5.2.3 Difference method

Let $f(x)$ be a function of x and the r-th term of the series $\sum_{r=1}^{n} u_r$ is of the form $u_r = f(r) - f(r-1)$, then

$$\begin{aligned}\sum_{r=1}^{n} u_r &= \sum_{r=1}^{n} \left[f(r) - f(r-1) \right] \\ &= \left[f(1) - f(0) \right] + \left[f(2) - f(1) \right] + \left[f(3) - f(2) \right] \\ &\quad + \left[f(4) - f(3) \right] + \ldots + \left[f(n) - f(n-1) \right] \\ &= -f(0) + f(n) \\ &= f(n) - f(0).\end{aligned}$$

To conclude,

If $u_r = \left[f(r) - f(r-1) \right]$, then $\sum_{r=1}^{n} u_r = \left[f(n) - f(0) \right]$.

Or equivalently

If $ku_r = \left[f(r) - f(r-1) \right]$, then $\sum_{r=1}^{n} u_r = \frac{1}{k}\left[f(n) - f(0) \right]$. where k is a constant.

Note: If we fail to express u_r into this form, $\left[f(r) - f(r-1) \right]$, then this method cannot be used.

Example 8:

Express the r-th term of the series $1\cdot 2+2\cdot 3+3\cdot 4+...+r\cdot(r+1)+...$ as the difference of two functions of r and r – 1. Hence find the sum of the first n terms of the series.

Solution:

Step 1: Find the general form of the r-th term:

Step 2: Form another sequence $f(r)$ by adding one more factor to the end of the general term u_r :

Step 3: Find $f(r-1)$:

Step 4: Form the difference:

$f(r)-f(r-1)=$

Step 5: Find the sum :

Tips:

(A) If the general term, u_r , of the series is in "product" form, you can **add** one more factor to the end of the general term u_r , so as to form a sequence $f(r)$ and then apply the difference method.

(B) If the general term, u_r , is in "quotient" form, you can **remove** one more factor at the end of the general term u_r , so as to form a sequence $f(r)$ and then apply the difference method.

Example 9:

By using the difference method, find the sum of the first n terms of the series

$$\frac{1}{2\cdot3}+\frac{1}{3\cdot4}+\frac{1}{4\cdot5}+...+\frac{1}{(n+1)(n+2)}.$$

Example (10):

Use the difference method; find the sum of the series

$$\sum_{r=1}^{n}\frac{2}{(r+2)(r+3)}.$$

5.3 Test of Convergence

5.3.1 Divergence Test

If $\sum_{r=1}^{\infty} a_r$ converges, then $\lim_{r\to\infty} a_r = 0$. Equivalently, if $\lim_{r\to\infty} a_r \neq 0$, or $\lim_{r\to\infty} a_r$ does not exist, then the series is diverges.

Example (11):

Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

Example 12:

Use Divergence Test to determine whether $\sum_{r=1}^{\infty} \frac{r}{\ln r}$ diverges or not.

5.3.2 The Integral Test

Suppose f is a continuous, positive, decreasing function on $[1,\infty)$ and let $a_r = f(r)$. Then the series $\sum_{r=1}^{\infty} a_r$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words

(a) If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{r=1}^{\infty} a_r$ is convergent.

(b) If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{r=1}^{\infty} a_r$ is divergent.

Note: Use this test when $f(x)$ is easy to integrate.

Example 13:

Use the Integral Test to determine whether the following series converges or diverges.

(a) $\displaystyle\sum_{r=2}^{\infty} \frac{1}{r \ln r}$.

(b) $\displaystyle\sum_{r=1}^{\infty} \frac{r}{\sqrt{r^2+4}}$.

5.3.3 Ratio Test

Let $\sum_{r=1}^{\infty} a_r$ be an infinite series with positive terms and let

$$\rho = \lim_{r \to +\infty} \frac{a_{r+1}}{a_r}.$$

a) If $0 \le \rho < 1$, the series converges.

b) If $\rho > 1$, or $\rho = +\infty$, the series diverges.

c) If $\rho = 1$, the test is inconclusive.

Example 14:

Use the Ratio Test to determine whether the following series converges or diverges.

(a) $\sum_{r=1}^{\infty} \frac{r^2}{4^r}$.

(b) $\sum_{r=1}^{\infty} r\, e^{-r}$.

