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# Circuit Theory (SKEE 1023)

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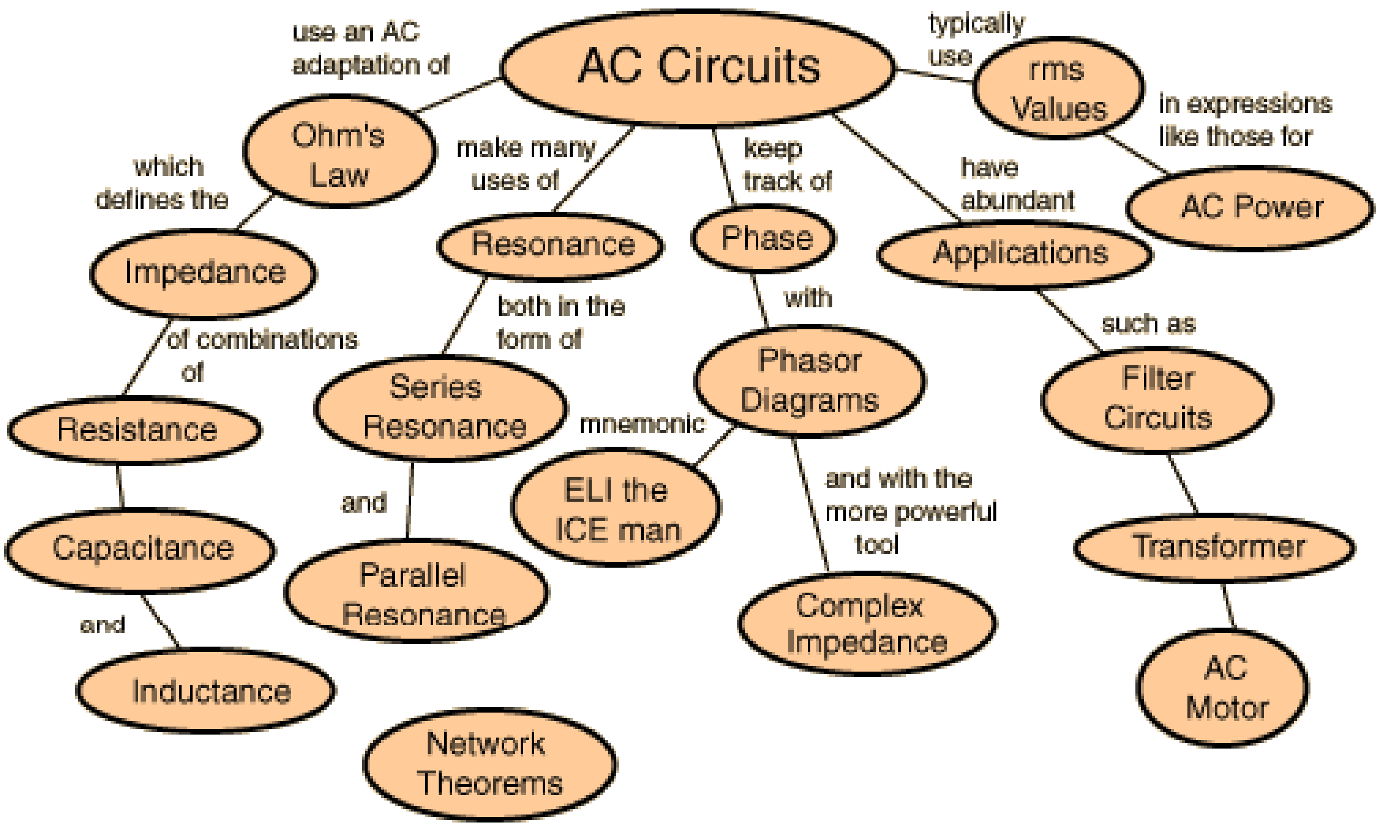
## Topics

- Generation of AC source.
- Characteristics of an AC waveform
- Sinusoids, average and effective values, phasors.
- Impedance, admittance, Kirchhoff's laws in the frequency domain

## Introduction

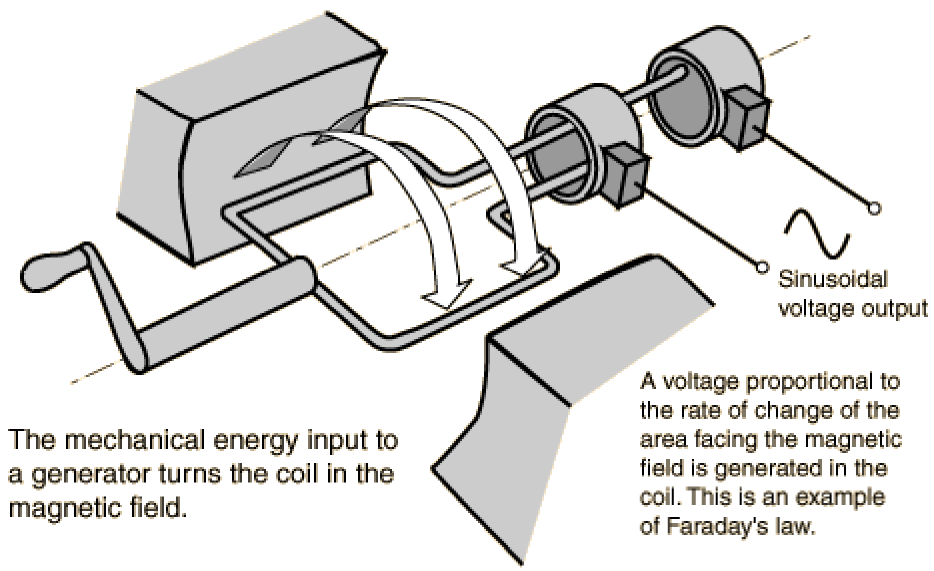
- The analysis of circuits in which the source voltage or current is time-varying.
- Interested in sinusoidally time-varying excitation (excitation by a sinusoid).
- A sinusoid is a signal that has the form of the sine or cosine function.
- A sinusoidal current is usually referred to as alternating current (ac) $\Rightarrow$ ac circuits are driven by sinusoidal current or voltage.

# AC Circuits Analysis – Part I

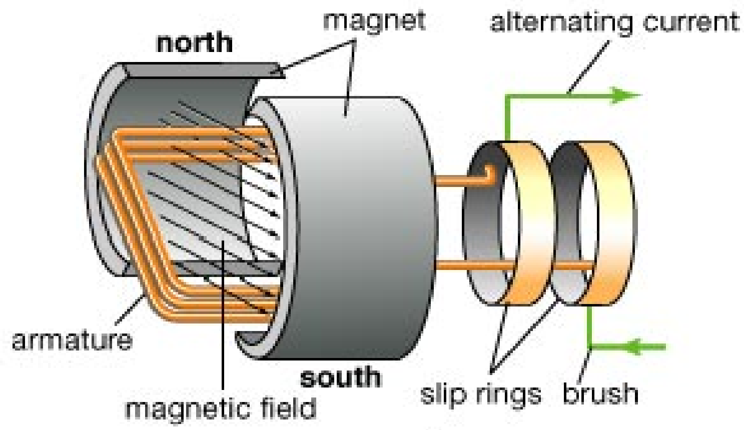


## Generation of ac source (current or voltage)

- The turning of a coil in a magnetic field produces motional emfs in both sides of the coil which add. Since the component of the velocity perpendicular to the magnetic field changes sinusoidally with the rotation, the generated voltage is sinusoidal or AC.

