Infinite Sequences

Defn: An **infinite sequence** $\{a_n\}$ is a function whose domain is the set of positive integers.

Example: $\left\{\frac{1}{n}\right\} = 1, \frac{1}{2}, \frac{1}{3}, ...$

Defn: We say the sequence $\{a_n\}$ converges to L if $\lim_{n\to\infty} a_n$ = L.

Special facts about determining if a *sequence* converges:

1) $\lim_{n\to\infty} r^n = 0$ if $|r| < 1$ Ex. $\lim_{n\to\infty}(-2/3)^n = 0$

$\lim_{r\to\infty} r^n$ does not exist if $|r| > 1$ Ex. $\lim_{n\to\infty}(-4)^n$ does not exist

2) For a rational expression (the quotient of two polynomials p and q)

a) if the degree of the numerator is greater than the degree of the denominator, $\lim_{n\to\infty} \frac{p}{q}$ does not exist. Ex. $\lim_{n\to\infty} \frac{1-n^3}{n^2+1}$ does not exist.

b) if the degree of the numerator is the same as the degree of the denominator, the $\lim_{n\to\infty} \frac{p}{q}$ equals the quotient of the coefficients of the highest degree terms.

Ex. $\lim_{n\to\infty} \frac{1-n^3}{2n^3+1} = -\frac{1}{2}$

c) if the degree of the numerator is less than the degree of the denominator, the $\lim_{n\to\infty} \frac{p}{q}$ is equal to 0. Ex. $\lim_{n\to\infty} \frac{1-n^2}{2n^3+1} = 0$

3) If $\lim_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty}(-1)^{n+1} a_n = 0$. If $\lim_{n\to\infty} a_n = L$ where $L \neq 0$, then $\lim_{n\to\infty}(-1)^{n+1} a_n$ does not exist.

Ex. $\lim_{n\to\infty}(-1)^{n+1} \frac{1}{n+1} = 0$; $\lim_{n\to\infty}(-1)^{n+1} \frac{n}{n+1}$ does not exist.

Infinite Series

Defn: Let $\{a_n\}$ be an infinite sequence. Then an expression of the form

$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + ...$ is an **infinite series**.

Each infinite series has a *sequence* $\{S_n\}$ called the *sequence of partial sums* associated with it.

Defn: For the series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + ...$, the **sequence of partial sums** $\{S_n\}$ associated with it is the sequence in which for each n, S_n is the sum of the first n terms of the series; that is, $S_n = a_1 + a_2 + a_3 + ... + a_n$.

Defn: The series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + ...$ **converges and has sum** S if its sequence of partial sums converges to S; that is, if $\lim_{n\to\infty} S_n = S$.

Nth Term Test for Divergence: If $\lim_{n\to\infty} a_n \neq 0$, then the infinite series $\sum_{n=1}^{\infty} a_n$ must diverge. (**Note**: Just because $\lim_{n\to\infty} a_n = 0$ **does not mean** that the series converges.)

Usually we cannot get S_n in a form where we can directly find whether or not $\lim_{n\to\infty} S_n$ exists, so we use certain tests for convergence of an infinite series to determine whether or not a series converges. These tests may tell us that the series converges without telling us the actual sum. That is, they may prove that $\lim_{n\to\infty} S_n$ exists but not tell us what the limit is equal to. Two types of series in which we can directly consider $\lim_{n\to\infty} S_n$ are:

1) telescoping (collapsing) series Ex. $\sum_{n=1}^{\infty} [\frac{1}{n+1} - \frac{1}{n}]$

2) geometric series $\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + ... = a(1 + r + r^2 + r^3 + ...)$

A geometric series converges and has sum $S = \frac{a}{1-r}$ if and only if $|r| < 1$. The geometric series diverges if $|r| \geq 1$.

Be sure to write out the first few terms of the series, and if the first term is not already a 1, factor it out to correctly identify the values of a and r, as we did in class. Notice particularly whether n (or whatever variable is used) begins with 0,1,2, etc.

Defn: A **p-series** is a series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$, where p > 0.

Theorem: We can show by the Integral Test that a p-series converges if $p > 1$ and diverges if $p \le 1$.

Ex.1) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges

Ex. 2) $\sum_{n=1}^{\infty} \frac{1}{n}$ (called the harmonic series) diverges

Ex. 3) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges.

Direct Comparison Test: Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be **positive term** infinite series.

1) If $a_n \le b_n$ for all n and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also must converge.

2) If $b_n \ge a_n$ for all n and $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ must also diverge.

