CHAPTER 4: IMPROPER INTEGRALS

4.0 INTRODUCTION

Up to now, the definite integral $\int_a^b f(x)\,dx$ has two properties:

(i) the domain of integration a to b is finite and

(ii) the range of the integrand in this domain is also finite.

However, if one or both of the conditions are not met, how can we define the resulting integral?

4.1 BASIC CONCEPTS

Definition 4.1 (Improper Integrals)

The definite integral $\int_a^b f(x)\,dx$ is called an **improper integrals** if

(1) $a = -\infty$ or $b = \infty$ or both

i.e one or both limits of integration is infinite (**Type I**).

(2) $f(x)$ is unbounded at one or more points of $a \le x \le b$. $f(x)$ has an infinite discontinuity at

these points and such points are called *singularities* of $f(x)$ **(Type II)**.

For example,

Type 1: $a = -\infty$ or $b = \infty$ or both

$$\int_{1}^{\infty} \frac{1}{1+x^2}\, dx; \quad \int_{-\infty}^{-1} x e^{-x^2}\, dx; \quad \int_{-\infty}^{\infty} x e^{-x^2}\, dx$$

Type II: $f(x)$ is unbounded at one or more points of $a \le x \le b$

$$\int_{0}^{2} \frac{1}{\sqrt{4-x^2}}\, dx; \quad \int_{0}^{2} \frac{1}{x-1}\, dx$$

Type III: Integrals with both conditions (1) and (2).

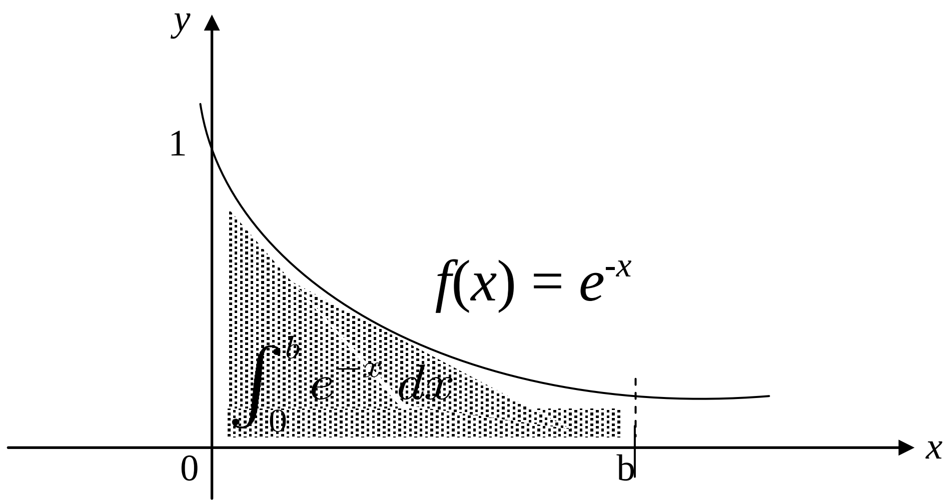
$$\int_{-\infty}^{1} \frac{1}{x^2-1}\, dx$$

4.2 INFINITE LIMITS OF INTEGRATION

To understand how an improper integral may be evaluated, consider the following example:

Figure 4.1 gives the graph of $f(x) = e^{-x}$ in the interval $[0,\infty)$.

Figure 4.1



We can find $\int_0^b e^{-x}\,dx$.

Let $b = T$, then, $\int_0^T e^{-x}dx = -e^{-x}\Big|_0^T = -e^{-T} + 1$

If $T \to \infty$, then $1 - e^{-T} \to 1$ i.e $\lim_{T\to\infty} 1 - e^{-T} = 1$

Thus, we can now define

$$\int_0^\infty e^{-x}\,dx = \lim_{T\to\infty}\int_0^T e^{-x}\,dx = 1$$

Definition 4.2 (Improper Integrals of Type I)

1. If $f(x)$ is continuous in the interval a, ∞ , then

$$\int_a^{\infty} f(x)\,dx = \lim_{T\to\infty} \int_a^{T} f(x)\,dx$$

2. If $f(x)$ is continuous in the interval $-\infty, b$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{T\to-\infty} \int_T^{b} f(x)\,dx$$

3. If $f(x)$ is continuous in the interval $-\infty, +\infty$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx$$

(c is any real number)

The improper integral is said to **converge** if the limit is finite (exist) and to **diverge**, otherwise.

