v** and **w.**

Example:

Let A=(4,1,2), B=(3,4,5) and C=(5,3,1) are the vertices of a triangle. Find the angle at vertex A.

Theorem 6.2:

The nature of an angle θ, between two vectors **u** and **v**.

1. θ is an acute angle if and only if $\mathbf{u}\cdot\mathbf{v} > 0$

2. θ is an obtuse angle if and only if $\mathbf{u}\cdot\mathbf{v} < 0$.

3. $\theta = 90^\circ$ if and only if $\mathbf{u}\cdot\mathbf{v} = 0$.

Example: Show that the given vectors are perpendicular to each other.

(a) **i** and **j**

(b) 3**i**-7**j**+2**k** and **i**+4**j**-**k**

Properties of scalar product

1. $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$

2. $\mathbf{u}\cdot\ \ \mathbf{v}+\mathbf{w}\ \ = \mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$

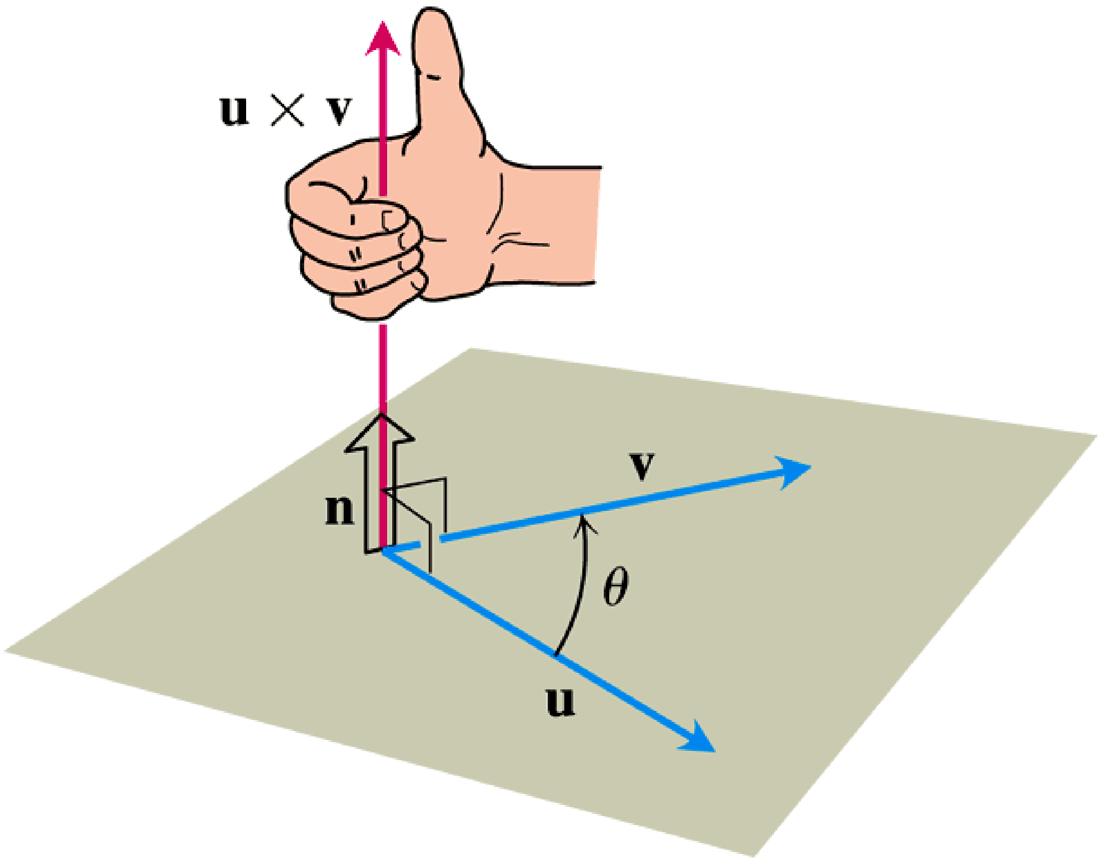
3. $k\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot k\mathbf{v}$

4. $\mathbf{v}\cdot\mathbf{v} = |\mathbf{v}|^2$

5. $\mathbf{u}\cdot\mathbf{0} = \mathbf{0}\cdot\mathbf{u} = 0$

6.3 Vector Products (Cross Products)

The cross product (vector product) $\mathbf{u}\times\mathbf{v}$ is a vector perpendicular to **u** and **v** whose direction is determined by the right hand rule and whose length is determined by the lengths of **u** and **v** and the angle between them.

