# <u>CHAPTER 3</u>

# INTEGRATION

3.1 Integration of hyperbolic functions

3.2 Integration of inverse trigonometric functions

3.3 Integration of inverse hyperbolic functions

*Recall*: Methods involved:

- Substitution of $u$
- By parts
- Tabular method
- Partial fractions

REVISION: **Techniques of integration**

(a) Integration by substitution

Example:

1. $\int \dfrac{\sin x}{1-\cos x}\, dx$

2. $\int \sin x \cos^4 x \; dx$

3. $\int x \cos x^2 \, e^{\sin x^2} \; dx$

(b) Integration by parts

Example:

1. $\int x \cos x \, dx$

2. $\int x \sin 2x \, dx$

(c) Tabular methods

Example:

1. $\int x \sec^2 x \, dx$

2. $\int e^{3x} \cos 2x \, dx$

(d) Integration using partial fractions

Example:

1. $\int \frac{3x+2}{x^2+3x+2} dx$

2. $\int \frac{1}{x^3+2x^2+x} dx$

## 3.1 Integrals of Hyperbolic Functions

Integral Formulae

1. $\int \sinh x dx = \cosh x + C$

2. $\int \cosh x dx = \sinh x + C$

3. $\int \sec h^2 x dx = \tanh x + C$

4. $\int \cos \mathrm{ech}^2 \mathrm{x} \mathrm{dx} = -\coth \mathrm{x} + \mathrm{C}$

5. $\int \sec hx \tanh x dx = -\sec hx + C$

6. $\int \cos echx \coth x dx = -\cos echx + C$

**Example 3.1 :** Evaluate the following integrals

a) $\int \sinh x \cosh x \, dx$

b) $\int \sqrt{\tanh x} \, \text{sec}h^2 x \, dx$

c) $\int x \cosh x \, dx$

d) $\int x^3 \cosh x \, dx$

## 3.2 Integration of Inverse Trigonometric Functions

Integration formulae of the Inverse Trigonometric Functions

| **Differentiation** | **Integration** |
|---|---|
| $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ | $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$ |
| $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$ | $\int \frac{-dx}{\sqrt{1-x^2}} = \cos^{-1} x + C$ |
| $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ | $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$ |
| $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$ | $\int \frac{-dx}{1+x^2} = \cot^{-1} x + C$ |
| $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$ | $\int \frac{dx}{\lvert x\rvert\sqrt{x^2-1}} = \sec^{-1} x + C$ |
| $\frac{d}{dx}(\csc^{-1} x) = \frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$ | $\int \frac{-dx}{\lvert x\rvert\sqrt{x^2-1}} = \csc^{-1} x + C$ |

**Example 3.2 :** Evaluate the following integrals

a) $\int_{0}^{1} \tan^{-1} x \, dx$

b) $\int \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}} \, dx$

c) $\int \frac{\sqrt{\tan^{-1} x}}{1+x^2} \, dx$

**What about**

$$\int \frac{dx}{\sqrt{4-x^2}},\ \int \frac{dx}{9+x^2},\ \int \frac{dx}{|x|\sqrt{x^2-10}}\ \ldots?$$

To find the answer for this question, lets try

to solve $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}}$.

***Solution:***

Let $x = au$, then $dx = adu$. Hence

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{du}{\sqrt{1-u^2}}$$

$$= \sin^{-1} u + C$$

$$= \sin^{-1}\left(\frac{x}{a}\right) + C$$

Using the same method, we can find the solution for

$$\int \frac{dx}{a^2+x^2},\ \int \frac{dx}{|x|\sqrt{x^2-a^2}},\ \ldots$$

From the above discussions, we obtain the general integration formulae as follows:

| |
|---|
| $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$ |
| $\int \frac{-dx}{\sqrt{a^2 - x^2}} = \cos^{-1}\left(\frac{x}{a}\right) + C$ |
| $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$ |
| $\int \frac{-dx}{a^2 + x^2} = \frac{1}{a}\cot^{-1}\left(\frac{x}{a}\right) + C$ |
| $\int \frac{dx}{\lvert x\rvert\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$ |
| $\int \frac{-dx}{\lvert x\rvert\sqrt{x^2 - a^2}} = \frac{1}{a}\csc^{-1}\left(\frac{x}{a}\right) + C$ |

**Example 3.3 :** Evaluate the following integrals

1. a) $\int \frac{dx}{\sqrt{16-x^2}}$

   b) $\int \frac{2\,dx}{3+x^2}$

2. a) $\int \frac{dx}{\sqrt{1-4x^2}}$

   b) $\int \frac{dx}{4+3x^2}$

3. a) $\int \frac{dx}{\sqrt{-x^2+2x+3}}$

   b) $\int \frac{dx}{x^2-2x+2}$

## 3.3 Integration involving Inverse Hyperbolic Functions

Integration formulae of the Inverse Hyperbolic Functions:

| **Differentiation** | **Integration** |
|---|---|
| $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$ | $\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x + C$ |
| $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$ | $\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x + C$ |
| $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$ | $\int \frac{dx}{1-x^2} = \tanh^{-1} x + C$ |

**What about**

$$\int \frac{dx}{\sqrt{4+x^2}},\ \int \frac{dx}{x^2-8},\ \int \frac{dx}{25-x^2}\ ...?$$

To find the answer for this question, lets try

to solve $\displaystyle\int \frac{dx}{\sqrt{a^2+x^2}}$.

***Solution:***

Let $x = au$, then $dx = adu$. Hence

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \int \frac{du}{\sqrt{1+u^2}}$$

$$= \sinh^{-1} u + C$$

$$= \sinh^{-1}\left(\frac{x}{a}\right) + C$$

Using the same method, we can find the solution for

$$\int \frac{dx}{a^2-x^2},\ \int \frac{dx}{\sqrt{x^2-a^2}}, \cdots$$

From the above discussions, we obtain the general integration formulae as follows:

| |
|---|
| $\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C$ |
| $\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C$ |
| $\int \frac{dx}{a^2-x^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) + C, & \lvert x \rvert < a \\ \frac{1}{a}\coth^{-1}\left(\frac{x}{a}\right) + C, & \lvert x \rvert > a \end{cases}$ |
| $\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\operatorname{sec}h^{-1}\left(\frac{x}{a}\right) + C$ |
| $\int \frac{dx}{x\sqrt{a^2+x^2}} = -\frac{1}{a}\cos ech^{-1}\left(\frac{x}{a}\right) + C$ |

**Examples 3.4:**

1. Solve the following:

a) $\int \frac{dx}{\sqrt{3x^2+2}}$

b) $\int \frac{dx}{x\sqrt{9-4x^2}}$

c) $\int \frac{dx}{\sqrt{2(x-3)^2+1}}$

d) $\int \frac{dx}{\sqrt{x^2+4x+3}}$

2. Show that $\int \frac{x+1}{\sqrt{x^2+1}}\,dx = \sqrt{x^2+1} + \sinh^{-1} x + C$.