AC CIRCUITS

Sinusoids and phasors

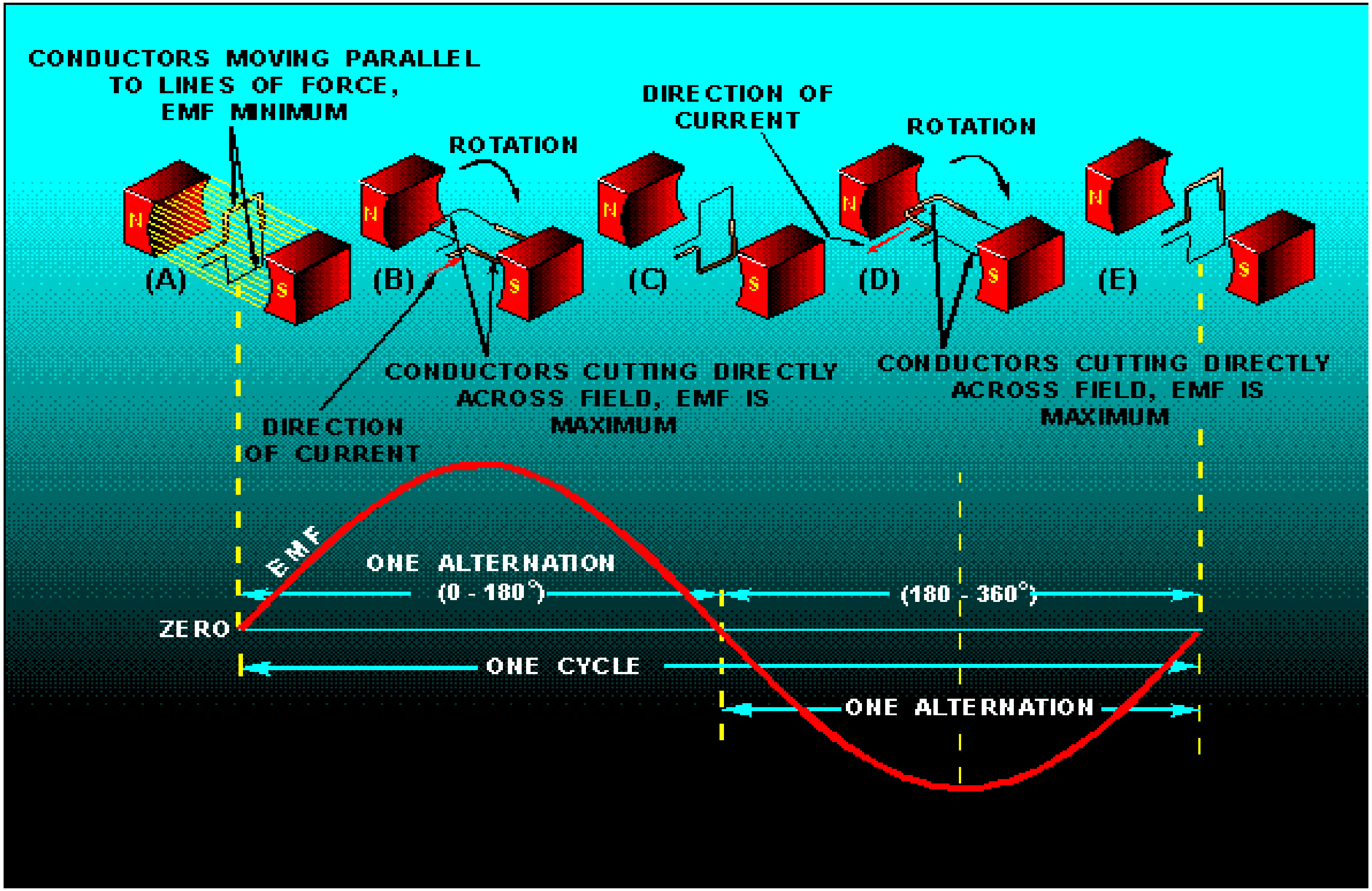
SINUSOIDS

- A sinusoid is a signal that has the form of the sine or cosine function.

- A sinusoidal current is called alternating current (ac). It reverses at regular time internals and has alternately positive and negative values

- AC circuits are circuits driven by sinusoidal current or voltages.

Sinusoids



CONDUCTORS MOVING PARALLEL TO LINES OF FORCE, EMF MINIMUM
ROTATION
DIRECTION OF CURRENT
ROTATION
N
S
(A)
(B)
(C)
(D)
(E)
CONDUCTORS CUTTING DIRECTLY ACROSS FIELD, EMF IS MAXIMUM
CONDUCTORS CUTTING DIRECTLY ACROSS FIELD, EMF IS MAXIMUM
DIRECTION OF CURRENT
EMF
ONE ALTERNATION (0 - 180°)
(180 - 360°)
ZERO
ONE CYCLE
ONE ALTERNATION

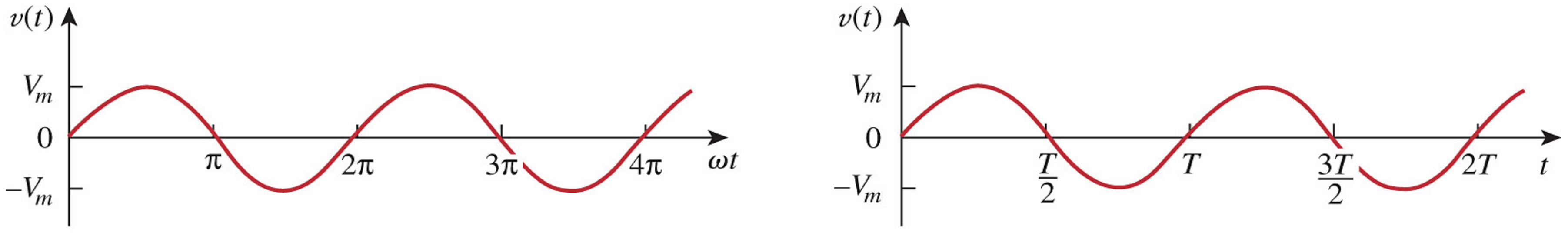
Sinusoids

- Considering a sinusoidal voltage,

$$v(t) = V_m \sin \omega t$$

where

Vm = the amplitude of the sinusoid
ω = the angular frequency in radians/s
ωt = the argument of the sinusoid



(a) (b)

$$\omega T = 2\pi$$

$$T = \frac{2\pi}{\omega}$$

Sinusoids

A periodic function is one that satisfies $v(t) = v(t + nT)$, for all t and for all integers n

$$f = \frac{1}{T}$$

F is in Hertz (Hz)

$$\omega = 2\pi f$$

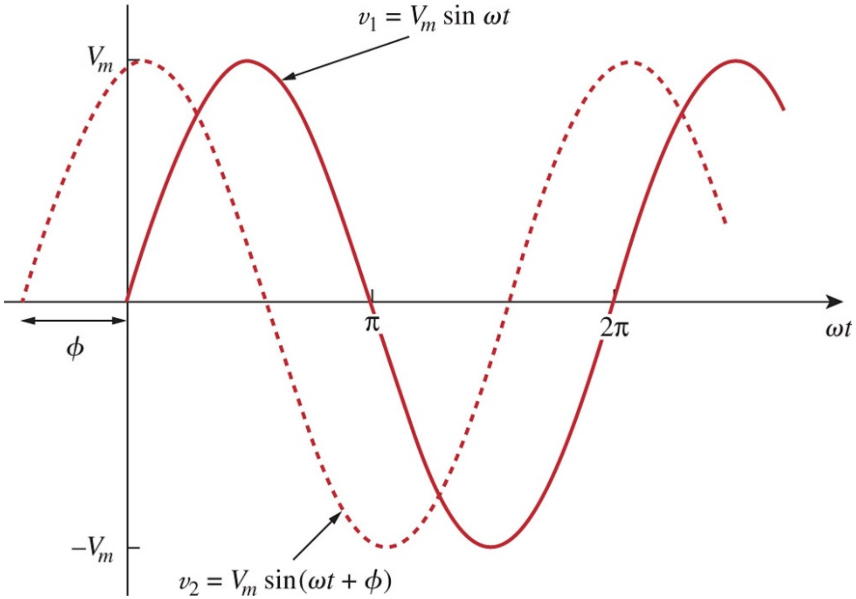
ω is in radians per second

Sinusoids

A more general expression for a sinusoid

$$v(t) = V_m \sin(\omega t + \phi)$$

Φ = the phase



- Only two sinusoidal values with the <u>same frequency</u> can be compared by their amplitude and phase difference.
- If phase difference is zero, they are in phase; if phase difference is not zero, they are out of phase.

