Part A (55%)

Answer All Questions.

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1. Let L_1 and L_2 be the lines whose symmetric equations are

$$L_1: \quad \frac{x-1}{2} = \frac{y-2}{-1} = \frac{z-4}{-2}$$
$$L_2: \quad \frac{x-9}{1} = \frac{y-5}{3} = \frac{z+4}{-1}.$$

Find a parametric equation for the line that is perpendicular to L_1 and L_2 and passes through their point of intersection.

[6 Marks]

2. By using Gauss elimination method solve the system of linear equations given by $AX = B$ where

$$A = \begin{pmatrix} 3 & -3 & 6 \\ 3 & 2 & -8 \\ 1 & 2 & -2 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \text{ and } \quad B = \begin{pmatrix} 11 \\ 22 \\ -330 \end{pmatrix}$$

[6 marks]

3. Replace the polar equation

$$r = 4\tan\theta\sec\theta$$

by an equivalent Cartesian equation and hence sketch the graph.

[5 Marks]

4. Find the intersection points between the cardioid $r = 3(1-\cos\theta)$ and the line $\theta = \frac{2\pi}{3}$.

[4 Marks]

5. Given that $z_1 = 2+i$ and $z_2 = -2+4i$, find z such that $\frac{1}{z} = \frac{1}{z_1} + \frac{1}{z_2}$. Give your answer in the form of $a+bi$. Hence, find the modulus and argument of z, such that $-\pi \leq \arg z \leq \pi$.

[6 marks]

6. Show that $\operatorname{cosech}^{-1}x = \ln\left[\dfrac{1+\sqrt{1+x^2}}{x}\right]$. Hence solve

$$1 + \ln x = \operatorname{cosech}^{-1}x$$

and give your answer in terms of e.

[5 Marks]

7. If $y = (\cos x)^x$ and $-\frac{1}{2}\pi \leq x \leq \frac{1}{2}\pi$, show that

$$\frac{dy}{dx} = y\left(\ln(\cos x) - x\tan x\right).$$

Hence obtain the expansion of y in ascending power of x up to x^3 and find the value of $(\cos\frac{1}{4})^{\frac{1}{4}}$

[6 Marks]

8. Show that $\displaystyle\int_1^{\infty} \frac{x^2}{(1+x^2)^2}\,dx = \frac{1}{8}(\pi+2)$.

[6 Marks]

9. Use the Integral Test to determine whether the following series converges or diverges.

$$\sum_{r=2}^{\infty} \frac{1}{r\ln r}.$$

[6 Marks]

10. Given that $y = \tan^{-1}\left(\dfrac{x^3}{a}\right)$ for $a > 0$, find $\dfrac{dy}{dx}$. Hence or otherwise, evaluate

$$\int_0^2 \frac{x^2}{16+x^6}dx.$$

[5 Marks]

Part B (45%)

Answer Any Three Questions.

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11. Given the lines l_1 and l_2;

$$l_1: \quad \frac{x-1}{1} = \frac{2-y}{4} = \frac{z}{2} \qquad l_2: \quad \frac{4-x}{1} = \frac{y-3}{1} = \frac{z+2}{3}.$$

(a) Show that l_1 and l_2 are skewed by showing that they do not intersect and not parallel. [4 Marks]

(b) Find the equation of both planes containing the line l_1 and parallel to the plane containing the line l_2. Hence obtain the shortest distance between the lines l_1 and l_2.

[7 Marks]

(c) Find the acute angle between the line l_1 and the plane

$$3x + 5y - 4z = 6.$$

[4 Marks]

12. (a) Given $z = -1 + \sqrt{3}\,i$,

(i) find z^5 in the form of $a + i\,b$. [3 Marks]

(ii) find all the roots of $z^5 = -1 + \sqrt{3}\,i$ in the form of $a + i\,b$. Show all the roots on an Argand diagram.

[6 Marks]

(b) Using de Moivre's theorem, or otherwise, show that

$$\sin 5\theta = 16 \sin^5\theta - 20 \sin^3\theta + 5 \sin\theta.$$

[6 marks]

13. Given that $r^2 = 16\sin(2\theta)$ with $0 \le \theta \le 2\pi$.

    (i) Test the symmetries of the above equation. [3 Marks]

    (ii) Construct a table for (r, θ) with the following values and sketch the graph of $r^2 = 16\sin(2\theta)$.

    [6 Marks]

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
r	0			4			0

    (iii) Sketch the graph of the circle $r = 2\sqrt{2}$ on the same diagram.

    [2 Marks]

    (iv) Find the intersection points between the curves $r^2 = 16\sin(2\theta)$ and the circle $r = 2\sqrt{2}$. [4 Marks]

14. Determine the eigenvalues of the matrix $A = \begin{pmatrix} 3 & -3 & 6 \\ 0 & 2 & -8 \\ 0 & 0 & -2 \end{pmatrix}$

    (i) Show that $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector of A.

    [4 Marks]

    (ii) Show that $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ is also an eigenvector of $B = \begin{pmatrix} 7 & -6 & 2 \\ 1 & 2 & 3 \\ 1 & -3 & 2 \end{pmatrix}$ and write down the corresponding eigenvalues.

    [5 Marks]

    (iii) Hence, or otherwise, write down an eigenvector of matrix AB and state the corresponding eigenvalue.

    [6 Marks]
