

# CHAPTER 4:

## IMPROPER INTEGRALS

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## 4.1 L'Hopital Rule

If you are doing any limit and you get something in the form of  $0/0$  or  $\infty/\infty$ , then you should probably try to use L'Hopital rule. The basic idea of L'Hopital rule is simple.

Consider the limit

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}.$$

If both the numerator and the denominator are finite at  $a$  and  $g(a) \neq 0$ , then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)}.$$

*Example 1:*

$$\lim_{x \rightarrow 3} \frac{x^2 + 1}{x + 2} = \frac{10}{5} = 2.$$

But what happen if both the numerator and the denominator tend to zero???

It is not clear what the limit is. In fact, depending on what functions  $f(x)$  and  $g(x)$  are, the limit can be anything at all!

### 4.1.1 *L'Hopital Rule for 0/0*

Suppose  $\lim f(x) = \lim g(x) = 0$ . Then

1. If  $\lim \frac{f'(x)}{g'(x)} = L,$

then  $\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)} = L.$

2. If  $\lim \frac{f'(x)}{g'(x)}$  tends to  $+\infty$  or  $-\infty$  in the limit, then

so does  $\lim \frac{f(x)}{g(x)}.$

*Example 2:*

Find  $\lim_{x \rightarrow 0} \frac{\sin x}{x}$  by L'Hopital rule.

*Example 3:*

Find  $\lim_{x \rightarrow 1} \frac{2 \ln x}{x - 1}$ .

*Example 4:*

Find  $\lim_{x \rightarrow 1} \frac{e^x - 1}{x^2}$ .

*Example 5:*

Find  $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x}$ .

Note: If the numerator and the denominator both tend to  $+\infty$  or  $-\infty$ , L'Hopital rule still applies.

### 4.1.2 *L'Hopital Rule for $\infty/\infty$*

Suppose  $\lim f(x)$  and  $\lim g(x)$  are both infinite. Then

1. If  $\lim \frac{f'(x)}{g'(x)} = L,$

then  $\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)} = L.$

2. If  $\lim \frac{f'(x)}{g'(x)}$  tends to  $+\infty$  or  $-\infty$  in the limit, then so

does  $\lim \frac{f(x)}{g(x)}.$

*Example 6:*

Find  $\lim_{x \rightarrow \infty} \frac{x}{e^x}.$

*Example 7:*

Find  $\lim_{x \rightarrow \infty} \frac{\ln(\ln x^{1000})}{\ln x}.$

## 4.2 Improper Integrals

The definite integral

$$\int_a^b f(x) dx$$

is known as *improper integral* if either

- 1) one or both limits are infinite, or
- 2)  $f(x)$  is undefined at certain points on/in the interval.

Note: We called case: 1) as Type I  
2) as Type II

### 4.2.1 *Improper Integral Type 1*

If  $f(x)$  is continuous in the interval  $[a, \infty)$ ,

$$\text{then } \int_a^{\infty} f(x) dx = \lim_{T \rightarrow \infty} \int_a^T f(x) dx.$$

1) If  $f(x)$  is continuous in the interval  $(-\infty, b]$ ,

$$\text{then } \int_{-\infty}^b f(x) dx = \lim_{T \rightarrow -\infty} \int_T^b f(x) dx.$$

Note: the improper integrals in 1) and 2) is said to *converge* if the limit exists and *diverge* if the limit does not exist.

2) If  $f(x)$  is continuous in the interval  $(-\infty, \infty)$ ,

$$\text{then } \int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^{\infty} f(x) dx$$

with any real number  $c$ .

Note: the improper integrals in 3) is said to *converge* if both terms converge and *diverge* if either term diverges.

*Example 8:*

Determine whether the following integrals are convergent or divergent:

$$1) \int_1^{\infty} \frac{1}{x} dx$$

$$2) \int_0^{\infty} x e^{-x} dx$$

$$3) \int_{-\infty}^{\infty} \frac{x}{1+x^2} dx$$

*Example 9:*

For what values of  $p$  is the integral  $\int_1^{\infty} \frac{1}{x^p} dx$  convergent?



### 4.2.2 *Improper Integral Type 2*

- 1) If  $f(x)$  is continuous on  $[a, b)$ , and discontinuous at  $b$ , then

$$\int_a^b f(x) dx = \lim_{T \rightarrow b^-} \int_a^T f(x) dx.$$

- 2) If  $f(x)$  is continuous on  $(a, b]$ , and discontinuous at  $a$ , then

$$\int_a^b f(x) dx = \lim_{T \rightarrow a^+} \int_T^b f(x) dx.$$

Note: the improper integrals in 1) and 2) is said to *converge* if the limit exists and *diverge* if the limit does not exist.

- 3) If  $f(x)$  has discontinuity at  $c$ , where  $a < c < b$ , then

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

Note: the improper integrals in 3) is said to *converge* if both terms converge and *diverge* if either term diverges.

*Example 10:*

Determine whether  $\int_1^9 \frac{1}{\sqrt[3]{x-9}} dx$  converge or diverge.

*Example 11:*

Find  $\int_0^3 \frac{1}{x-1} dx$  if possible.