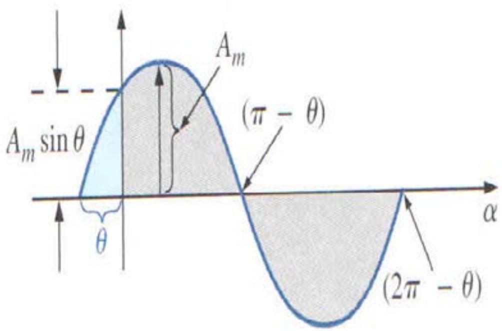
Sinusoids

- A sinusoid can be expressed in either sine or cosine form.
- When comparing two sinusoids, it is better to express both as either sine or cosine with positive amplitudes.
- To achieve this, 2 approaches can be used:
    1. Trigonometric identities
    2. Graphical approach

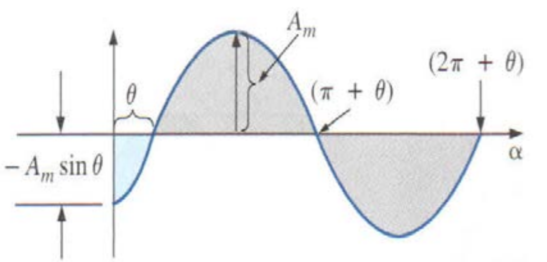
$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \sin B \mp \sin A \cos B$$
$$\sin(\omega t \pm 180^o) = -\sin \omega t$$
$$\cos(\omega t \pm 180^o) = -\cos \omega t$$
$$\sin(\omega t \pm 90^o) = \pm cos\ \omega t$$
$$\cos(\omega t \pm 90^o) = \mp sin\ \omega t$$

Sinusoids

Phase Relations



$$A_m \sin(\omega t + \theta)$$



$$A_m \sin(\omega t - \theta)$$

Sinusoids

Practice Problem 9.1

Given the sinusoid $5\sin(4\pi t - 60^0)$, calculate its amplitude, phase, angular frequency, period and frequency.

Practice Problem 9.2

Find the phase angle between $i_1 = -4\sin(377t + 25^o)$ and $i_2 = 5\cos(377t - 40^o)$. Does i_1 lead or lag i_2?

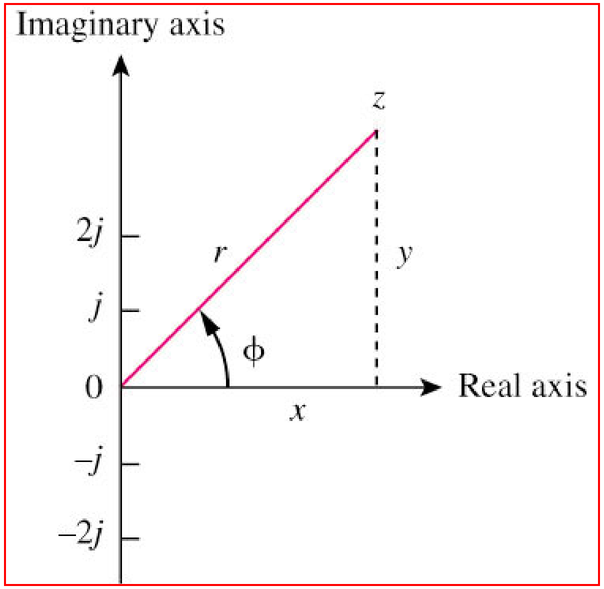
PHASORS

- A phasor is a complex number that represents the amplitude and phase of a sinusoid.
- It can be represented in one of the following three forms:

a. Rectangular $z = x + jy = r(\cos\phi + j\sin\phi)$

b. Polar $z = r\angle\phi$

c. Exponential



where $r = \sqrt{x^2 + y^2}$

$\phi = \tan^{-1}\frac{y}{x}$

Phasors

Mathematic operation of complex number:

1. Addition $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$
2. Subtraction $z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$
3. Multiplication $z_1 z_2 = r_1 r_2 \angle \phi_1 + \phi_2$
4. Division $\frac{z_1}{z_2} = \frac{r_1}{r_2} \angle \phi_1 - \phi_2$
5. Reciprocal $\frac{1}{z} = \frac{1}{r} \angle -\phi$
6. Square root $\sqrt{z} = \sqrt{r} \angle \phi/2$
7. Complex conjugate $z^* = x - jy = r \angle -\phi = re^{-j\phi}$
8. Euler's identity $e^{\pm j\phi} = \cos\phi \pm j\sin\phi$

Phasors

- Transform a sinusoid to and from the time domain to the phasor domain:

$$v(t) = V_m \cos(\omega t + \phi) \longleftrightarrow V = V_m \angle \phi$$

(time domain) (phasor domain)

- Amplitude and phase difference are two principal concerns in the study of voltage and current sinusoids.
- Phasor will be defined from the cosine function in all our proceeding study. If a voltage or current expression is in the form of a sine, it will be changed to a cosine by subtracting from the phase.

Phasors

Sinusoidal–Phasor Transformation

Time domain representation	Phasor domain representation
$V_m = \cos\,(\omega t + \varphi)$	$V_m \angle\phi$
$V_m = \sin\,(\omega t + \varphi)$	$V_m \angle\phi - 90^0$
$I_m = \cos(\omega t + \varphi)$	$I_m \angle\phi$
$I_m = \sin(\omega t + \varphi)$	$I_m \angle\phi - 90^0$

Phasors

The differences between v(t) and V:

- v(t) is instantaneous or <u>time-domain</u> representation
  <u>V is the frequency</u> or phasor-domain representation.
- v(t) is time dependent, V is not.
- v(t) is always real with no complex term, V is generally complex.

<u>Note</u>: Phasor analysis applies only when frequency is constant; when it is applied to <u>two or more</u> sinusoid signals only if they have the <u>same frequency</u>.

Phasors

Relationship between differential, integral operation in phasor listed as follow:

$$v(t) \longleftrightarrow \mathbf{V} = V\angle\phi$$

$$\frac{dv}{dt} \longleftrightarrow j\omega\mathbf{V}$$

$$\int v\,dt \longleftrightarrow \frac{\mathbf{V}}{j\omega}$$

Phasors

Practice Problem 9.4

Express these sinusoids as phasors:

a. $v = 7\cos(2t + 40^o)\,V$

b. $i = -4\sin(10t + 10^0) A$

Practice Problem 9.5

Find the sinusoids corresponding to these phasors:

a. $V = -10\angle 30$ V

b. $j(5 - j12)$ A

Practice Problem 9.6

If $v_1 = -10\sin(\omega t - 30^0)\,V$ and $v_2 = 20\cos(\omega t + 45^0)\,V$.

