Chapter 9: Complex Numbers

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9.1 Imaginary numbers

Consider:

$$x^2 = -4$$

This equation has no real solution. To solve the equation, we will introduce an imaginary number.

Definition 9.1 (Imaginary Number)

> The imaginary number ***i*** is defined as:
>
> $$i^2 = -1$$

Therefore, using the definition, we will get,

$$\begin{aligned} x^2 &= -4 \\ x &= \sqrt{-4} \\ &= \sqrt{4(-1)} \\ &= \sqrt{4i^2} \\ &= \pm 2i \end{aligned}$$

Example: Express the following as imaginary numbers

a) $\sqrt{-25}$ b) $\sqrt{-8}$

9.2 Complex Numbers

Definition 9.2 (Complex Numbers)
If z is a **complex number**, then it can be expressed in the form : $$z = x + iy,$$ where $x, y \in R$ and $i = \sqrt{-1}$.
x : **real part**
y : **imaginary part**
Or frequently represented as :
$Re(z) = x$ and $Im(z) = y$

Example:

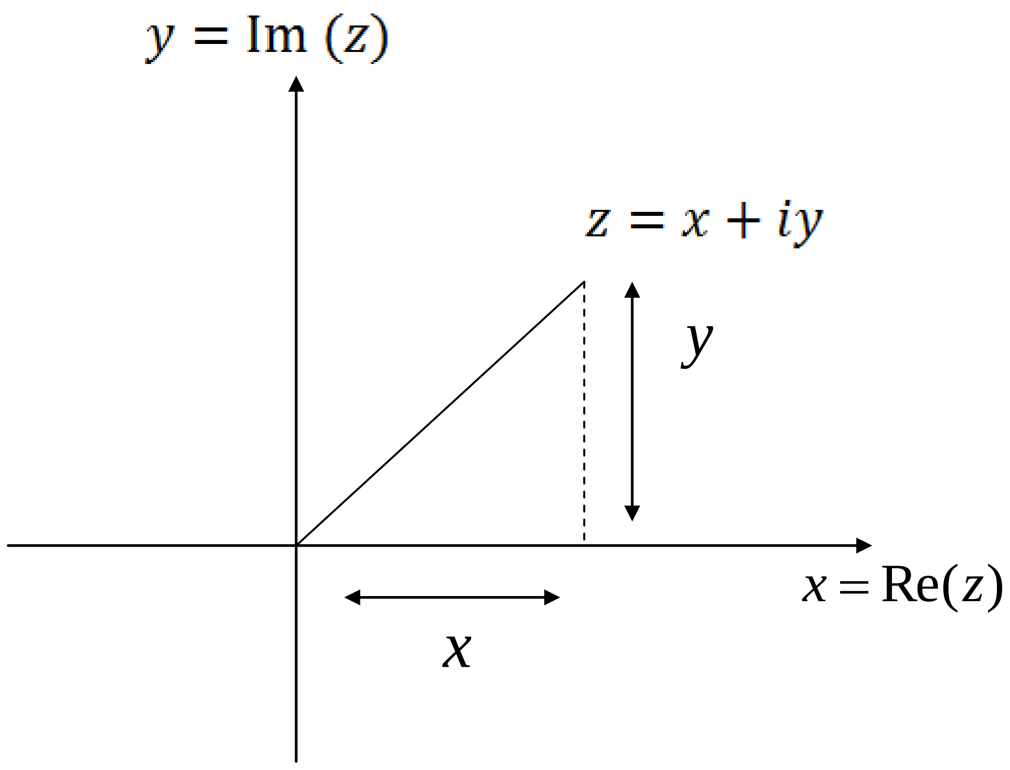
Find the real and imaginary parts of the following complex numbers

(a) $z_1 = 2 + 3i$

(b) $4i^2 + i - 2i^3$

9.2.1 Argand Diagram

We can graph complex numbers using an **Argand Diagram**.



Example:

Sketch the following complex numbers on the same axes.

(a) $z_1 = 3 + 2i$ (b) $z_2 = 3 - 2i$

(c) $z_3 = -3 - 2i$ (d) $z_4 = -3 + 2i$

9.2.2 Equality of Two Complex Numbers

Given that $z_1 = a + bi$ and $z_2 = c + di$

where $z_1, z_2 \in C$.