Example 1

Evaluate the following improper integrals. Determine if the integrals converges or diverges.

(a) $\displaystyle\int_1^{+\infty} \frac{1}{x^2}\, dx$

(b) $\displaystyle\int_1^{\infty} \frac{1}{x}\, dx$

(c) $\displaystyle\int_{-\infty}^{1} x e^{-x^2}\, dx$

(d) $\displaystyle\int_0^{\infty} x e^{-2x}\, dx$

(e) $\displaystyle\int_0^{\infty} \sin x\, dx$

(f) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$

(g) $\displaystyle\int_{-\infty}^{\infty} \frac{2x}{1+x^2}\, dx$

(a) $\int_{1}^{+\infty} \frac{1}{x^2}\, dx$

Solution

The region under the curve $y = \frac{1}{x^2}$ for $x \geq 1$ is shown in Figure 4.2:

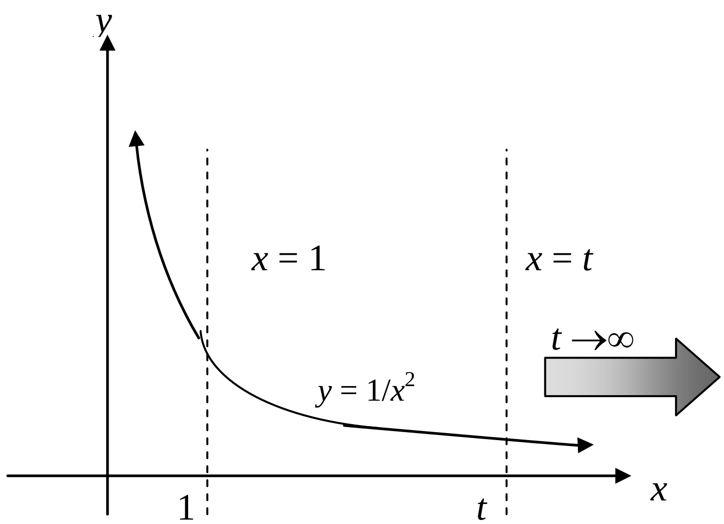


Fig 4.2 $y = \frac{1}{x^2}$

This region is unbounded, will the area be unbounded as well?

Consider,

$$\int_{1}^{T} \frac{1}{x^2}\, dx = -\frac{1}{x}\Big|_{1}^{T} = -\frac{1}{T} + 1$$

However, if $T \to +\infty$, then

$$\int_1^{+\infty} \frac{1}{x^2}\,dx = \lim_{T\to+\infty} \int_1^{T} \frac{1}{x^2}\,dx$$

$$= \lim_{T\to+\infty} \left[-\frac{1}{x}\right]_1^T = \lim_{T\to+\infty} \left[-\frac{1}{T}+1\right] = 1$$

Thus, the improper integral converges and has the value 1.

(b) $\int_1^{\infty} \frac{1}{x}\,dx$

Solution

Figure 4.3 gives the graph of $y = \dfrac{1}{x}$.

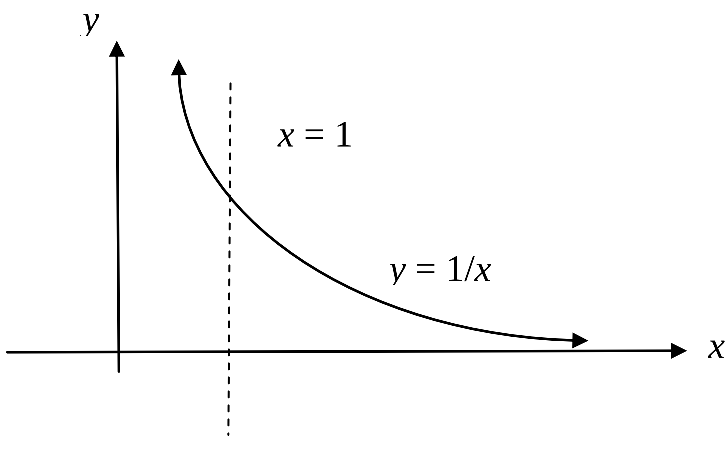


Fig 4.3 $y = \dfrac{1}{x}$

$$\int_{1}^{+\infty} \frac{1}{x}\,dx = \lim_{T\to+\infty} \int_{1}^{T} \frac{1}{x}\,dx$$

$$= \lim_{T\to+\infty} \ln x\,\Big|_{1}^{T} = \lim_{N\to+\infty} \ln T - \ln 1 = +\infty$$

Thus the improper integral diverges. What does this say about the area under the curve?