Find $v = v_1 + v_2$

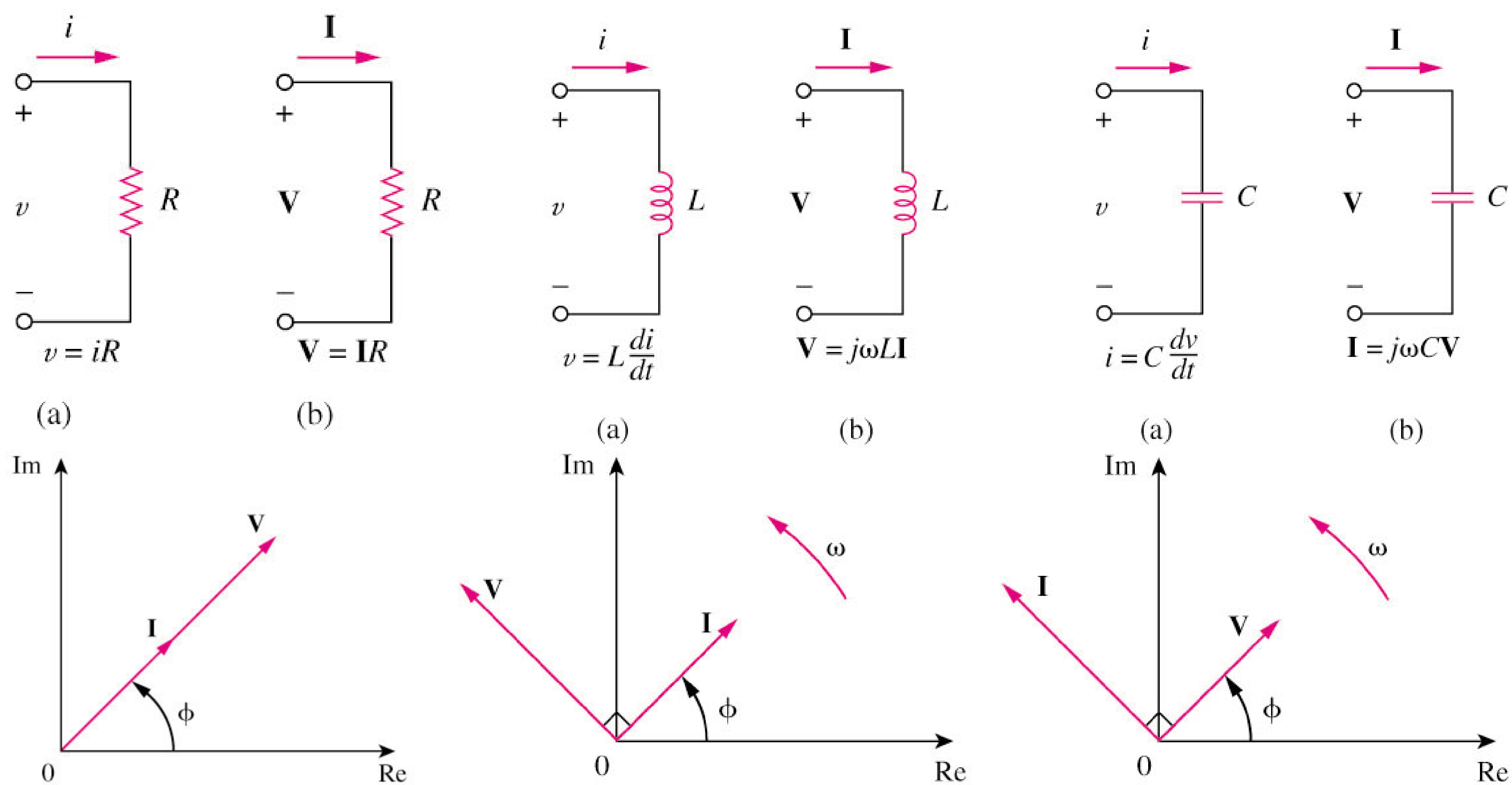
Phasors

Practice Problem 9.7

Find the voltage v(t) in a circuit described by the integrodifferential equation

$$2\frac{dv}{dt} + 5v + 10\int v\,dt = 50\cos(5t - 30^0)$$

PHASORS RELATIONSHIPS FOR CIRCUIT ELEMENTS



PHASORS RELATIONSHIPS FOR CIRCUIT ELEMENTS

Summary of voltage-current relationship		
Element	Time domain	Frequency domain
R	$v = Ri$	$V = RI$
L	$v = L\frac{di}{dt}$	$V = j\omega LI$
C	$i = C\frac{dv}{dt}$	$V = \frac{I}{j\omega C}$

Phasors

Practice Problem 9.8

If $v = 10\cos(100t + 30^o)$ is applied to a $50\mu F$ capacitor, calculate the current through the capacitor.

Example 9.8

The voltage $v = 12\cos(60t + 45^o)$ is applied to a 0.1-H inductor. Find the steady-state current through the inductor.

Practice Problem 9.3

Evaluate the following complex numbers:

(a) $[(5 + j2)(-1 + j4) - 5\angle 60^o)]^*$

(b) $\frac{10+j5+3\angle 40^o}{-3+j4} + 10\angle 30^o + j5$

IMPEDANCE & ADMITTANCE

- The impedance Z of a circuit is the ratio of the phasor voltage V to the phasor current I, measured in ohms Ω. It is a frequency-dependent quantity.
- It represents the opposition that the circuit exhibits to the flow of sinusoidal current.

$$Z = \frac{V}{I} \quad = R + jX \quad = |Z| \angle \theta$$

where R = Re Z is the resistance and X = Im Z is the reactance.

- The admittance Y is the reciprocal of impedance, measured in siemens (S).

$$Y = \frac{1}{Z} = \frac{I}{V} \quad = G + jB$$

where G = Re Y is the conductance and B = Im Y is the susceptance.

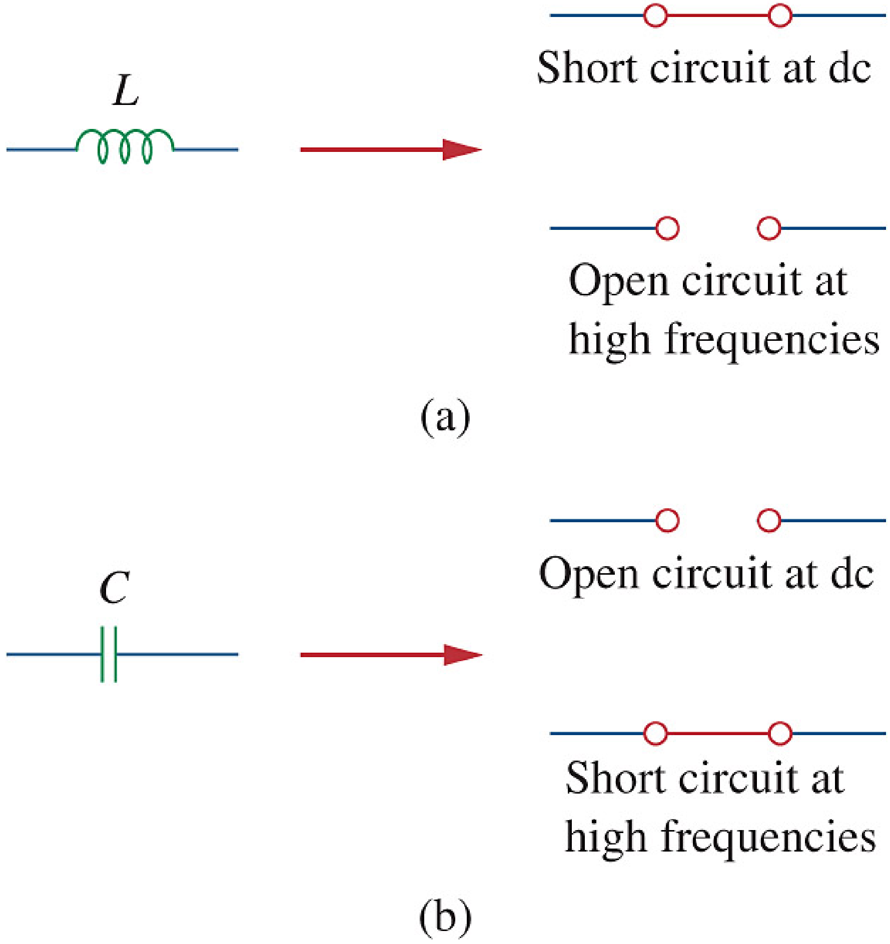
Impedance & Admittance

Impedances and admittances of passive elements		
Element	**Impedance**	**Admittance**
R	$Z = R$	$Y = \frac{1}{R}$
L	$Z = j\omega L$	$Y = \frac{1}{j\omega L}$
C	$Z = \frac{1}{j\omega C}$	$Y = j\omega C$