Since we know exactly when p-series and geometric series converge and when they diverge, we are most often comparing to these types of series, usually to p-series. We will most often use the Comparison Test when a_n is a quotient of terms of the form n to a constant power. Ex. $\sum_{n=1}^{\infty} \frac{n + 2n + 1}{n^{3/2} + 2}$

Limit Comparison Test: Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be **positive term** infinite series. Then if $\lim_{n \to \infty} \frac{a_n}{b_n} = k$ for some real number $k > 0$, then either both series converge or both series diverge.

The Limit Comparison Test may be easier to use if you are not sure what inequalities you need for the direct Comparison Test. To use the Limit Comparison Test, let a_n be the nth term of the series in question, and generally we choose b_n by taking the one term from the numerator and the one term from the denominator that gets largest as n gets large. In this way you should know whether $\sum_{n=1}^{\infty} b_n$ converges or diverges and $\lim_{n \to \infty} \frac{a_n}{b_n}$ should equal a positive real number.

Defn: An **alternating series** is a series of the form

$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - ......$ where $a_n > 0$ for all n.

Alternating Series Test: The <u>alternating series</u> $\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + a_5 - \ldots\ldots$

converges if: 1) $\lim_{n\to\infty} a_n = 0$

2) $a_{n+1} < a_n$ for all n

Ex. $\sum_{n=1}^{\infty}(-1)\frac{1}{\sqrt{n}}$ converges by the Alternating Series Test.

<u>Do not</u> try to apply the Alternating Series Test to a positive term series.

We also know the following about the **sum of an alternating series**:
If S_n (the sum of the first n terms) is used to approximate the sum of a convergent alternating series, the error will be less than the absolute value of the $(n+1)^{st}$ term of the series. That is, with a convergent alternating series $|S - S_n| < a_{n+1}$.

The Ratio Test: Let $\sum_{n=1}^{\infty} a_n$ be an infinite series. Then

1) if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, the series converges.

2) if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1 \quad or \quad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$ the series diverges.

3) if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, then no conclusion can be reached by the Ratio Test.

The Ratio Test is used if a_n contains a factor of the form "a constant" to the n power or factorial expressions.

Ex. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n^{10}}$ or $\sum_{n=1}^{\infty}\frac{n}{n!}$

The Ratio Test is also used to find the <u>radius of convergence</u> of a power series.

Defn: A **<u>power series in (x - c)</u>** is a series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \ldots$$

For a power series in x - c, exactly one of the following statements is true:
1) The power series converges only for x = c.
2) The power series converges for all real numbers x.
3) The power series converges on an interval of real numbers centered at c; that is, it converges for x between c - r and c + r for some number r called the radius of convergence.

Doing Problems on Convergence/Divergence

Step 1. Ask yourself: Do I want to determine if a sequence converges or if a series converges? For a sequence, you just want to know if $\lim_{n\to\infty} a_n$ exists, and if so, what is it?
For a series, go on to Step 2.

Step 2: If you are trying to determine if $\sum_{n=1}^{\infty} a_n$ converges, check, if it is relatively easy to do so, $\lim_{n\to\infty} a_n$. If this limit is not 0, by the Nth Term Test for Divergence, the series $\sum_{n=1}^{\infty} a_n$ diverges. If this limit is 0, go on; you need to use another test for convergence or divergence. (If $\lim_{n\to\infty} a_n$ does not seem easy to find, there may be an easier way to determine convergence or divergence.)

Step 3: If the exact sum of the series is asked for at this point in the course it is probably either a collapsing series or a geometric series. In the case of a collapsing series, your write-up should show exactly what S_n equals, then you should find $\lim_{n\to\infty} S_n$, and you should use the definition of convergence of an infinite series. For finding the sum of a geometric series, identify a and r correctly and use the formula $S = \dfrac{a}{1-r}$ if $|r| < 1$. If not one of these kinds of problems, go to:

Step 4. Decide if you have an alternating series or a positive term series. If you have an alternating series, use the Alternating Series Test. If you have a positive term series, go to step ~~4~~. 5

Step 5. If you have a positive term series, use the Direct Comparison Test, the Limit Comparison Test or the Ratio Test. If there is a trig expression or log expression, etc. as a factor of a_n, you may want to make a comparison such as one of the following:
e.g. $\dfrac{\sin^2 n}{n^2} \le \dfrac{1}{n^2}$, $\quad \dfrac{1}{\ln n} > \dfrac{1}{n}$ (since $\ln n < n$), $\quad \dfrac{1}{n \sin n} \ge \dfrac{1}{n}$ since $\sin n \le 1$

A comparison test is usually used if each term in the numerator and denominator is a term of the form n^c where c is a constant. For a kind of series where you would use a Comparison Test, you should be able to tell whether you are pretty sure the series converges or diverges before starting your "proof" by taking the quotient of the "dominant" terms in the numerator and the denominator. If either the numerator or denominator contains factorial expressions or a term of the form c^n, you often want to use the Ratio Test, but you may want to make a comparison first. For example, $\dfrac{1}{n!+n} < \dfrac{1}{n!}$ or $\dfrac{n}{n+2^n} < \dfrac{n}{2^n}$. It is easiest to use the Ratio Test when there is not a sum in either the numerator or the denominator.