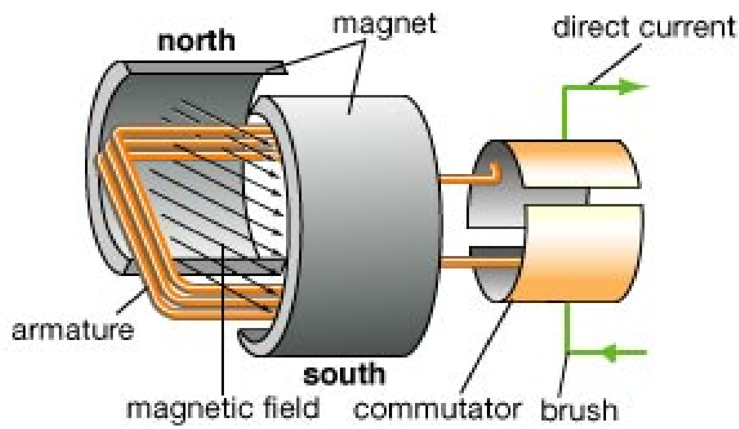


# AC Circuits Analysis – Part I




AC generator

DC generator



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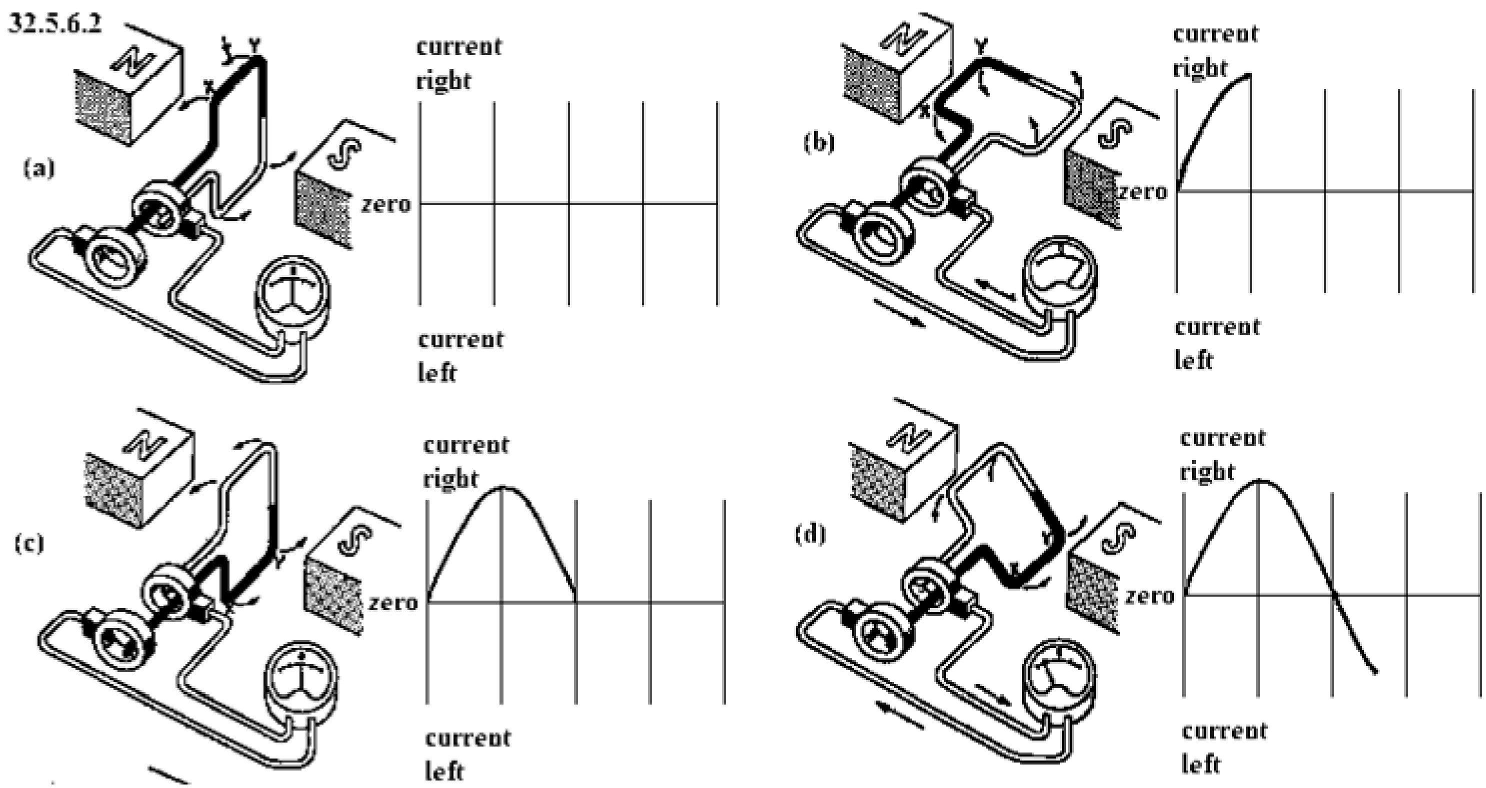
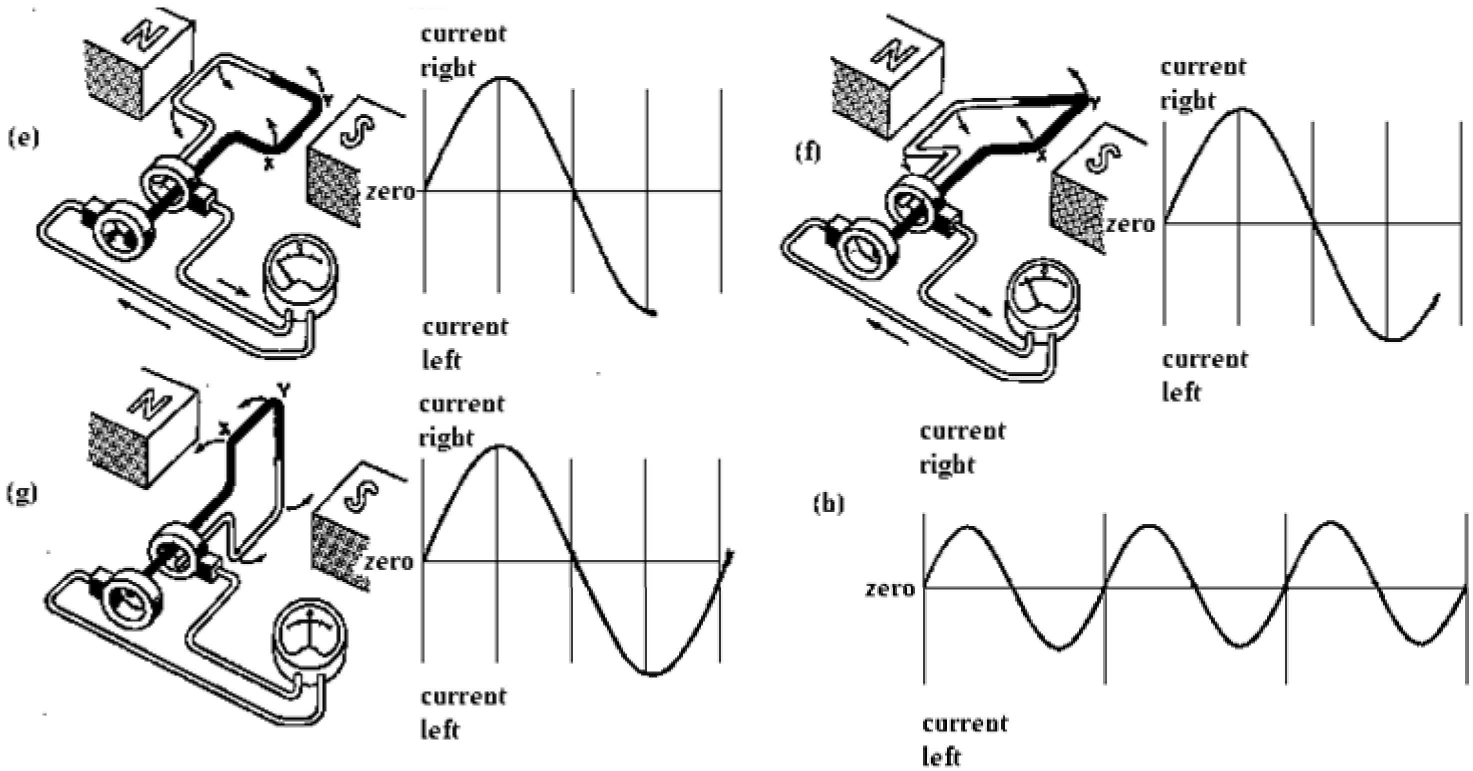
# AC Circuits Analysis – Part I