5.4 Power Series

Definition

A power series about $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots$$

A power series about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \ldots + a_n(x-a)^n + \ldots$$

in which the center a and the coefficients $a_0, a_1, a_{2,} \ldots, a_n, \ldots$ are constants.

5.4.1 Expansion of Exponent Function

The power series of the exponent function can be written as

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots$$

The expansion is true for all values of x**.** In general,

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n .$$

Example (15):

Given

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots + \frac{1}{n!}x^n + \dots$$

Write down the first five terms of the expansion of the following functions

(a) e^{2x}

(b) e^{x-1}

Example (16):

Write down the first five terms on the expansion of the function, $(1+x)^2 e^{-x}$ in the form of power series.

5.4.2 Expansion of Logarithmic Function

The expansion of logarithmic function can be written as

$$\ln(1+\mathrm{x}) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5$$

$$-\frac{1}{6}x^6 + \frac{1}{7}x^7 - \ldots$$

The series converges for $-1 < x \leq 1$. Thus the series $\ln\left(1+x\right)$ is valid for $-1 < x \leq 1$.

By assuming x with $-x$, we obtain

$$\ln(1-\mathrm{x}) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{5}x^5$$

$$-\frac{1}{6}x^6 - \frac{1}{7}x^7 - \ldots$$

Thus, this result is true for $-1 < -x \leq 1$ or $-1 \leq x < 1$.

Example (17):

Write down the first five terms of the expansion of the following functions

(a) $\ln(1+3x)$

(b) $3\ln(1-2x^2)(1+3x)$

Example (18):

Find the first four terms of the expansion of the function,
$(1+x)^2 \ln(1+2x)^3$.

5.4.3 Expansion of Trigonometric Function

The power series for trigonometric functions can be written as

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$$

Both series are valid for all values of x.

Example (19):

Given

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$$

Find the expansion of $\cos(2x)$ and $\cos(3x)$. Hence, by using an appropriate trigonometric identity find the first four terms of the expansion of the following functions:

(a) $\sin^2(x)$

(b) $\cos^3(x)$

5.5 Taylor and the Maclaurin Series

Definition 5.9 (TAYLOR AND MACLAURIN SERIES)

If $f(x)$ has a derivatives of all orders at $x = a$, then we call the series as **Taylor's Series** for $f(x)$ about $x = a$ and is given by

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots + \frac{(x-a)^r}{r!}f^r(a) + \cdots.$$

or

$$f(x+a) = f(a) + x\,f'(a) + \frac{x^2}{2!}f''(a) + \frac{x^3}{3!}f'''(a) + \cdots + \frac{x^r}{r!}f^r(a) + \cdots.$$

In the special case where $a = 0$, this series becomes the **Maclaurin Series** for $f(x)$ and is given by

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x}{r!}f^r(0) + \cdots \quad \diamond$$

Example 20:

Obtain the Taylor series for $f(x) = 3x^2 - 6x + 5$ around the point $x = 1$.

Ans: $2 + 3(x-1)^2$

Example 21:

Obtain Maclaurin series expansion for the first four terms of e^x and five terms of $\sin x$. Hence, deduct that Maclaurin series for $e^x \sin x$ is given by $x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 + \ldots$

Example 22:

Use Taylor's theorem to obtain a series expansion of first five terms for $\cos\left(x + \frac{\pi}{3}\right)$. Hence find $\cos 62^0$ correct to 4 dcp.

Ans: 0.4695

Example 23:

If $y = \ln \cos x$, show that

$$\frac{d^2y}{dx^2} + 1 + \left(\frac{dy}{dx}\right)^2 = 0$$

Hence, by differentiating the above expression several times, obtain the Maclaurin's series of $y = \ln \cos x$ in the ascending power of x up to the term containing x^4.

Solution:

Finding Limits with Taylor Series and Maclaurin Series.

Example 24:

Find $\lim_{x\to 0}\frac{e^x-1-x}{x^2}$.

Ans: ½

Example 25:

Evaluate $\lim_{x\to 0}\frac{x^2+2\cos x-2}{3x^4}$.

Ans: 1/36

Evaluating Definite Integrals with Taylor Series and Maclaurin Series.

Example 26:

Use Maclaurin series to approximate the following definite integral.

a) $\int_0^1 e^{-x^2}\,dx$

b) $\int_0^1 x\cos(x^3)\,dx$

Ans: 0.747, 0.440