Two complex numbers are equal iff the real parts and the imaginary parts are respectively equal.

So, if $z_1 = z_2$, then $a = c$ and $b = d$.

Example 1:

Solve for x and y if given $3x + 4i = (2y + x) + xi$.

Example 2:

Solve $(3 + 4i)^2 - 2(x - iy) = x + iy$ for real numbers x and y.

Example 3:

Solve the following equation for x and y where

$$xy - 2i + x + 2xyi - 5 = \frac{3}{2} - 3i$$

9.3 Algebraic Operations on Complex Numbers

9.3.1 Addition and subtraction

If $z_1 = a + bi$ and $z_2 = c + di$ are two complex numbers, then

$$z_1 \pm z_2 = (a + c) \pm (b + d)i$$

Example:

Given $Z_1 = -2 + 2i$, $Z_2 = 1 - \frac{\sqrt{3}}{2}i$ and $Z_3 = 4 - 6i$. Find

a) $Z_1 - Z_2$ b) $Z_1 + Z_3$

9.3.2 Multiplication

If $z_1 = a + bi$ and $z_2 = c + di$ are two complex numbers, and k is a constant, then

(i) $z_1 \cdot z_2 = (a + bi) \cdot (c + di)$

$$= (ac - bd) + (ad + bc)i$$

(ii) $kz_1 = ka + kbi$

Example:

Given $Z_1 = -2 + 2i$, and $Z_2 = 4 - 6i$. Find $Z_1 Z_2$.

9.3.3 *Complex Conjugate*

If $z = a + bi$ then the **conjugate** of z is denoted as $\bar{z} = a - bi$.

Note that $z.\bar{z} = a^2 + b^2$

9.3.4 *Division*

If we are dividing with a complex number, the denominator must be converted to a real number. In order to do that, multiply both the denominator and numerator by complex conjugate of the denominator.

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \times \frac{x_2 - iy_2}{x_2 - iy_2}$$

Example 1:

Given that $z_1 = 1 - 2i$, $z_2 = -3 + 4i$. Find $\frac{z_1}{z_2}$, and express it in $a + bi$ form.

Example 2:

Given $z_1 = 2 + i$ and $z_2 = 3 - 4i$, find $\frac{1}{z_1} + \frac{1}{z_2}$ in the form of $a + ib$.

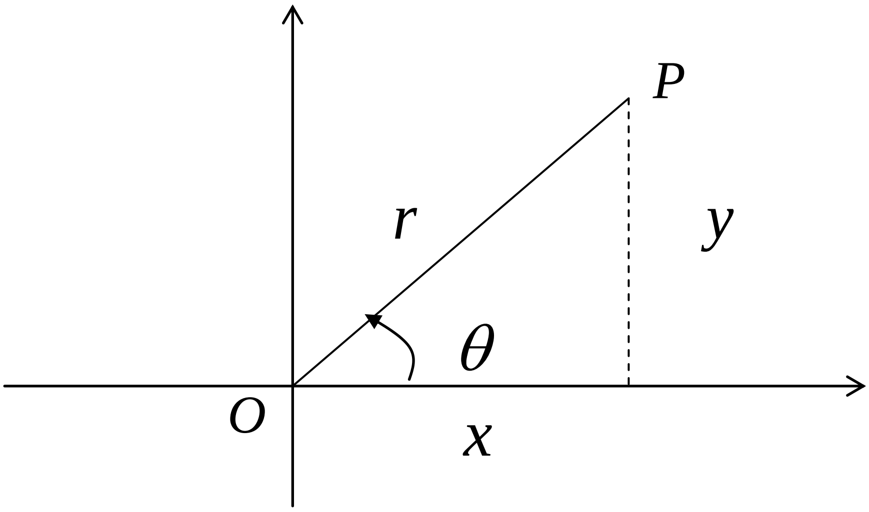
Example 3:

Given $Z = \frac{-2+3i}{3-2}$. Find the complex conjugate, $\bar{Z}$. Write your answer in $a + ib$ form.

Example 4:

Given $Z_1 = -2 + 2i$, and $Z_2 = 4 - 6i$. Find $\frac{2}{\overline{Z_1} + \overline{Z_2}}$.

9.4 Polar Form of Complex Numbers



Modulus of *z*, $|z| = r = \sqrt{x^2 + y^2}$.