(e) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

Solution

The graph of $f(x) = \dfrac{1}{1+x^2}$ is given in Figure 4.4.

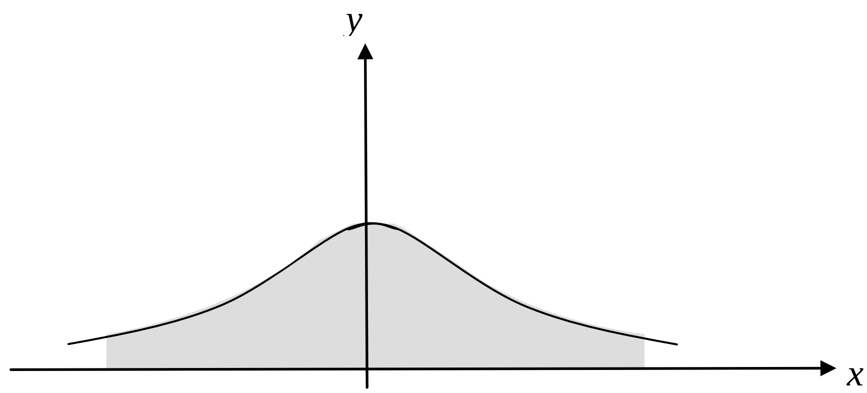


Fig 4.4

The function $f(x) = \dfrac{1}{1+x^2}$ is known as the Cauchy Density Function.

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{a} \frac{1}{1+x^2}\,dx + \int_{a}^{\infty} \frac{1}{1+x^2}\,dx.$$ We have,

$$\int_{-\infty}^{a} \frac{1}{1+x^2}\,dx = \lim_{T\to-\infty} \int_{T}^{a} \frac{1}{1+x^2}\,dx$$

$$= \lim_{T\to-\infty} \left[\tan^{-1} x\right]_{T}^{a} = \lim_{T\to-\infty} \left[\tan^{-1} a - \tan^{-1} T\right]$$

$$= \tan^{-1} a - \ -\frac{\pi}{2} \ = \tan^{-1} a + \ \frac{\pi}{2}$$

and similarly,

$$\int_{a}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{T\to\infty} \int_{a}^{T} \frac{1}{1+x^2}\,dx$$

$$= \lim_{T\to\infty} \left[\tan^{-1} x\right]_{a}^{T} = \frac{\pi}{2} - \tan^{-1} a$$

Therefore, $$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{a} \frac{1}{1+x^2}\,dx + \int_{a}^{\infty} \frac{1}{1+x^2}\,dx$$

$$= \ \tan^{-1} a + \frac{\pi}{2} \ + \ \frac{\pi}{2} - \tan^{-1} a \ = \pi.$$

The improper integral is convergent.

4.3 INFINITE DISCONTINUITY

If the integrand is unbounded at a limit of integration or at some point between the limits of integration then it is said to have an infinite discontinuity or there is a vertical asymptote at these points.

For example, consider $\displaystyle\int_0^1 \frac{1}{(x-1)^{2/3}}\,dx$.

When $x \to 1^-, \dfrac{1}{x-1\ {}^{2/3}} \to +\infty$.

Thus the integrand is infinite at $x = 1$.