Half cycle completed



# AC Circuits Analysis – Part I

One cycle completed



# Sinusoids

- Sinusoidal voltage ⇨ $v(t) = V_m \sin \omega t$

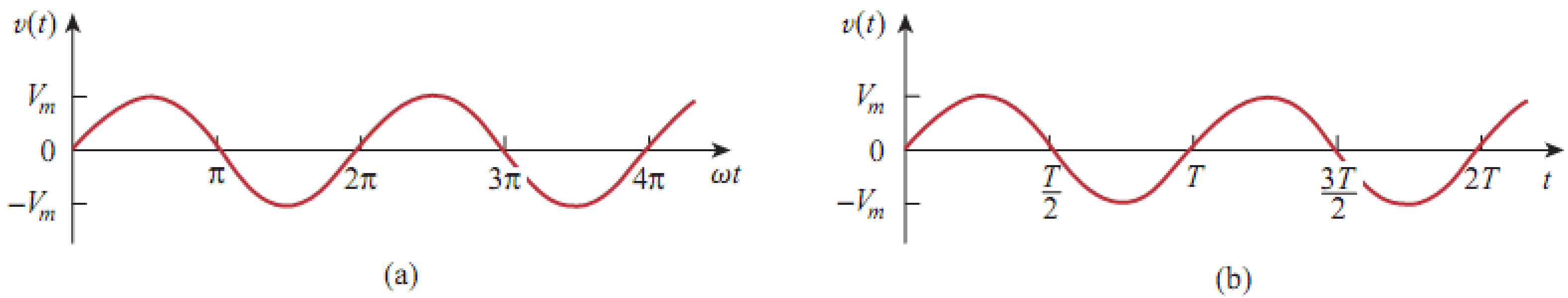
where,

$V_m$ = the amplitude of the sinusoid

$\omega$ = the angular frequency in radian/s

$\omega t$ = the argument of the sinusoid

- Sinusoidal voltage as a function of $\omega t$ (a); and a function of $t$ (b).



## Sinusoids (Cont.)

- The sinusoid repeats itself every $T$ seconds; $\Rightarrow$ $T$ is called *period* of the sinusoid (waveform).

$$T = \frac{2\pi}{\omega}$$

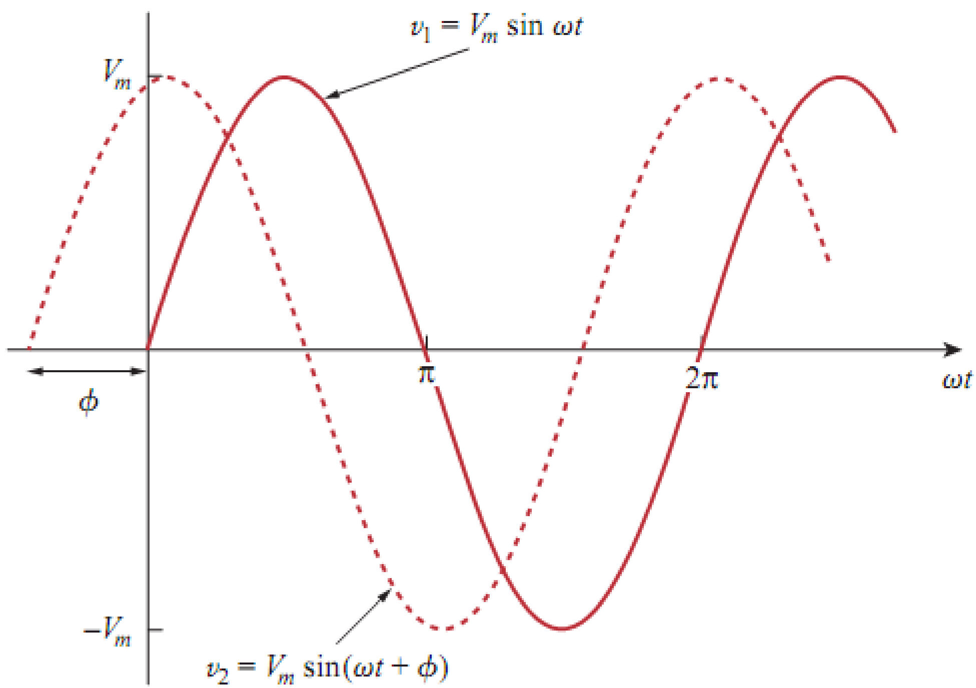
- The signal (waveform) is said to be *periodic* if the signal repeats itself every $T$ seconds, and satisfies with the following expression;

$$f(t) = f(t + nT)$$

- The reciprocal of the $T$ (period) is the number of cycles per second, known as the cyclic *frequency*, $f$ of the sinusoid.

$$f = \frac{1}{T}$$

## Two sinusoids with different phases




$$v_1(t) = V_m \sin \omega t$$
$$v_2(t) = V_m \sin(\omega t + \phi)$$

$\therefore\ v_2$ leads $v_1$ by $\phi$ OR $v_1$ lags $v_2$ by $\phi$

If $\phi \neq 0$, $\Rightarrow$ $v_1$ and $v_2$ are out of phase.

If $\phi = 0$, $\Rightarrow$ $v_1$ and $v_2$ are in phase.

- When comparing two sinusoids, it is expedient to express both as either sine or cosine with positive amplitudes, and must have the same frequency.

## Average Value

- The average value of the sinusoidal waveform is the average of the instantaneous value over one period of waveform. The average value is the equivalent of dc value.
- There are 2 methods on determining the average value, ie;

$$X_{avg} = \frac{\text{Algebraic sum of areas covered by the waveform } x(t) \text{ over one period}}{\text{one period}}$$

$$X_{avg} = \frac{1}{T}\int_0^T x(t)\,dt$$

If the waveform is symmetry, take the integration process up to half-cycle, T/2 only.