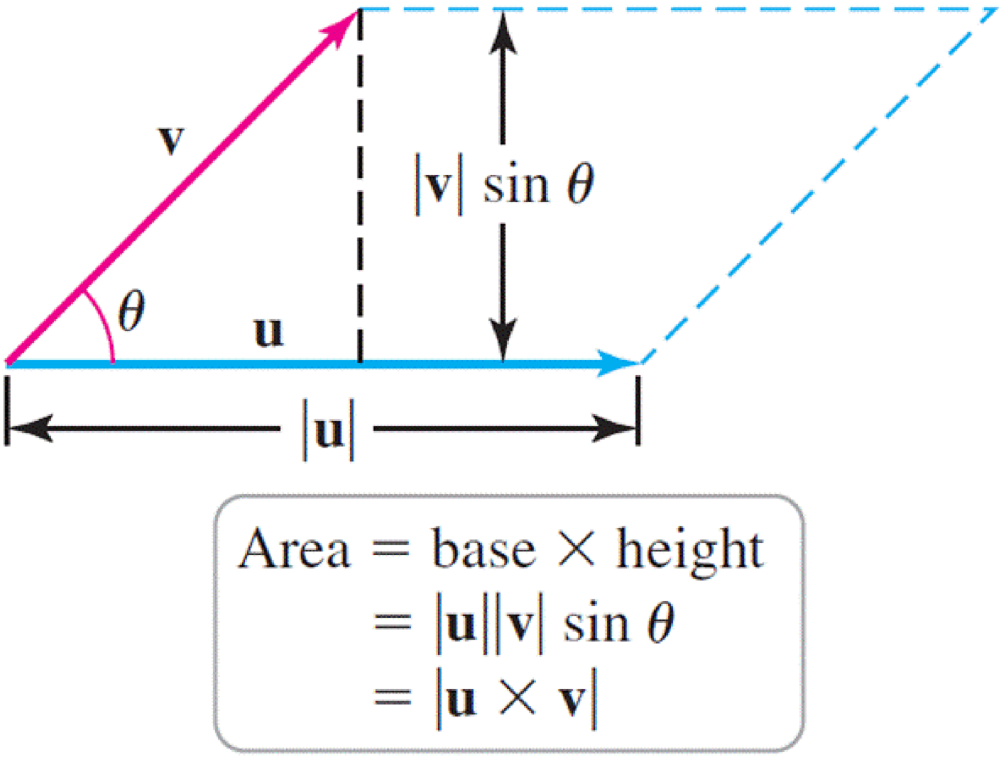


Definition:

If **u** and **v** are nonzero vectors, and θ $(0 < \theta < \pi)$ is the angle between **u** and **v**, then

$$|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta.$$

The length of the cross product as as an area



Theorem 6.3

If **u** and **v** are two sides of a parallelogram, then the area of the parallelogram is

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta.$$

The cross product obeys the laws

(a) $\mathbf{u} \times \mathbf{v} = - \ \mathbf{v} \times \mathbf{u}$

(b) $k\mathbf{u} \ \times \mathbf{v} = \mathbf{u} \times \ k\mathbf{v} \ = k \ \mathbf{u} \times \mathbf{v}$

(c) $\mathbf{u} \times \ \mathbf{v} + \mathbf{w} \ = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$

(d) $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$

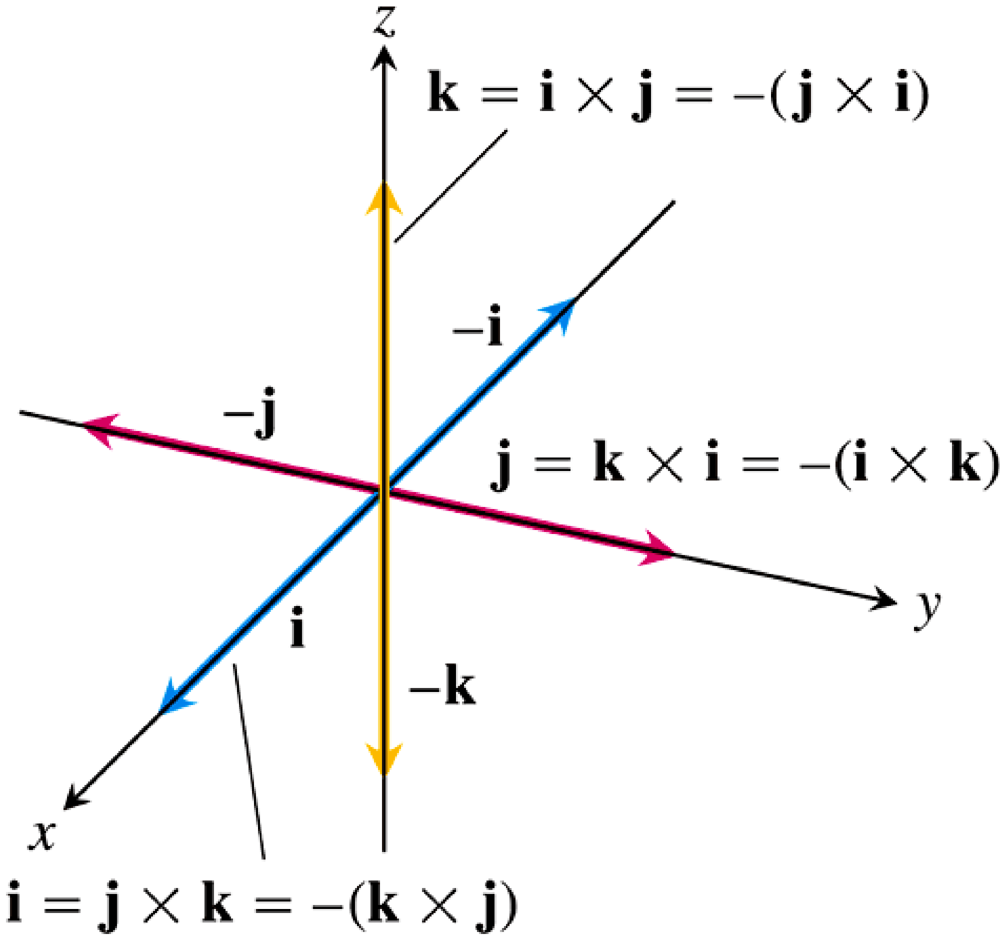
(e) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