Summary

Test	*When to Use*	*Conclusions*
Geometric Series	$\sum_{k=0}^{\infty} ar^k$	Converges to $\frac{a}{1-r}$ if $\lvert r\rvert < 1$; diverges if $\lvert r\rvert \geq 1$.
kth-Term Test	All series	If $\lim_{k\to\infty} a_k \neq 0$, the series diverges.
Integral Test	$\sum_{k=1}^{\infty} a_k$ where $f(k) = a_k$, f is continuous and decreasing and $f(x) \geq 0$	$\sum_{k=1}^{\infty} a_k$ and $\int_1^{\infty} f(x)\,dx$ *both* converge or *both* diverge.
***p*-series**	$\sum_{k=1}^{\infty} \frac{1}{k^p}$	Converges for $p > 1$; diverges for $p \leq 1$.
Comparison Test	$\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$, where $0 \leq a_k \leq b_k$	If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges. If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.
Limit Comparison Test	$\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$, where $a_k, b_k > 0$ and $\lim_{k\to\infty} \frac{a_k}{b_k} = L > 0$	$\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ *both* converge or *both* diverge.
Alternating Series Test	$\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ where $a_k > 0$ for all k	If $\lim_{k\to\infty} a_k = 0$ and $a_{k+1} \leq a_k$ for all k, then the series converges.
Absolute Convergence	Series with some positive and some negative terms (including alternating series)	If $\sum_{k=1}^{\infty} \lvert a_k\rvert$ converges, then $\sum_{k=1}^{\infty} a_k$ converges absolutely.
Ratio Test	Any series (especially those involving exponentials and/or factorials)	For $\lim_{k\to\infty} \left\lvert \frac{a_{k+1}}{a_k} \right\rvert = L$, if $L < 1$, $\sum_{k=1}^{\infty} a_k$ converges absolutely if $L > 1$, $\sum_{k=1}^{\infty} a_k$ diverges, if $L = 1$, no conclusion.
Root Test	Any series (especially those involving exponentials)	For $\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} = L$, if $L < 1$, $\sum_{k=1}^{\infty} a_k$ converges absolutely if $L > 1$, $\sum_{k=1}^{\infty} a_k$ diverges, if $L = 1$, no conclusion.

MACLAURIN SERIES	INTERVAL OF CONVERGENCE
$\dfrac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots$	$-1 < x < 1$
$\dfrac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k} = 1 - x^2 + x^4 - x^6 + \cdots$	$-1 < x < 1$
$e^x = \sum_{k=0}^{\infty} \dfrac{x^k}{k!} = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots$	$-\infty < x < +\infty$
$\sin x = \sum_{k=0}^{\infty} (-1)^k \dfrac{x^{2k+1}}{(2k+1)!} = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$	$-\infty < x < +\infty$
$\cos x = \sum_{k=0}^{\infty} (-1)^k \dfrac{x^{2k}}{(2k)!} = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$	$-\infty < x < +\infty$
$\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} \dfrac{x^k}{k} = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$	$-1 < x \le 1$
$\tan^{-1} x = \sum_{k=0}^{\infty} (-1)^k \dfrac{x^{2k+1}}{2k+1} = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots$	$-1 \le x \le 1$
$\sinh x = \sum_{k=0}^{\infty} \dfrac{x^{2k+1}}{(2k+1)!} = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \dfrac{x^7}{7!} + \cdots$	$-\infty < x < +\infty$
$\cosh x = \sum_{k=0}^{\infty} \dfrac{x^{2k}}{(2k)!} = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \dfrac{x^6}{6!} + \cdots$	$-\infty < x < +\infty$
$(1+x)^m = 1 + \sum_{k=1}^{\infty} \dfrac{m(m-1)\cdots(m-k+1)}{k!} x^k$	$-1 < x < 1^*$ $(m \ne 0, 1, 2, \ldots)$

*The behavior at the endpoints depends on m: For $m > 0$ the series converges absolutely at both endpoints; for $m \le -1$ the series diverges at both endpoints; and for $-1 < m < 0$ the series converges conditionally at $x = 1$ and diverges at $x = -1$.