- The average value of the sinusoidal waveform is 0.637 of the maximum/peak value.

For sinusoidal waveform,

$$X_{avg} = \frac{2X_m}{\pi}$$

## Effective or rms Value

- The effective value of a periodic signal is its *root mean square (rms)* value.
- The effective value of a periodic current is equivalent to the dc current that delivers the same average power to a resistor as the periodic current.

$$X_{rms} = \sqrt{\frac{1}{T}\int_0^T x^2\,dt}$$

- The rms value of the sinusoidal waveform is 0.707 of the maximum/peak value.

For sinusoidal waveform,

$$X_{rms} = \frac{X_m}{\sqrt{2}}$$

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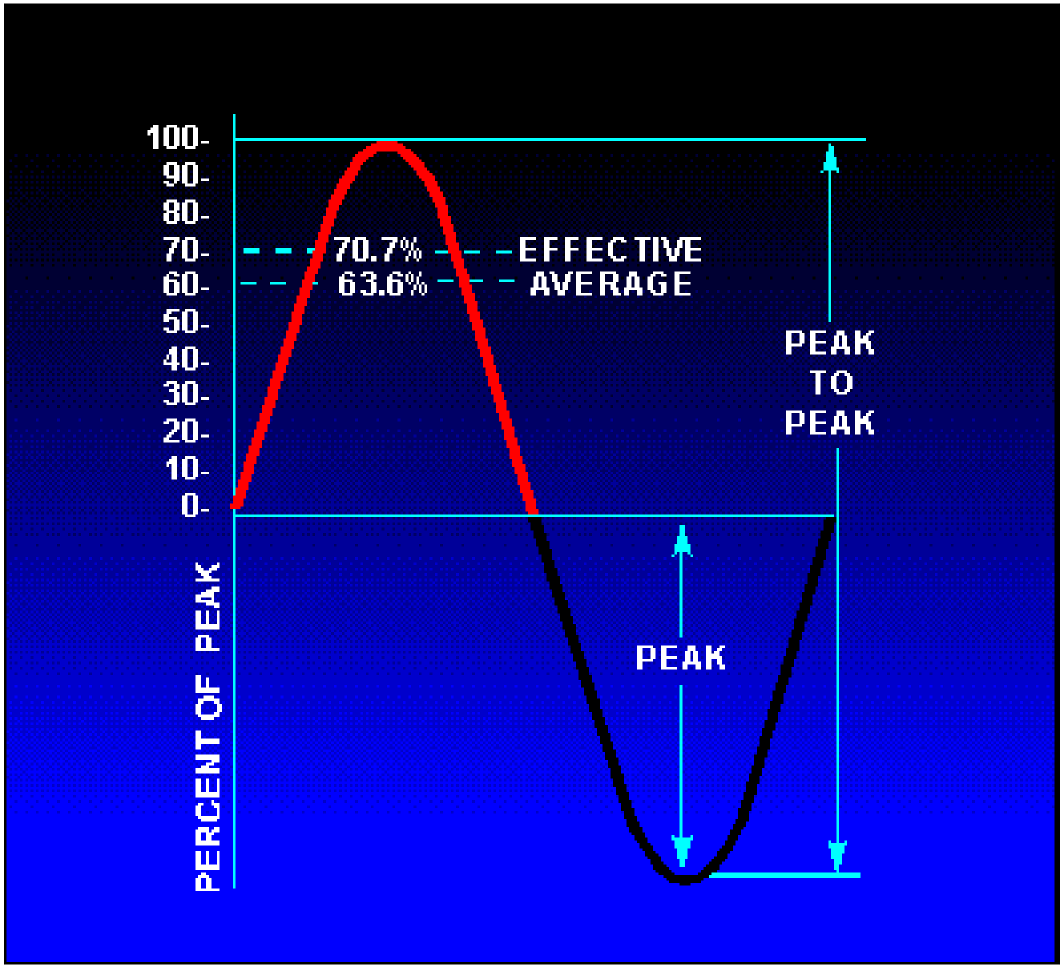
# AC Circuits Analysis – Part I



## Phasors

- Is a complex number that represents the amplitude and phase of a sinusoid.
- Provide a simple means of analyzing linear circuits excited by sinusoidal sources.
- A complex number can be represented in three ways:

| | |
|---|---|
| $z = x + jy$ | : Rectangular form |
| $z = r\angle\phi$ | : Polar form |
| $z = re^{j\phi}$ | : Exponential form |

Given $x$ and $y$, we can get $r$ and $\phi$ as

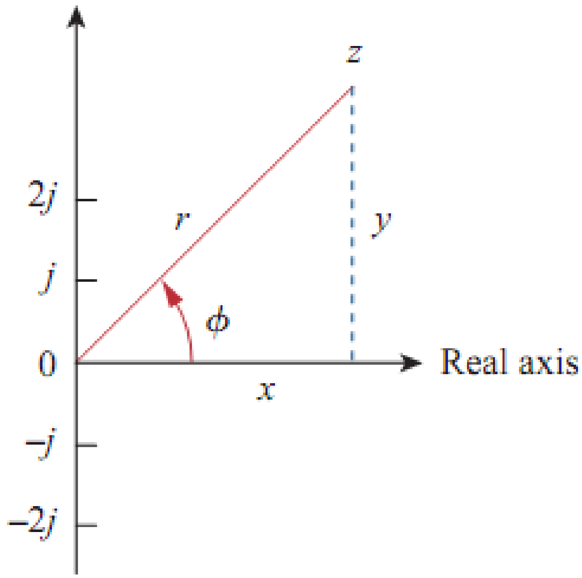
$$r = \sqrt{x^2 + y^2}, \qquad \phi = \tan^{-1}\frac{y}{x}$$

Given $r$ and $\phi$, we can get $x$ and $y$ as

$$x = r\cos\phi, \qquad y = r\sin\phi$$

# AC Circuits Analysis – Part I

$$z = x + jy = r\angle\phi = r(\cos\phi + j\sin\phi)$$




- The idea of phasor representation is based on Euler's identity.