$Z = R + jX$ inductive/lagging (*I* lags *V*)

$Z = R - jX$ capacitive/leading (*I* leads *V*)

Impedance & Admittance



$$\omega = 0; Z = 0$$

$$\omega \to \infty; Z \to \infty$$

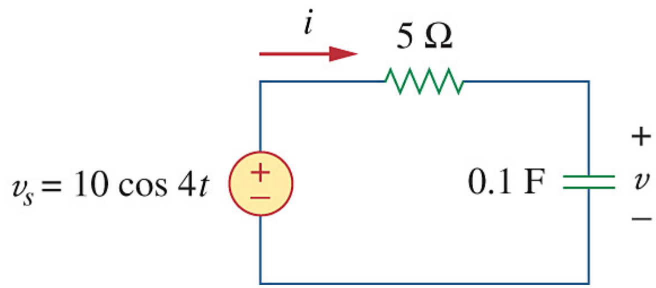
$$\omega = 0; Z \to \infty$$

$$\omega \to \infty; Z = 0$$

Impedance & Admittance

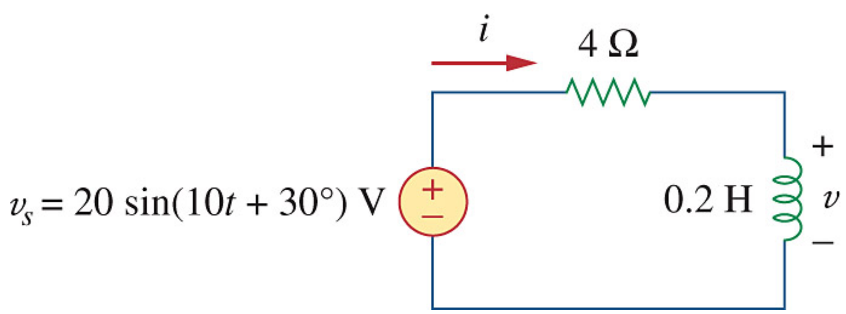
Example 9.9

Find $v(t)$ and $i(t)$ in the circuit below:



Practice Problem 9.9

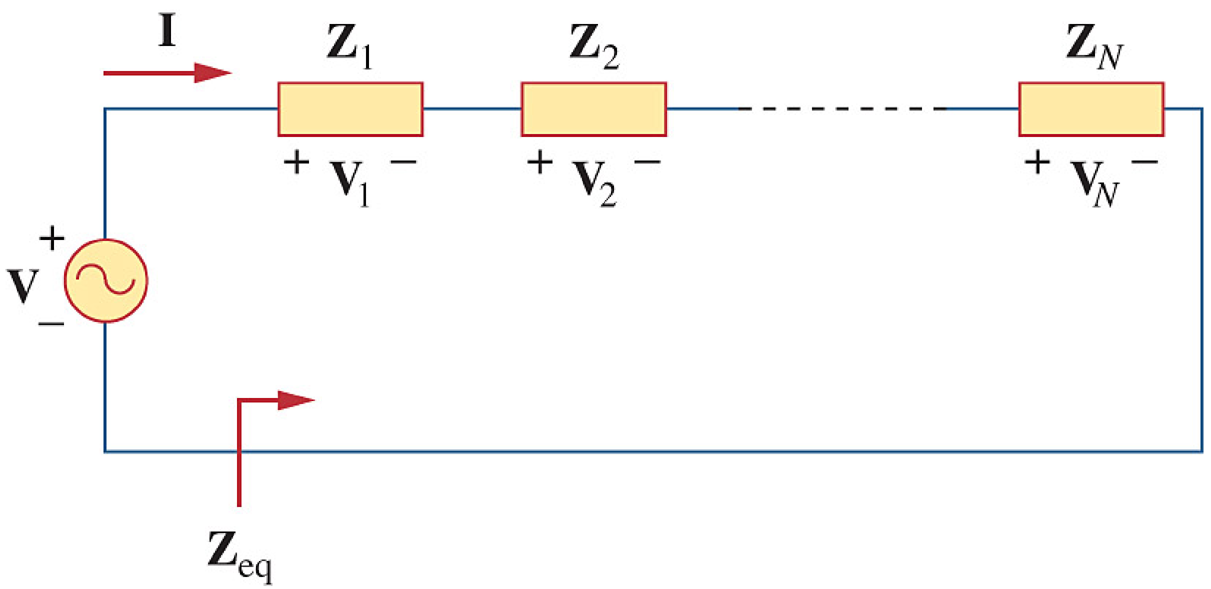
Determine $v(t)$ and $i(t)$ in the circuit below:



KIRCHHOFF'S LAWS IN THE FREQUENCY DOMAIN

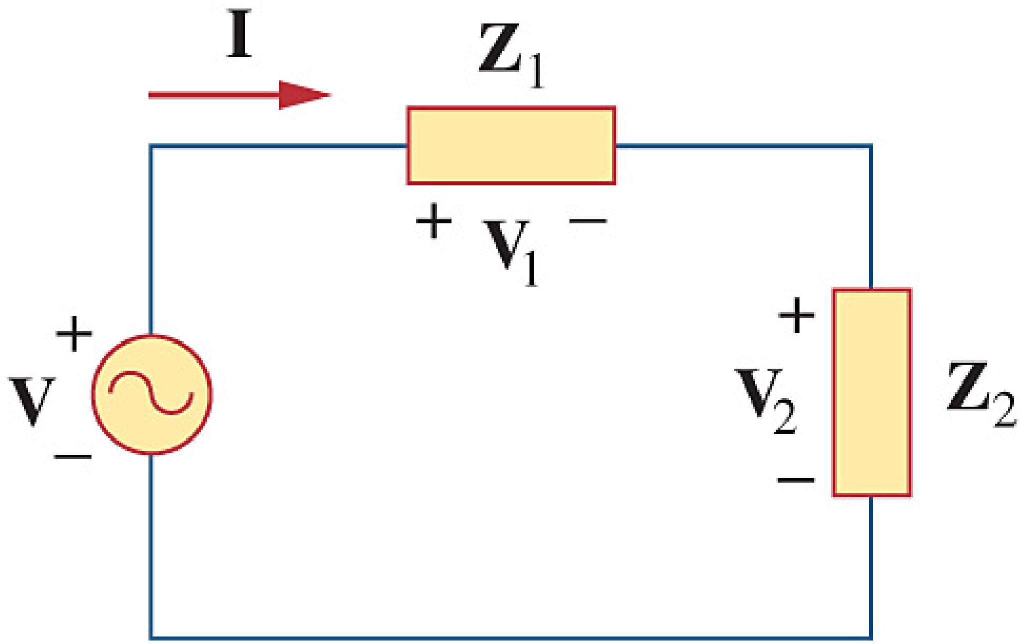
- Both KVL and KCL are hold in the phasor domain (frequency domain).
- Variables to be handled are phasors, which are complex numbers.
- All the mathematical operations involved are now in complex domain.
- The following principles used for DC circuit analysis all apply to AC circuit:
  - a. voltage division
  - b. current division
  - c. circuit reduction
  - d. impedance equivalence
  - e. Y-Δ transformation