Theorem 6.4

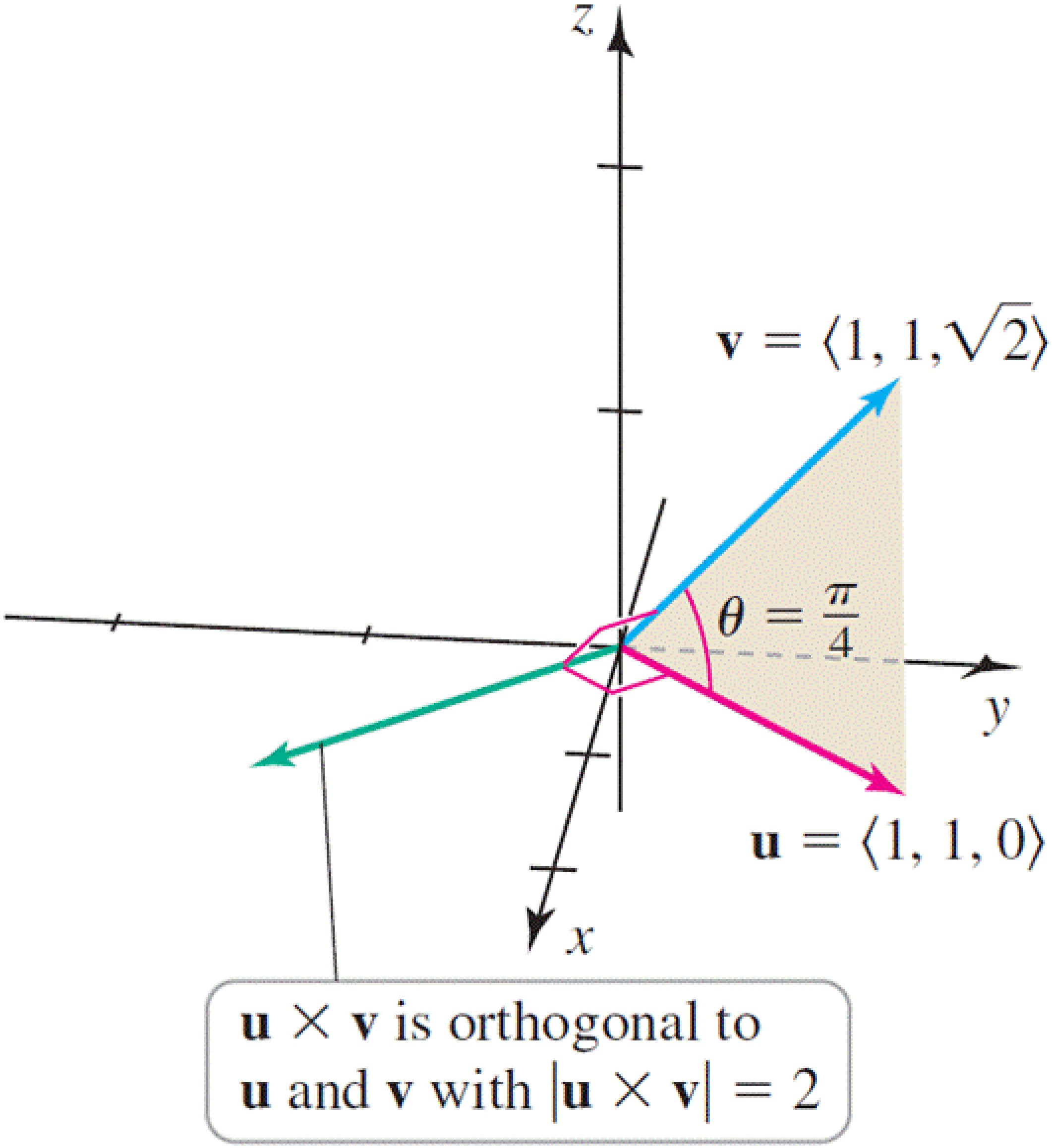
If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