$$e^{\pm j\phi} = \cos\phi \pm j\sin\phi$$

$$\cos\phi = \mathrm{Re}(e^{j\phi})$$

$$\sin\phi = \mathrm{Im}(e^{j\phi})$$

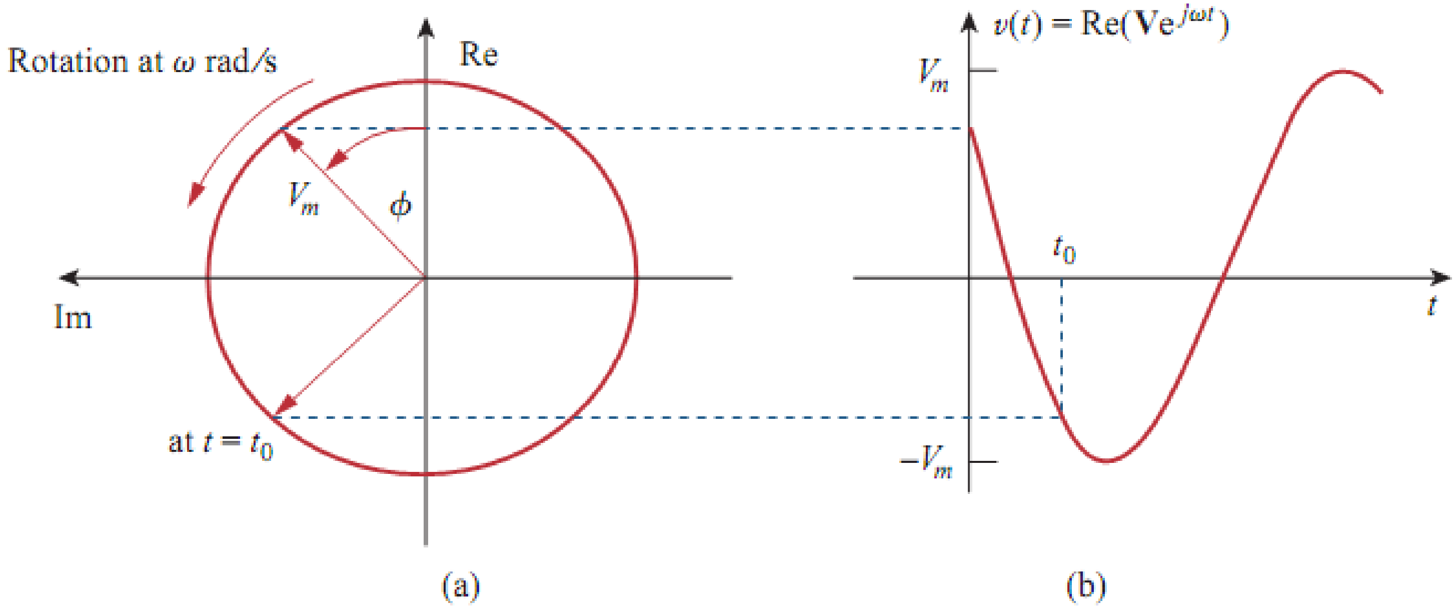
$$v(t) = V_m \cos(\omega t + \phi) = \mathrm{Re}(V_m e^{j(\omega t + \phi)})$$

$$v(t) = \mathrm{Re}(V_m e^{j\phi} e^{j\omega t})$$

Thus,

$$v(t) = \mathrm{Re}(\mathbf{V} e^{j\omega t})$$

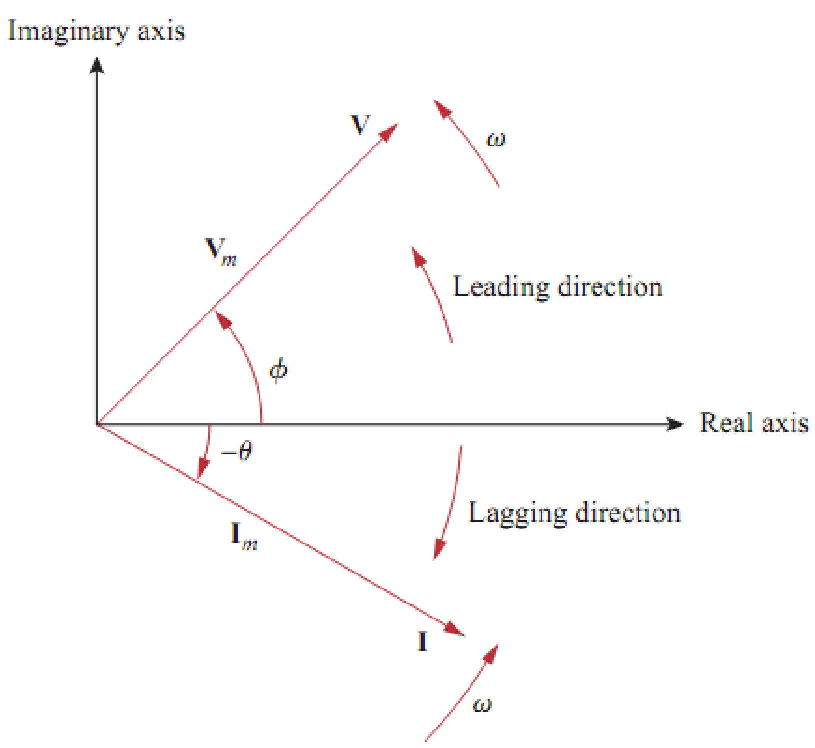
where $\mathbf{V} = V_m e^{j\phi} = V_m \angle\phi$

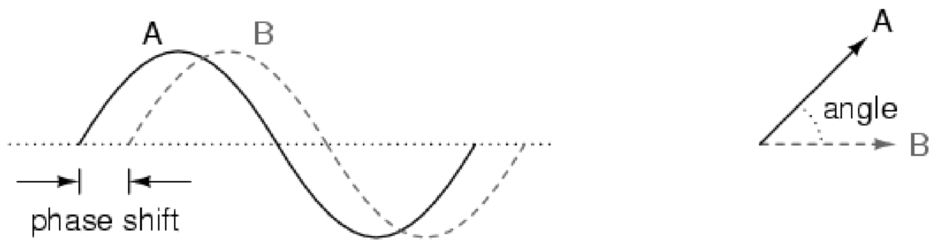



Representation of $\mathbf{V}e^{j\omega t}$ : (a) sinor rotating counterclockwise, (b) its projection on the real axis, as a function of time.

# AC Circuits Analysis – Part I

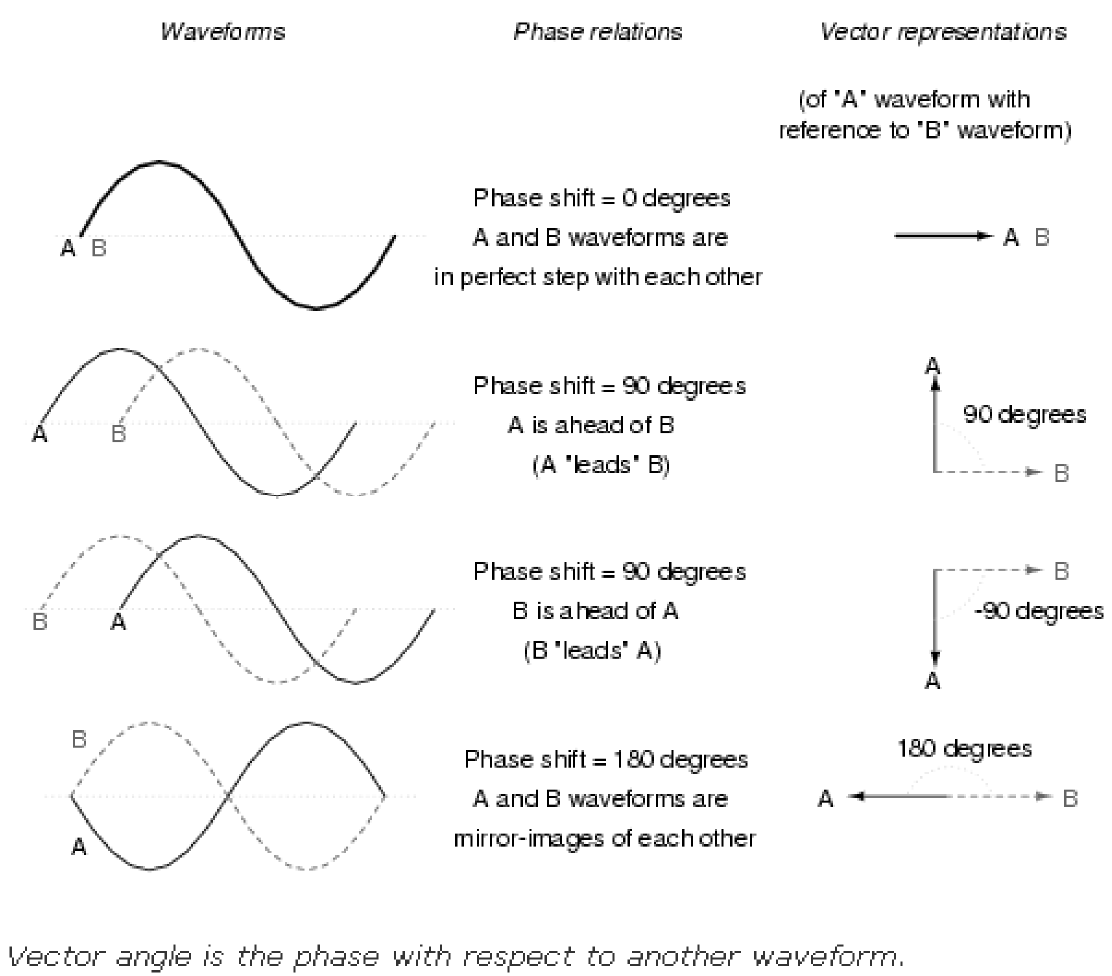
$$v(t) = \underbrace{V_m \cos(\omega t + \phi)}_{\text{(Time-domain representation)}} \Leftrightarrow \underbrace{\mathbf{V} = V_m \angle \phi}_{\text{(Phasor-domain representation)}}$$