KIRCHHOFF'S LAWS IN THE FREQUENCY DOMAIN



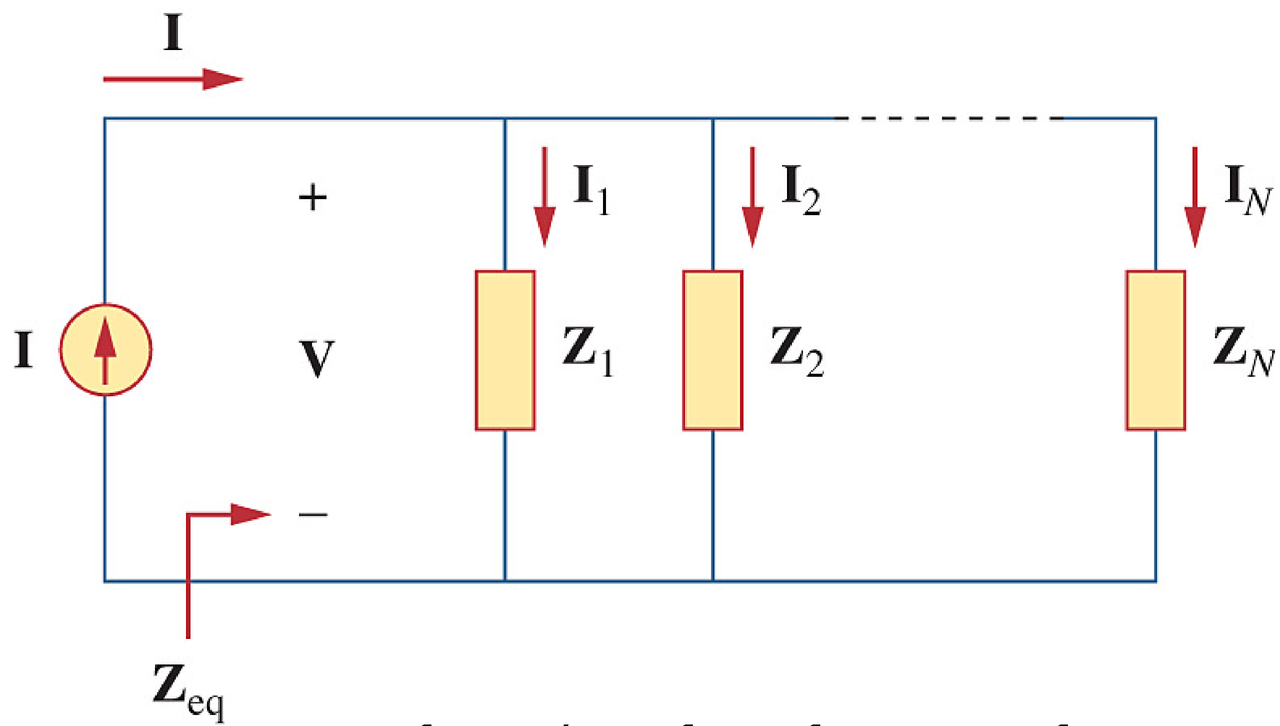
$$Z_{eq} = Z_1 + Z_2 + \ldots\ldots + Z_N$$

KIRCHHOFF'S LAWS IN THE FREQUENCY DOMAIN



$$V_1 = \frac{Z_1}{Z_1 + Z_2} V \qquad V_2 = \frac{Z_2}{Z_1 + Z_2} V$$

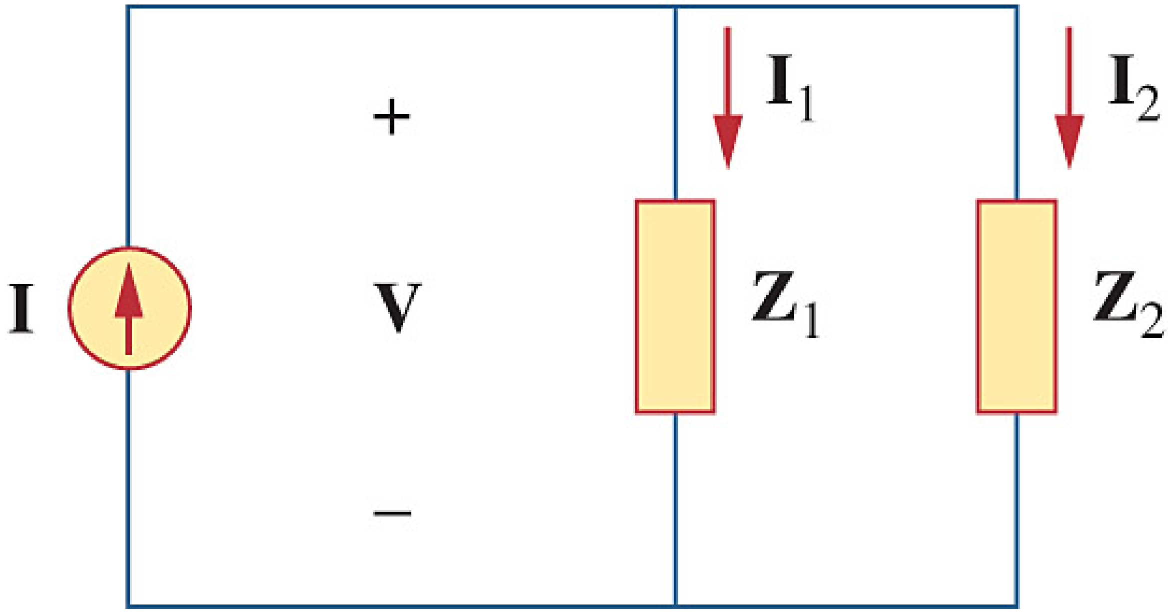
KIRCHHOFF'S LAWS IN THE FREQUENCY DOMAIN



$$\frac{1}{Z_{eq}} = \frac{I}{V} = \frac{1}{Z_1} + \frac{1}{Z_2} + \ldots\ldots\frac{1}{Z_N}$$

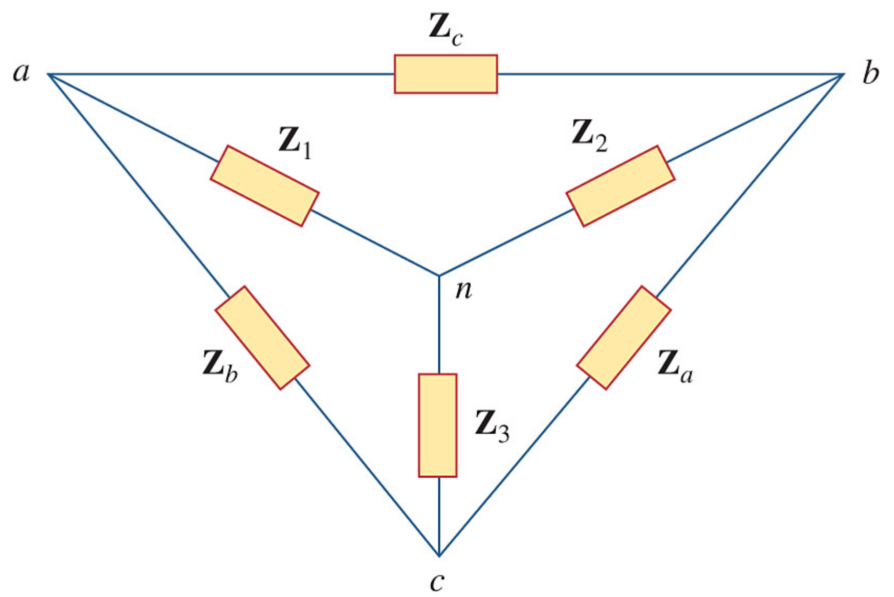
$$Y_{eq} = Y_1 + Y_2 + \ldots\ldots + Y_N$$