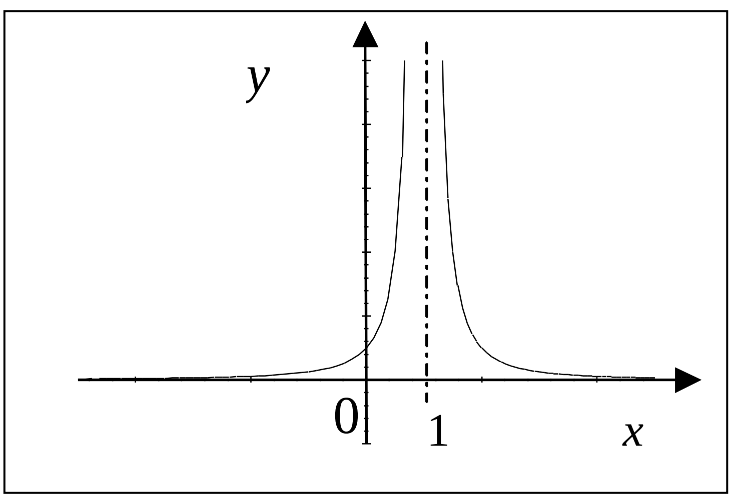


Figure 4.5
$y = \dfrac{1}{(x - 1)^{2/3}}$

Definition 4.3 (Improper Integrals of Type II)

1. If $f(x)$ is continuous in the interval a, b, and has infinite discontinuity at $x = b$, then

$$\int_a^b f(x)\,dx = \lim_{T \to b^-} \int_a^T f(x)\,dx$$

2. If $f(x)$ is continuous in the interval a, b, and has infinite discontinuity at $x = a$, then

$$\int_a^b f(x)\,dx = \lim_{T \to a^+} \int_T^b f(x)\,dx$$

3. If $f(x)$ is continuous in the interval a, b, and has infinite discontinuity at $x = c$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

(c is any real number)

The improper integral is said to **converge** if the limit is finite (exist) and to d**iverge**, otherwise.

Example 2

Evaluate

(a) $\displaystyle\int_0^1 \frac{1}{(x-1)^{2/3}}\,dx$

(b) $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$

(c) $\displaystyle\int_0^4 \frac{1}{4-x}\,dx$

(d) $\displaystyle\int_{-2}^1 \frac{1}{x^2}\,dx$

(d) $\int_{-2}^{1} \frac{1}{x^2}\, dx$

Figure 4.6 gives the graph of
$f(x) = \frac{1}{x^2}, -2 \le x \le 1$

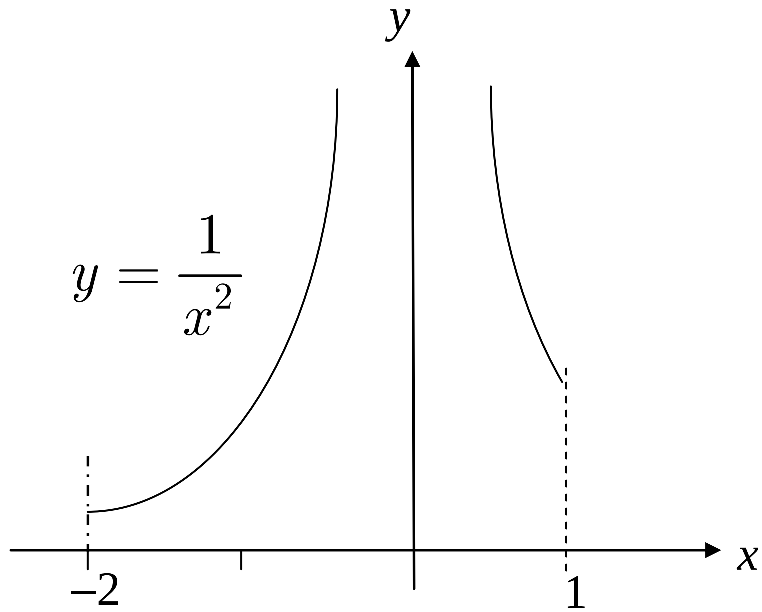


Fig 4.6

From the graph, we can see that as
$x \to 0^+, f(x) \to +\infty$ and as $x \to 0^-, f(x) \to +\infty$, i.e. the curve is unbounded at $x = 0$.