*Phase shift between waves and vector phase angle*

# AC Circuits Analysis – Part I




Vector angle is the phase with respect to another waveform.

See examples on page 381-384 (A. Sadiku)

## Phasor Relationships for Circuit Elements

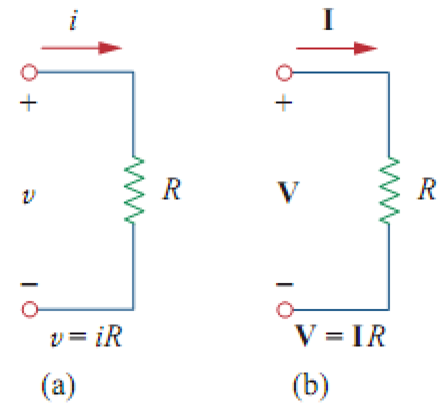
- **Resistor**

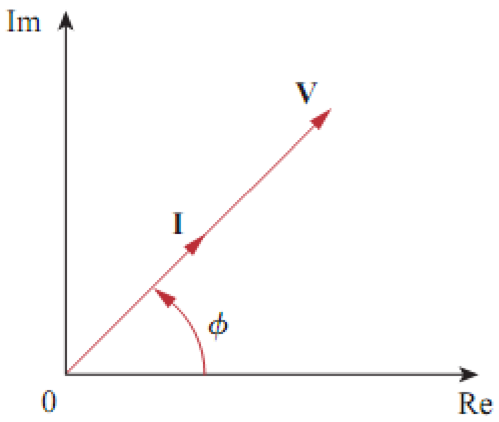
The current through a resistor $R$ is $i = I_m \cos(\omega t + \phi)$.
The voltage across it is given by Ohm's law as;
$v = iR = RI_m \cos(\omega t + \phi)$

In phasor form;

$\mathbf{V} = RI_m \angle \phi$ and $\mathbf{I} = I_m \angle \phi$

Hence, $\mathbf{V} = R\mathbf{I}$




→ Voltage and current are in phase

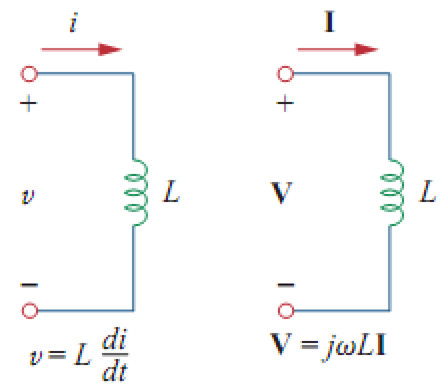
## Inductor

The current through the inductor, $i = I_m \cos(\omega t + \phi)$.

The voltage across the inductor is,

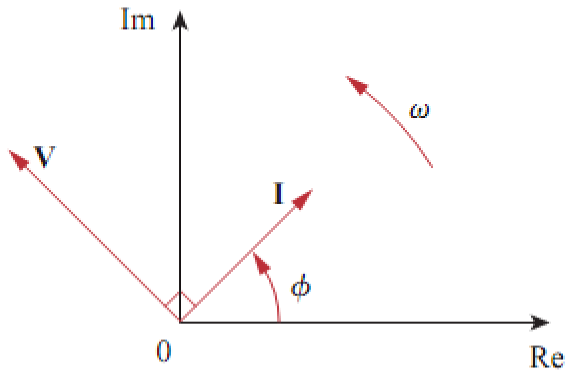
$$v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \phi) = \omega L I_m \cos(\omega t + \phi + 90^\circ)$$

In the phasor form : $\mathbf{V} = \omega L I_m \angle \phi + 90^\circ$



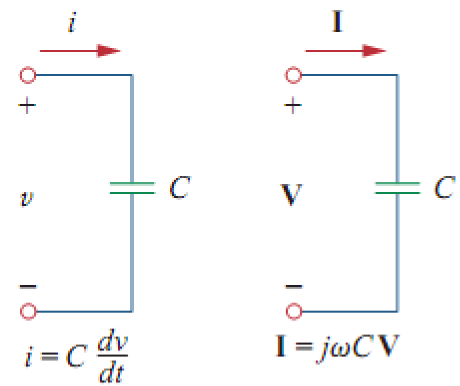

But $I_m \angle \phi = \mathbf{I}$, and $e^{j90^\circ} = j$.

Thus, $\mathbf{V} = j\omega L\mathbf{I}$




Voltage and current are 90º out of phase. The current lags the voltage by 90º.

## Capacitor

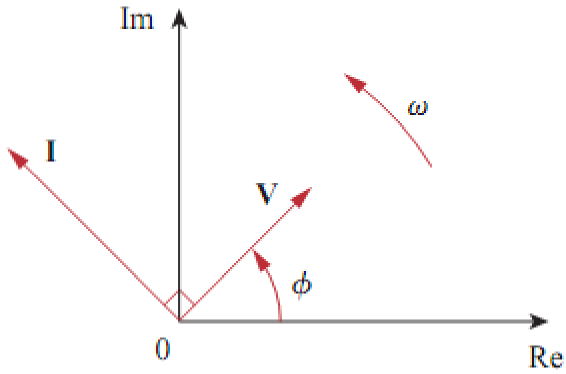



The voltage across the capacitor, $v = V_m \cos(\omega t + \phi)$.

The current through the capacitor is,

$$i = C\frac{dv}{dt} = -\omega C V_m \sin(\omega t + \phi) = \omega C V_m \cos(\omega t + \phi + 90^\circ)$$

In the phasor form: $\mathbf{I} = j\omega C\mathbf{V} \Rightarrow \mathbf{V} = \dfrac{\mathbf{I}}{j\omega C}$




Current and voltage are 90º out of phase. The current leads the voltage by 90º.