Argument of *z*, arg (z) = θ

where $\tan\theta = \dfrac{y}{x}$.

From the diagram above, we can see that

$$x = r\cos\theta \qquad y = r\sin\theta$$

Then, *z* can be written as

$$\begin{aligned} z &= x + iy \\ &= r\cos\theta + ir\sin\theta \\ &= r(\cos\theta + i\sin\theta) \\ &= r\text{cis}\theta \qquad (z \text{ in polar form}) \end{aligned}$$

Example 1:

Express $z = -2 - \sqrt{3}\, i$ in polar form.

Example 2:

Express $\frac{2+3i}{1-i}$ in polar form.

Example 3

Given that $z_1 = 2 + i$ and $z_2 = -2 + 4i$, find z such that

$\frac{1}{z} = \frac{1}{z_1} + \frac{1}{z_2}$. Give your answer in the form of $a + ib$. Hence,

find the modulus and argument of z.

9.5 De Moivre's Theorem

9.5.1 The *n*-th Power Of A Complex Number

Definition 9.5 (De Moivre's Theorem)

If $z = r(\cos\theta + i\sin\theta)$ and $n \in R$, then

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

Example 1:

a) Write z =$1-i$ in the polar form.

Then, using De Moivre's theorem, find z^4.

b) Use D'Moivre's formula to write $(-1 - i)^{12}$ in the form of $a + ib$.

9.5.2 The *n*-th Roots of a Complex Number

A complex number *w* is a *n*-th root of the complex number *z* if

$w^n = z$ or $w = z^{\frac{1}{n}}$. Hence

$$w = z^{\frac{1}{n}}$$

$$= [r(\cos\theta + i\sin\theta)]^{\frac{1}{n}},$$

$$= r^{1/n}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right) \text{ for } k = 0,1,2,\cdots,n-1$$

Substituting $k = 0,1,2,\cdots,n-1$ yields the *n*th roots of the given complex number.

Example 1:

Find all the roots for the following equations:

(a) $z^3 = 27$ (b) $z^4 = \left(\sqrt{3} + i\right)$.

Example 2:

Solve $z^4 + (-1 + i) = 0$ and express them in *a* + *ib* form.

Example 3:

Find all cube roots of $-26 - 8i$.

Example 4:

Solve $z^3 + 8 = 0$. Sketch the roots on the argand diagram.

9.5.3 De Moivre's Theorem to Prove Trigonometric Identities

De-Moivre's theorem can be used to prove some trigonometric identities. (with the help of Binomial theorem or Pascal triangle.)

Example:

Prove that

$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ and

$\sin 5\theta = 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta$.

Solution:

The idea is to write $(\cos\theta + i\sin\theta)^5$ in two different ways.

We use both the Pascal triangle and De Moivre's theorem, and compare the results.

From Pascal triangle,

$$(\cos\theta + i\sin\theta)^5 = \cos^5\theta + i\,5\cos^4\theta\sin\theta - 10\cos\theta\sin^2\theta$$

$$- i\,10\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta.$$

$$= (\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta) +$$

$$i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta).$$

Also, by De Moivre's Theorem, we have

$$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta.$$

and so

$$\cos 5\theta + i\sin 5\theta = (\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta) + i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta).$$

Equating the real parts gives

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta.$$

$$= \cos^5\theta - 10\cos^3\theta(1 - \cos^2\theta) + 5\cos\theta(1 - \cos^2\theta)^2$$

$$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta.$$

$$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta. \text{ (proved)}$$

Equating the imaginary parts gives

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$

$$= 5(1 - \sin^2\theta)^2\sin\theta - 10(1 - \sin^2\theta)\sin^3\theta + \sin^5\theta$$

$$= 5(1 - 2\sin^2\theta + \sin^4\theta)\sin\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$$

$$= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$$

$$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta \text{ (proved).}$$

9.6 Eulers's Formula

Definition 9.6

Euler's formula states that

$$e^{i\theta} = \cos\theta + i\sin\theta$$

It follows that

$$e^{in\theta} = \cos n\theta + i\sin n\theta$$

From the definition, if z is a complex number with modulus r and Arg(z), θ; then

$$z = r(\cos\theta + i\sin\theta)$$

$$= re^{i\theta} \qquad (z \text{ in euler form})$$

Example:

Express the following complex numbers in the form of $re^{i\theta}$

(a) $3 + i$ (b) $2 - 4i$

9.6.1 The *n*-th Power Of A Complex Number

We know that a complex number can be express as $z = re^{i\theta}$, then

$$z^2 = r^2 e^{i2\theta}$$

$$z^3 = r^3 e^{i3\theta}$$

$$z^4 = r^4 e^{i4\theta}$$

$$\vdots$$

$$z^n = r^n e^{in\theta}$$

Example 1:

Given $z = 2 + 2\sqrt{3}\, i$. Find the modulus and argument of z^5.

Example 2:

Find $(\sqrt{3} - i)^{40}$ in the form of $a + ib$.

Example 3:

Express the complex number $z = -1 + \sqrt{3}i$ in the form of $re^{i\theta}$.

Then find

(a) z^2 (b) z^3 (c) z^7

9.6.2 The *n*-th Roots Of A Complex Number

The *n*-th roots of a complex number can be found using the Euler's formula. Note that:

$$z = re^{i(\theta+2k\pi)}$$

Then,

$$z^{\frac{1}{2}} = r^{\frac{1}{2}} e^{\left(\frac{\theta+2k\pi}{2}\right)i}, \qquad k = 0,1$$

$$z^{\frac{1}{3}} = r^{\frac{1}{3}} e^{\left(\frac{\theta+2k\pi}{3}\right)i}, \qquad k = 0,1,2$$

$$\vdots$$

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{\left(\frac{\theta+2k\pi}{n}\right)i}, \qquad k = 0,1,2,\ldots,n-1$$

Example 1:

Find the cube roots of $z = 1+i$.

Example 2:

Given $z = -1 + i$. Find all roots of $z^{\frac{1}{3}}$ in Euler form.

Example 3:

Solve $z^3 + 8i = 0$ and sketch the roots on an Argand diagram.

9.6.3 Relationship between circular and hyperbolic functions.

Euler's formula provides the theoretical link between circular and hyperbolic functions. Since

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{and} \quad e^{-i\theta} = \cos\theta - i\sin\theta$$

we deduce that

$$\boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}} \quad \text{and} \quad \boxed{\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.} \qquad (1)$$

In Chapter 8, we defined the hyperbolic function by

$$\boxed{\cosh x = \frac{e^{x} + e^{-x}}{2}} \quad \text{and} \quad \boxed{\sinh x = \frac{e^{x} - e^{-x}}{2}} \qquad (2)$$

Comparing (1) and (2), we have

$$\cosh ix = \frac{e^{ix} + e^{-ix}}{2} = \cos x$$

$$\sinh ix = \frac{e^{ix} - e^{-ix}}{2i} = i\sin x$$

so that

$$\tanh ix = i\tan x.$$

Also,

$$\begin{aligned}\cos ix &= \frac{e^{i^2x}+e^{-i^2x}}{2} = \frac{e^{-x}+e^{x}}{2} = \cosh x \\ \sin ix &= \frac{e^{i^2x}-e^{-i^2x}}{2i} = \frac{e^{-x}-e^{x}}{2} = \sinh x\end{aligned} \qquad (3)$$

so that

$$\tan ix = i\tanh x\,.$$

Using these results, we can evaluate functions such as sin z, cos z, tan z, sinh z, cosh z and tanh z.

For example, to evaluate

$$\cos z = \cos(x+iy)$$

we use the identity

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

and obtain

$$\cos z = \cos x\cos iy - \sin x\sin iy$$

Using results in (3), this gives

$$\cos z = \cos x\cosh y - i\sin x\sinh y\,.$$

Example: Find the values of

a) $\sinh(3+4i)$

b) $\tan\left(\frac{\pi}{4}-3i\right)$

c) $\sin\left(\frac{\pi}{4}(1+i)\right)$

(Ans: a) -6.548-7.619i   b) 0.005-1.0i   c) 0.9366+0.6142i)

Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 200 252 200 120 45 10 1

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3,$$

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4,$$

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5,$$

$$(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6,$$

$$(x+y)^7 = x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7.$$

In general:

$$(x+y)^n = c_1 x^n y^0 + c_2 x^{n-1} y^1 + \ldots + c_{n+1} x^0 y^n$$