Using definition 4.2:

$$\int_{-2}^{1} \frac{1}{x^2}\, dx = \int_{-2}^{0} \frac{1}{x^2}\, dx + \int_{0}^{1} \frac{1}{x^2}\, dx$$

However, $\int_{-2}^{0} \frac{1}{x^2}\,dx = \lim_{T\to 0^-} \int_{-2}^{T} \frac{1}{x^2}\,dx \ldots$ *Complete the solution*

$$\int_{0}^{1} \frac{1}{x^2}\,dx = \lim_{T\to 0^+} \int_{0}^{1} \frac{1}{x^2}\,dx \ldots$$

Therefore:

$$\int_{-2}^{1} \frac{1}{x^2}\,dx = \ldots$$

Example 3: Improper integral of Type III

Evaluate $\int_0^\infty \frac{x}{\sqrt{1-x^2}}\,dx$.

Solution

$$\int_0^\infty \frac{x}{\sqrt{1-x^2}}\,dx = \int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx + \int_1^\infty \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= \int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx + \int_1^a \frac{x}{\sqrt{1-x^2}}\,dx + \int_a^\infty \frac{x}{\sqrt{1-x^2}}\,dx$$

Using Definitions 4.1 and 4.2:

$$\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx = \lim_{T\to 1^-} \int_0^T \frac{x}{\sqrt{1-x^2}}\,dx \qquad (1)$$

$$\int_0^\infty \frac{x}{\sqrt{1-x^2}}\,dx = \lim_{T\to 1^+} \int_T^a \frac{x}{\sqrt{1-x^2}}\,dx \qquad (2)$$

$$\int_a^\infty \frac{x}{\sqrt{1-x^2}}\,dx = \lim_{T\to \infty} \int_a^T \frac{x}{\sqrt{1-x^2}}\,dx \qquad (3)$$

$$\int \frac{x}{\sqrt{1-x^2}}\, dx = -\sqrt{1-x^2} + C$$

Evaluate the limits for each expressions (1), (2) and (3). If all the limits exist then the integral converges, otherwise, it diverges.

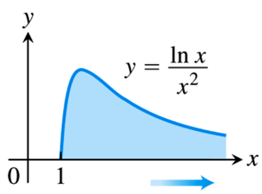
Summary

Types of Improper Integrals Discussed in This Section

Infinite Limits of Integration: Type I

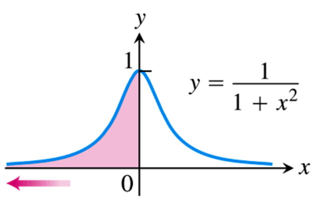
1. Upper limit

$$\int_1^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{\ln x}{x^2}\,dx$$



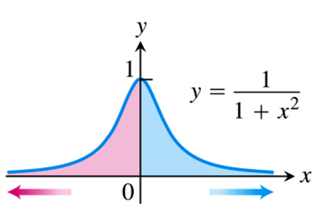
2. Lower limit

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{a\to-\infty} \int_a^{0} \frac{dx}{1+x^2}$$



3. Both limits

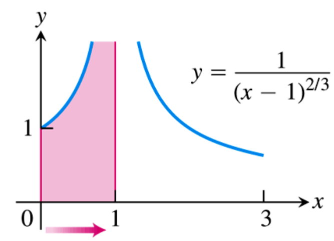
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{b\to-\infty} \int_b^{0} \frac{dx}{1+x^2} + \lim_{c\to\infty} \int_0^{c} \frac{dx}{1+x^2}$$



Integrand Becomes Infinite: Type II

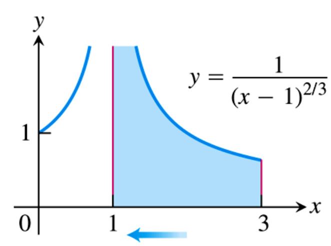
4. Upper endpoint

$$\int_0^{1} \frac{dx}{(x-1)^{2/3}} = \lim_{b\to1^-} \int_0^{b} \frac{dx}{(x-1)^{2/3}}$$



5. Lower endpoint

$$\int_1^{3} \frac{dx}{(x-1)^{2/3}} = \lim_{d\to1^+} \int_d^{3} \frac{dx}{(x-1)^{2/3}}$$



6. Interior point

$$\int_0^{3} \frac{dx}{(x-1)^{2/3}} = \int_0^{1} \frac{dx}{(x-1)^{2/3}} + \int_1^{3} \frac{dx}{(x-1)^{2/3}}$$