## Summary of voltage-current relationships

| Element | Time domain | Frequency domain |
|---|---|---|
| $R$ | $v = Ri$ | $\mathbf{V} = R\mathbf{I}$ |
| $L$ | $v = L\dfrac{di}{dt}$ | $\mathbf{V} = j\omega L\mathbf{I}$ |
| $C$ | $i = C\dfrac{dv}{dt}$ | $\mathbf{V} = \dfrac{\mathbf{I}}{j\omega C}$ |

## Impedance and Admittance

Voltage - current relations for the three passive elements;

$$\mathbf{V} = R\mathbf{I}, \qquad \mathbf{V} = j\omega L\mathbf{I}, \qquad \mathbf{V} = \frac{\mathbf{I}}{j\omega C}$$

In terms of the ratio of the phasor voltage to the phasor current;

$$\frac{\mathrm{V}}{\mathrm{I}} = R, \qquad \frac{\mathrm{V}}{\mathrm{I}} = j\omega L, \qquad \frac{\mathrm{V}}{\mathrm{I}} = \frac{1}{j\omega C}$$

$$\mathrm{Z} = \frac{\mathrm{V}}{\mathrm{I}} \quad \text{or} \quad \mathrm{V} = \mathrm{ZI}$$

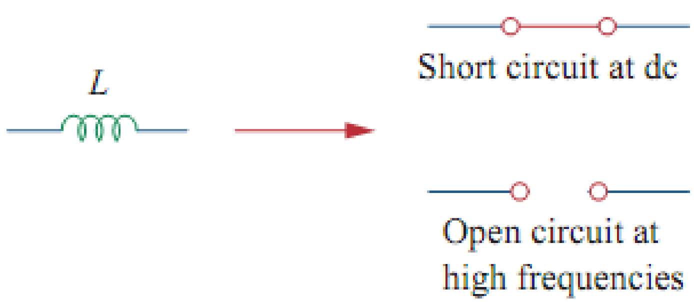
The impedance Z of a circuit is the ratio of the phasor voltage V to the phasor current I, measured in ohms (Ω)

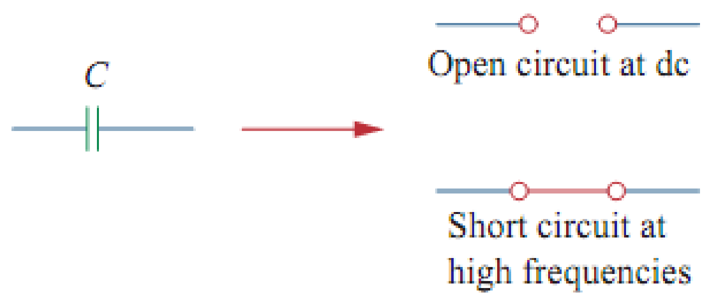
# AC Circuits Analysis – Part I

- Impedance is not a phasor, because it does not correspond to a sinusoidal varying quantity.

| Element | Impedance | Admittance |
|---|---|---|
| $R$ | $\mathbf{Z} = R$ | $\mathbf{Y} = \frac{1}{R}$ |
| $L$ | $\mathbf{Z} = j\omega L$ | $\mathbf{Y} = \frac{1}{j\omega L}$ |
| $C$ | $\mathbf{Z} = \frac{1}{j\omega C}$ | $\mathbf{Y} = j\omega C$ |

$$\Rightarrow \; Z_R = R \; ; \quad Z_L = j\omega L \; ; \quad Z_C = \frac{-j}{\omega C}$$



# AC Circuits Analysis – Part I

- As a complex quantity, the impedance may be expressed in rectangular form as;

$$Z = R + jX$$

$R$ : resistance, $X$ : reactance

- The impedance is inductive (or lagging) when *X* is positive or capacitive (or leading) when *X* is negative.
- The impedance may also be expressed in polar form as;

$$Z = |Z|\angle\theta$$

$$\Rightarrow \quad |Z| = \sqrt{R^2 + X^2}, \quad \theta = \tan^{-1}\frac{X}{R}$$

$$R = |Z|\cos\theta, \quad X = |Z|\sin\theta$$

# AC Circuits Analysis – Part I

- The admittance *Y* is the reciprocal of impedance, measured in siemens (S).

$$Y = \frac{1}{Z} = \frac{\mathbf{I}}{\mathbf{V}}\ ; \quad \Rightarrow \ Y = G + jB$$

$$G : \text{conductance}, \quad B : \text{susceptance}$$

- The equivalent impedance connected in series,

$$Z_{eq} = Z_1 + Z_2 + ..... + Z_N$$

- The equivalent impedance connected in parallel,

$$\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2} + ..... + \frac{1}{Z_n}$$

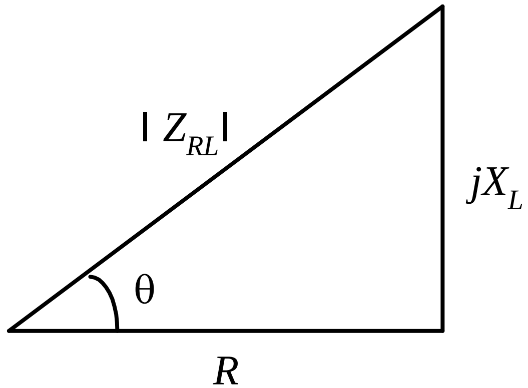
and the equivalent admittance is

$$Y_{eq} = Y_1 + Y_2 + ..... + Y_N$$

## Impedance Triangle

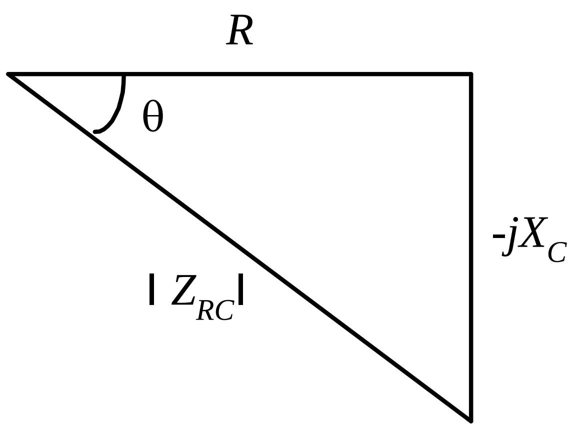
Inductive Impedance ($Z_{RL}$)

$$Z_{RL} = R + j\omega L$$




Capacitive Impedance ($Z_{RC}$)

$$Z_{RC} = R - j\frac{1}{\omega C}$$



## Kirchhoff's Law in the Frequency Domain

- KVL and KCL can be applied as well in the frequency domain circuits.
- For the circuit which have many voltages around a closed loop; $v_1 + v_2 + v_3 + \ldots. + v_n = 0 \Leftrightarrow \mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_3 + \ldots. + \mathbf{V}_n = 0$.
- For the circuit where the currents entering and leaving a closed surface (node); $i_1 + i_2 + i_3 + \ldots. + i_n = 0 \Leftrightarrow \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 + \ldots. + \mathbf{I}_n = 0$.

## Example 4.1

Find *v(t)* and *i(t)* in the circuit shown in Fig. E4.1.