KIRCHHOFF'S LAWS IN THE FREQUENCY DOMAIN



$$I_1 = \frac{Z_2}{Z_1 + Z_2} I \qquad I_2 = \frac{Z_1}{Z_1 + Z_2} I$$

KIRCHHOFF'S LAWS IN THE FREQUENCY DOMAIN



$$Z_a = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_1}$$

$$Z_b = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_2}$$

$$Z_c = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_3}$$

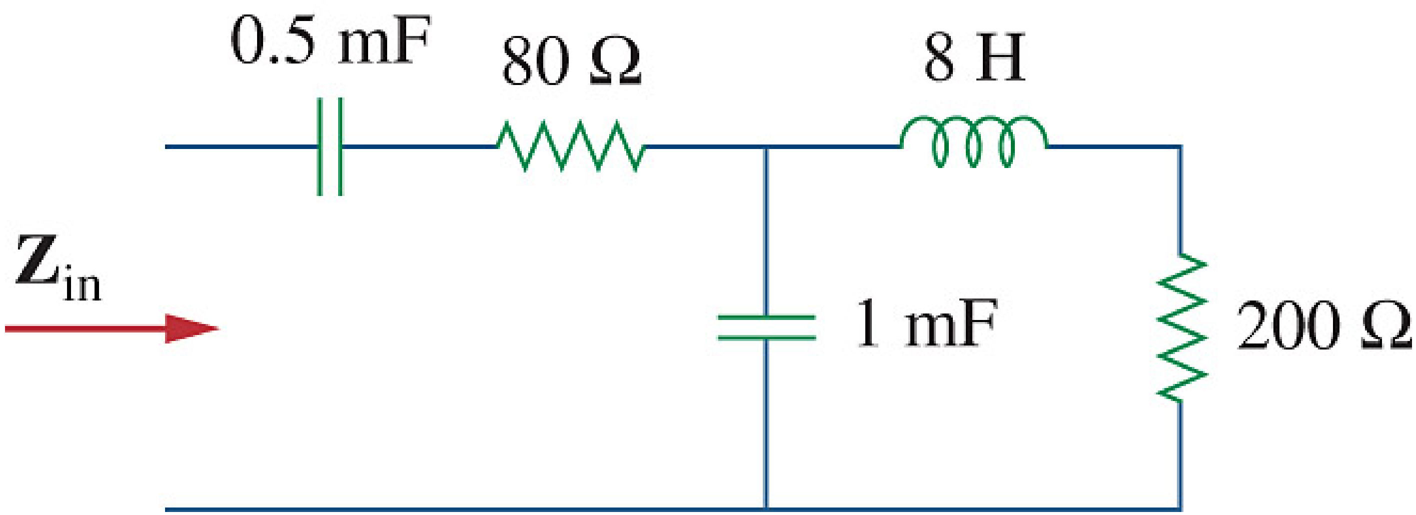
$$Z_1 = \frac{Z_bZ_c}{Z_a + Z_b + Z_c}$$

$$Z_2 = \frac{Z_cZ_a}{Z_a + Z_b + Z_c}$$

$$Z_3 = \frac{Z_aZ_b}{Z_a + Z_b + Z_c}$$

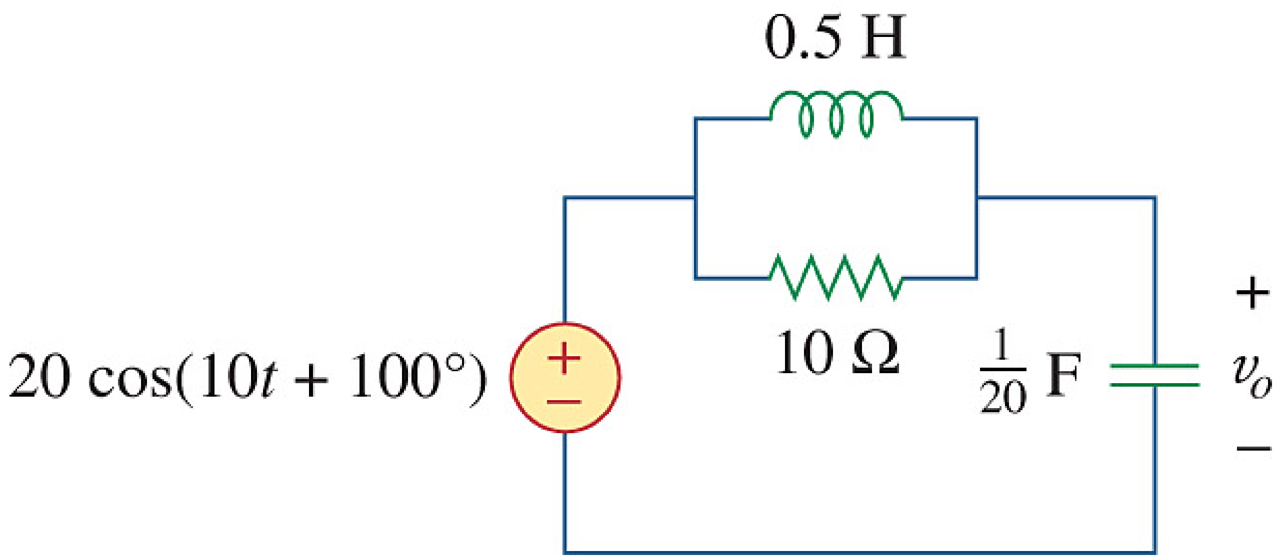
Practice Problem 9.10

Determine the input impedance of the circuit below at $\omega = 10\, rad/s$



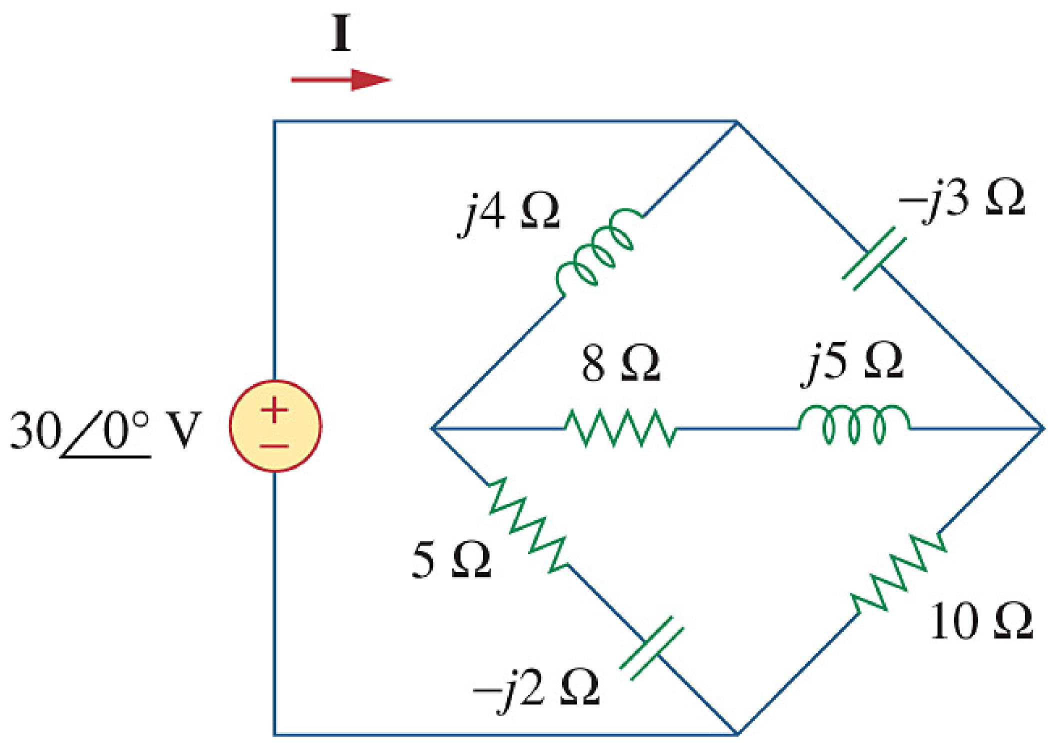
Practice Problem 9.11

Calculate v_o in the circuit below:



Practice Problem 9.12

Find **I** in the circuit below:



Problem 9.6

For the following pairs of sinusoids, determine which one leads and by how much.