The pairwise cross products of **i**, **j**, **k**:



Example:



Example:

Given that $\mathbf{u} = \langle 3,0,4 \rangle$ and $\mathbf{v} = \langle 1,5,-2 \rangle$, find

(a) $\mathbf{u} \times \mathbf{v}$ (b) $\mathbf{v} \times \mathbf{u}$

Example:

Find two unit vectors that are perpendicular to the vectors $\mathbf{u} = 2\mathbf{i}+2\mathbf{j}-3\mathbf{k}$ and $\mathbf{v} = \mathbf{i}+3\mathbf{j}+\mathbf{k}$.

Example:

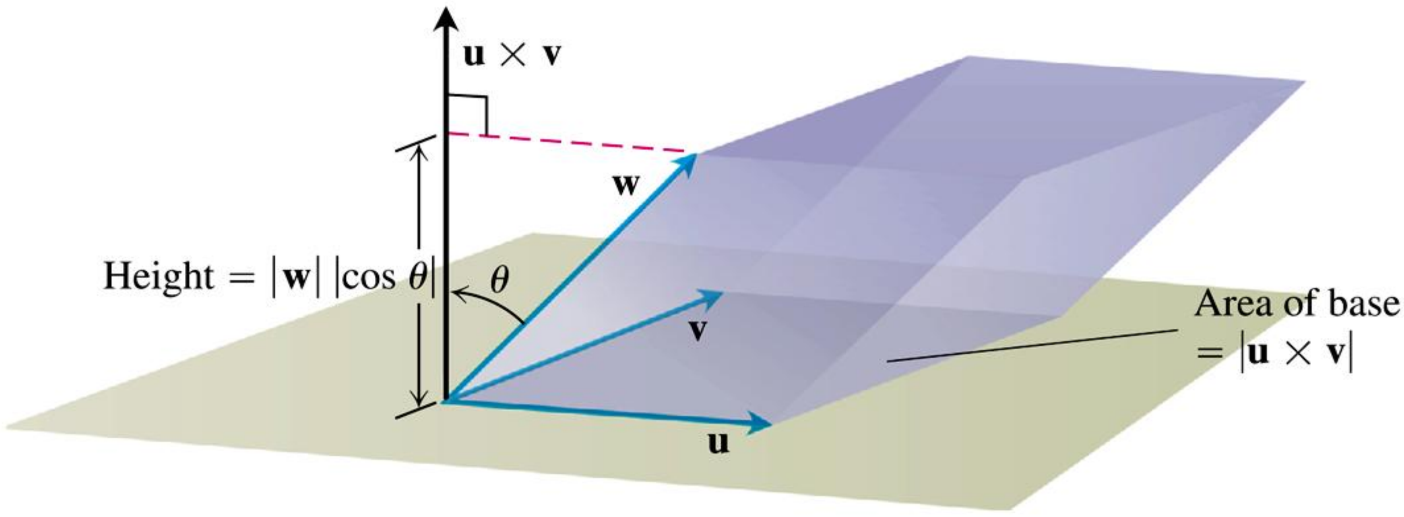
(a) Find an area of a parallelogram that is formed from vectors **u** = **i** + **j** - 3**k** and **v** = -6**j** + 5**k.**

(b) Find an area of a triangle that is formed from vectors **u** = **i** + **j** - 3**k** and **v** = -6**j** + 5**k.**

Triple Scalar Product

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

The number $|\mathbf{u} \times \mathbf{v} \cdot \mathbf{w}|$ is the volume of a parallelepiped.



$$\begin{aligned} \text{Volume} &= \text{area of base} \cdot \text{height} \\ &= |\mathbf{u} \times \mathbf{v}|\,|\mathbf{w}|\,|\cos\theta| \\ &= |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| \end{aligned}$$

Example

Find the volume of the parallelepiped determined by the vectors **a**; **b**; and **c**.

a = **i** + **j** – **k**, **b** = **i** – **j** + **k**, **c** = – **i** + **j** + **k**

Solution

$$\mathbf{a}\cdot\ \mathbf{b}\times\mathbf{c}\ =\begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix}=-4$$

So, the volume is 4.