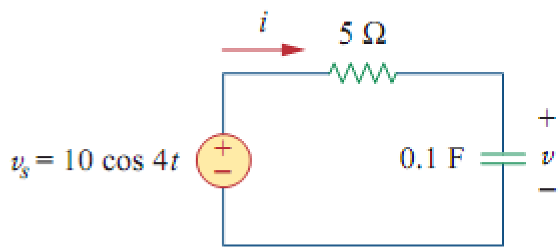



Fig. E4.1

## Solution

From the voltage source $10\cos 4t$, $\omega = 4$,

$$V_s = 10\angle 0^\circ \text{ V}$$

The impedance,

$$Z = 5 + \frac{1}{j\omega C} = 5 + \frac{1}{j4(0.1)} = 5 - j2.5\,\Omega$$

Hence the current, $I = \frac{V_s}{Z} = \frac{10\angle 0^\circ}{5 - j2.5} = 1.789\angle 26.57^\circ \text{ A}$

Voltage across the capacitor;

$$V = IZ_C = \frac{I}{j\omega C} = \frac{1.789\angle 26.57^\circ}{j4(0.1)} = 4.47\angle -63.43^\circ \text{ V}$$

Converting I & V to time domain;

$$i(t) = 1.789\cos(4t + 26.57^\circ) \text{ A}$$

$$v(t) = 4.47\cos(4t - 63.43^\circ) \text{ V}$$

$\Rightarrow \quad i(t)$ leads $v(t)$ by $90^\circ$ as expected

# AC Circuits Analysis – Part I

**Example 4.2:** Determine $v_o(t)$ in the circuit of Fig. E4.2.

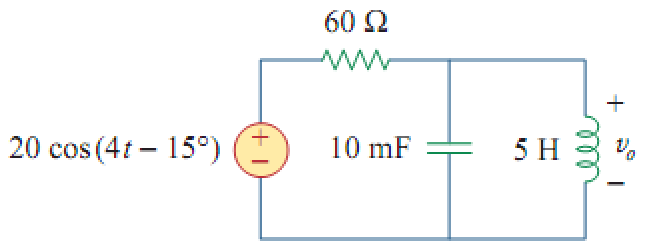
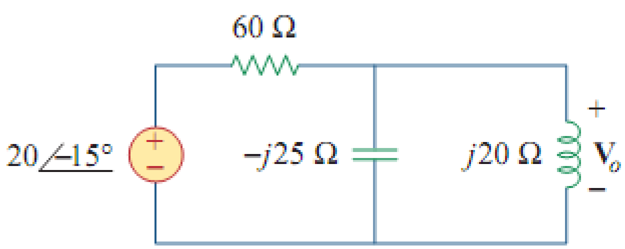



Fig. E4.2

## Solution

- Transform the time domain circuit to the phasor domain.




Let $Z_1 = 60\,\Omega$, and $Z_2 = -j25 // j20 = j100\,\Omega$

Using voltage division;

$$V_o = \frac{Z_2}{Z_1 + Z_2} V_s = \frac{j100}{60 + j100}(20\angle -15^\circ) = 17.15\angle 15.96^\circ \text{ V}$$

$\therefore$ In time domain;

$$v_o(t) = 17.15\cos(4t + 15.96^\circ) \text{ V}$$

**Problem 4.1:** Calculate $v_o(t)$ in the circuit of Fig. P4.1.

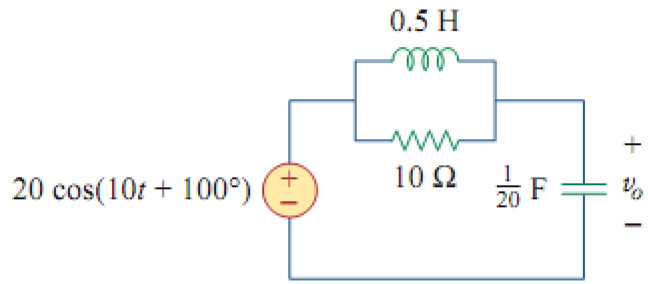



Fig. P4.1

## Example 4.3

Find current I in the circuit of Fig. E4.3.

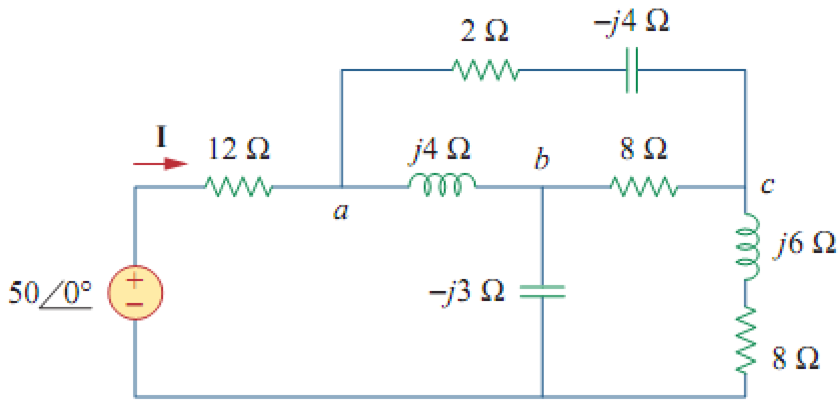



Fig. E4.3

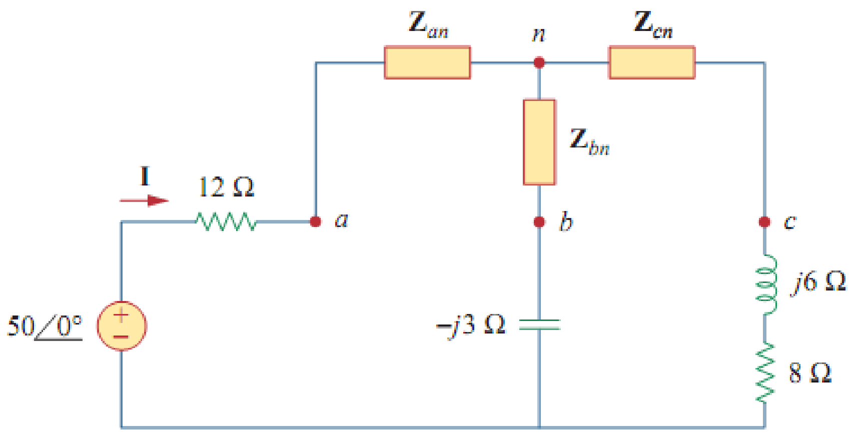
## Solution

- The delta network connected to nodes *a*, *b* and *c* can be converted to the Y network.

$$\therefore \ Z_{an} = \frac{j4(2-j4)}{j4+2-j4+8} = (1.6+j0.8)\,\Omega$$

$$Z_{bn} = \frac{j4(8)}{10} = j3.2\,\Omega$$

$$Z_{cn} = \frac{8(2-j4)}{10} = (1.6-j3.2)\,\Omega$$




Total impedance at the source terminals,

$$Z = 12 + Z_{an} + (Z_{bn} - j3)\,//\,(Z_{cn} + j6 + 8)$$

$$= 13.64\angle 4.204^{\circ}\ \Omega$$

$$\therefore \ I = \frac{V}{Z} = \frac{50\angle 0^{\circ}}{13.64\angle 4.204^{\circ}} = 3.666\angle -4.204^{\circ}\ A$$