1. $v(t) = 10 \cos(4t - 60°)$ and $i(t) = 4 \sin(4t + 50°)$
2. (b) $v_1(t) = 4 \cos(377t + 10°)$ and $v_2(t) = -20 \cos 377t$
3. (c) $x(t) = 13 \cos 2t + 5 \sin 2t$ and $y(t) = 15 \cos(2t - 11.8°)$

TUTORIALS (Phasors)

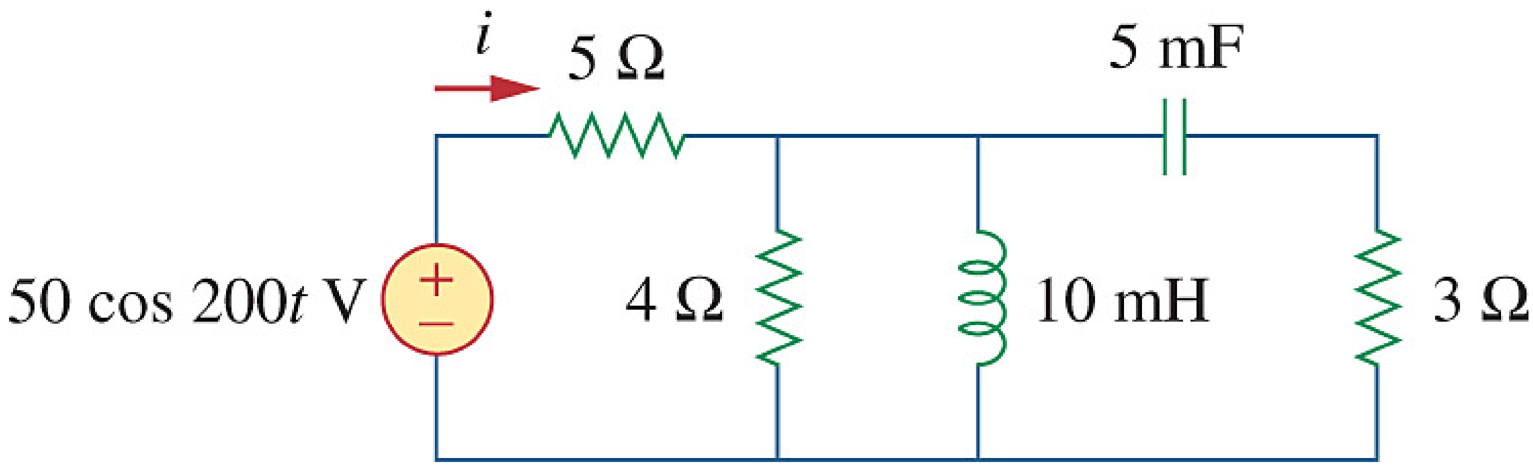
1. Find the phasors corresponding to the following signals.
   (a) $v(t) = 21\cos(4t - 15^o)$ V
   (b) $i(t) = -8\sin(10t + 70^o)$ mA
   (c) $v(t) = 120\sin(10t - 50^o)$ V
   (d) $i(t) = -60\cos(30t + 10^o)$ mA

2. Using phasors, find:
   (a) $3\cos(20t + 10^\circ) - 5\cos(20t - 30^\circ)$
   (b) $40\sin 50t + 30\cos(50t - 45^\circ)$
   (c) $20\sin 400t + 10\cos(400t + 60^\circ) - 5\sin(400t - 20^\circ)$

3. Find $v(t)$ in the following integrodifferential equations using the phasor approach:
   (a) $v(t) + \int v\,dt = 5\cos(t + 45^o)$ V
   (b) $\frac{dv}{dt} + 5v(t) + 4\int v\,dt = 20\sin(4t + 10^o)$ V

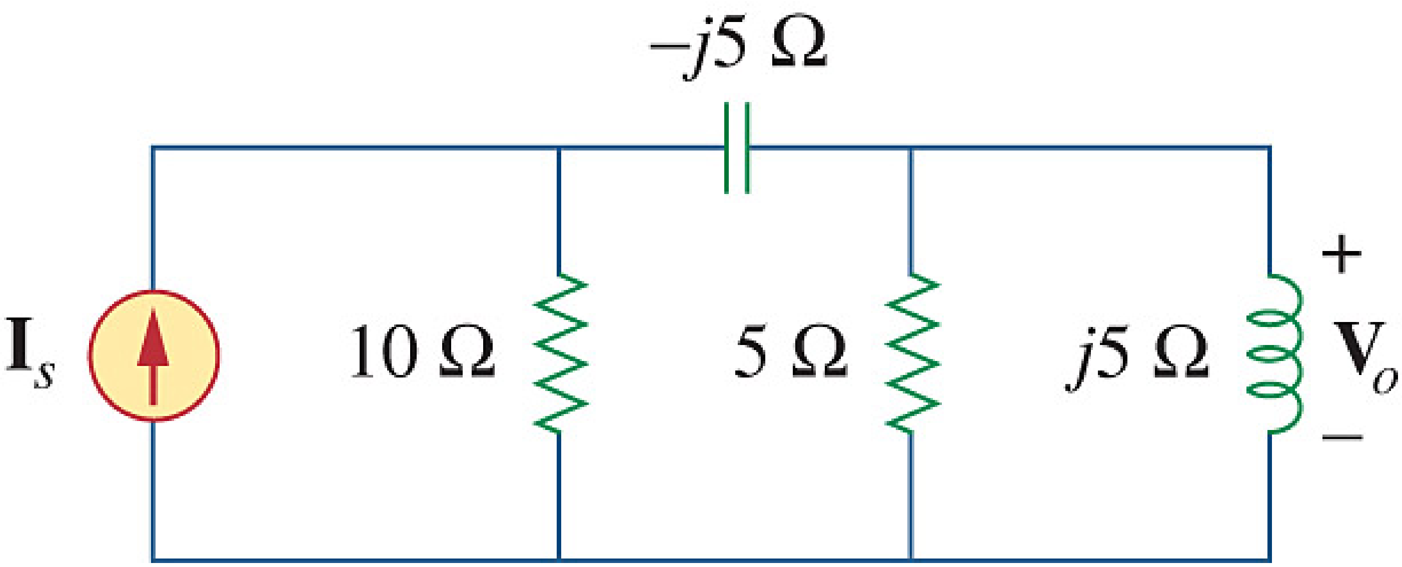
Problem 9.44

Calculate $i(t)$ in the circuit below:



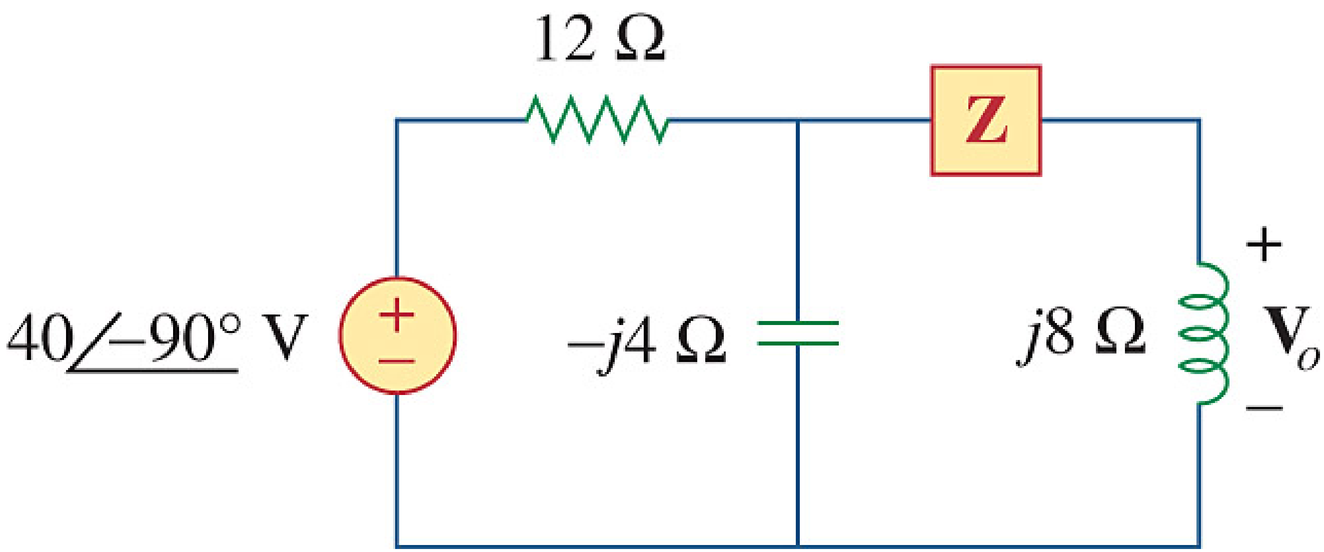
Problem 9.52

If $V_o = 20\angle 45^o$ V in the circuit, find I_s:



Problem 9.55*

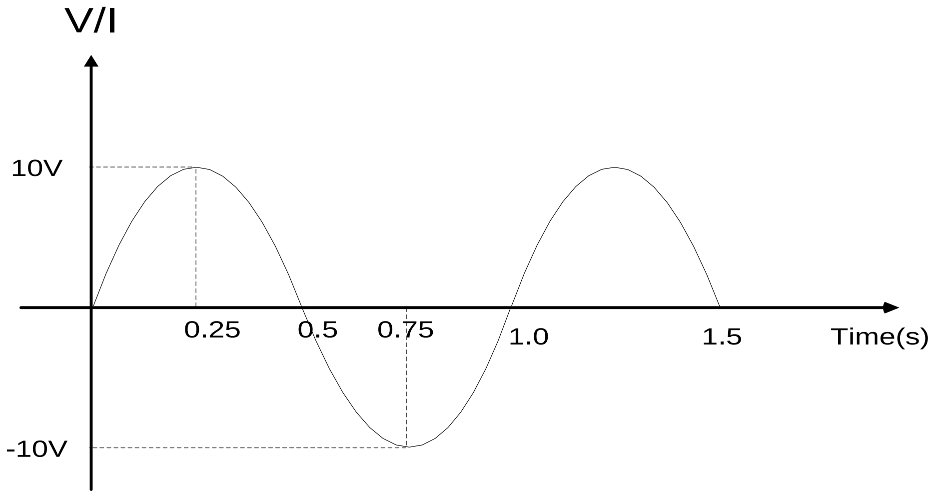
Given $V_o = 8\angle 0^o$ V, find the Z. What are the elements are contained in Z? Calculate the value of their resistances/reactances if the system frequency is 50 Hz.



Average & Effective (rms) Values

Instantaneous Value

- Instantaneous value is magnitude value of waveform at one specific time.
- Symbol for Instantaneous value of voltage is v(t) and current is i(t).
- Example of Instantaneous value for voltage is shown:



$$v(0.25) = 10V$$
$$v(0.5) = 0V$$
$$v(0.75) = -10V$$
$$v(1.0) = 0V$$

Average & Effective (rms) Values

Average Value

- Average value is average value for all instantaneous value in half or one complete waveform cycle.
- It can be calculate in two ways:

  1. Calculate the area under the graph:

  $$\text{Average value} = \frac{\text{area under the function in a period}}{\text{period}}$$

  2. Use integral method

$$average\_value = \frac{1}{T}\int_0^T v(t)dt$$

Average & Effective (rms) Values

Average Value

- For example

Instantaneous power:

$$p(t) = v(t)i(t) = V_m I_m \cos(\omega t + \theta_v)\cos(\omega t + \theta_i)$$

Average power:

$$P = \frac{1}{T}\int_0^T p(t)dt = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)$$

Average & Effective (rms) Values

Effective value

- The most common method of specifying the amount of sine wave of voltage or current by relating it into dc voltage and current that will produce the same heat effect.
- It is called root means square value, rms
- The formula of effective value for sine wave waveform is

$$V_{rms} = \frac{v_m}{\sqrt{2}} = 0.7071 v_m$$

where I_m & V_m are peak values

$$I_{rms} = \frac{i_m}{\sqrt{2}} = 0.7071 i_m$$

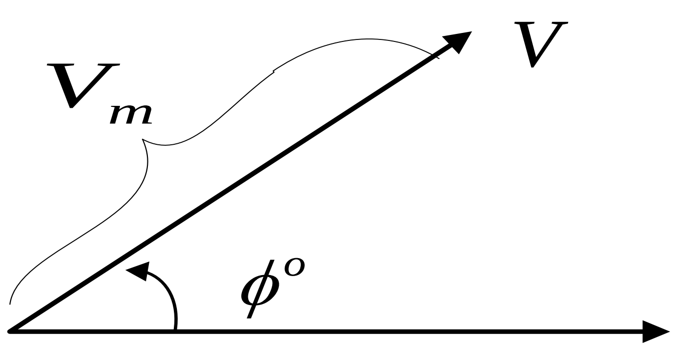
Phasor Diagram

- For example, if given a sine wave waveform

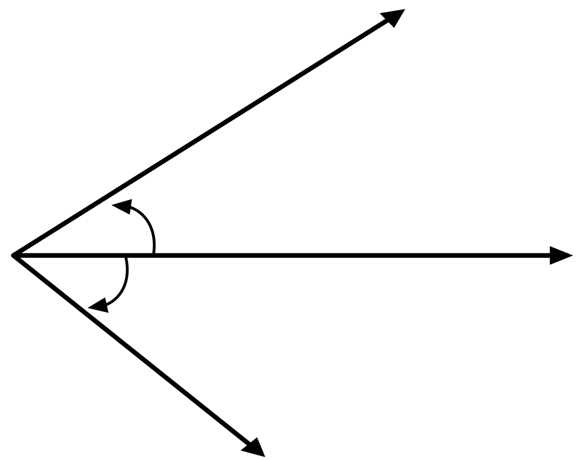
$$v(t) = V_m \cos(\omega t + \phi^0)V$$

- It can be represent by a phasor diagram

$$V = V_m \angle \phi^0$$



Phasor Diagram



From the phase diagram above, it can be conclude that:

i) I leading V for θ° degree or V lagging I for θ° degree

ii) V leading V1 for Φ° degree or V1 lagging V for Φ° degree

iii) I leading V1 for $(\Phi^\circ + \theta^\circ)$ degree or V1 lagging I for $(\Phi^\circ + \theta^\circ)$ degree

Example

Given the circuit below, sketch the phasor diagram of V_S, V_R, V_L, V_C, I_S.