Chapter 3: Integration of Further Transcendental Functions

Motivation – why study integration

The processes of integration are used in many applications.

The **Petronas Towers** in Kuala Lumpur experience high forces due to winds. **Integration** was used to design the building for strength.

Often we know the relationship involving **the rate of change** of two variables, but we want the **direct** relationship between the two variables.

- For example, we know the **velocity** of an object at a particular time, but we want to know the **position** of the object at that time.

  To find this direct relationship, we need to use the process which is **opposite** to differentiation. This is called **integration** (or **anti-differentiation**).

- **Integration are used to find:**
  - Area under a curve
    - Areas under curved surfaces
  - Volumes of solids of revolution
    - The volume of a solid obtained by revolving the graph of a function $y = f(x)$ around the x-axis
  - Displacement & velocity
    - The distance travelled by an object moving with a given velocity function over a given time interval
  - Mass, Moments & Centers of mass
  - Work done
    - The work done by a force applied over a certain distance
  - Fluid flow
  - Modelling the behaviour of objects under stress etc.

Area under a curve can be approximated without using calculus. We do it with **calculus** to find **exact area**

- **Anatomy of an Integral**
  - integral sign
  - [a, b] interval of integration
  - a, b limits of integration
  - a lower limit
  - b upper limit
  - $f(x)$ integrand
  - x variable of integration

Recap - Evaluation of Integral

- Standard Integrals
- Techniques of integration
  - u-substitution
  - integration by parts
  - partial fractions
  - trigonometric substitution

3.1 Standard Integrals for Hyperbolic Functions

Recall the derivatives of hyperbolic functions:
By reversing the derivative formulas, we can obtain a list of standard integrals.

Standard integrals involving hyperbolic functions:

Derivatives	**Integrals**
$\frac{d}{dx}(\cosh x) = \sinh x$	$\int \sinh x\, dx = \cosh x + C$
$\frac{d}{dx}(\tanh x) = \text{sech}^2 x$	$\int \text{sech}^2 x\, dx = \tanh x + C$
$\frac{d}{dx}(-\coth x) = \text{cosech}^2 x$	$\int \text{cosech}^2 x\, dx = -\coth x + C$
$\frac{d}{dx}(-\text{sech}\, x) = \text{sech}\, x \tanh x$	$\int \text{sech}\, x \tanh x\, dx = -\text{sech}\, x + C$
$\frac{d}{dx}(-\text{csch}\, x) = \text{csch}\, x \coth x$	$\int \text{csch}\, x \coth x\, dx = -\text{csch}\, x + C$

Example 1

(i) $$\int \sinh 3x\, dx = \frac{1}{3}\cosh 3x + C$$

(ii) $\int e^{-x}\sinh x\ dx$

- Compare this with $\int e^{-x}\sin x\ dx$. How is the integral evaluated?

However, notice that

$$e^{-x}\sinh x = e^{-x}\left(\frac{1}{2}\ e^{x}-e^{-x}\ \right)=\frac{1}{2}\ 1-e^{-2x}$$

Hence,

$$\int e^{-x}\sinh x\,dx=\frac{1}{2}\int\ 1-e^{-2x}\ \ dx=\frac{x}{2}+\frac{1}{4}e^{-2x}+C$$

- This integral is much easier and does not require using integration by parts.

Example 2: Evaluate the following integrals

(a) $\int 3\sinh 2x\ dx$ (b) $\int \sinh x\cosh x\ dx$

(c) $\int \cosh^{2} 2x\ dx$ (d) $\int x^{3}\cosh 2x\ dx$

(e) $\int \sin 3x\sinh 2x\ dx$

3.2 Integration of Inverse Trigonometric Functions

Recall derivatives of inverse trig functions

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}, \quad |x| < 1$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}, \quad |x| < 1$$

From the definition of antiderivatives, we can obtain the list of standard integrals.

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + C$$

$$\int \frac{dx}{1+x^2} = \tan^{-1}x + C$$

$$\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}x + C$$

Integral formulas involving Inverse Trigonometric Functions

Derivatives	Integrals
$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$
$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$	$\int \frac{-dx}{\sqrt{1-x^2}} = \cos^{-1} x + C$
$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$	$\int \frac{dx}{1+x^2} = \tan^{-1} x + C$
$\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$	$\int \frac{-dx}{1+x^2} = \cot^{-1} x + C$
$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\int \frac{dx}{\lvert x\rvert\sqrt{x^2-1}} = \sec^{-1} x + C$
$\frac{d}{dx}(\csc^{-1} x) = \frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$	$\int \frac{-dx}{\lvert x\rvert\sqrt{x^2-1}} = \csc^{-1} x + C$

- When given integral problems, look for these patterns.
- Many integrals look like the inverse trig forms.

Example 3

Which of the following <u>are</u> of the inverse trig forms?

(a) $\int \frac{x\,dx}{1+x^2}$ (b) $\int \frac{x\,dx}{\sqrt{1-x^2}}$

(c) $\int \frac{dx}{1+x^2}$ (d) $\int \frac{dx}{\sqrt{1-x^2}}$

If they are not, how are they integrated?

- What about $\int \frac{dx}{\sqrt{4-x^2}}, \int \frac{dx}{9+x^2},$ $\int \frac{dx}{|x|\sqrt{x^2-10}}$ ...?

To find the answer for this question, lets' try to solve $\int \frac{dx}{\sqrt{a^2-x^2}}$.

Example 4

Evaluate $\int \frac{dx}{\sqrt{a^2-x^2}}$.

Solution: Let $x = au$, then $dx = adu$. Hence

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{du}{\sqrt{1-u^2}}$$

$$= \sin^{-1} u + C$$

$$= \sin^{-1}\left(\frac{x}{a}\right) + C$$

Using the same method, we can find the solution for $\int \frac{dx}{a^2 + x^2}$, $\int \frac{dx}{|x|\sqrt{x^2 - a^2}}$, ...

From the above discussions, we obtain the general integral formulas as follows:

$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$
$\int \frac{-dx}{\sqrt{a^2 - x^2}} = \cos^{-1}\left(\frac{x}{a}\right) + C$
$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

$\int \frac{-dx}{a^2+x^2} = \frac{1}{a}\cot^{-1}\left(\frac{x}{a}\right)+C$
$\int \frac{dx}{\lvert x\rvert\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)+C$
$\int \frac{-dx}{\lvert x\rvert\sqrt{x^2-a^2}} = \frac{1}{a}\csc^{-1}\left(\frac{x}{a}\right)+C$

Example 5 : Evaluate the following integrals

(a) $\int \frac{5}{\sqrt{1-x^2}}\,dx$ (b) $\int \frac{3}{\sqrt{4-x^2}}\,dx$

(c) $\int_0^{1/4} \frac{dx}{\sqrt{1-16x^2}}$ (d) $\int \frac{1}{2+9x^2}\,dx$

(e) $\int \frac{3}{\lvert x\rvert\sqrt{2x^2-8}}\,dx$ (f) $\int \frac{t}{\sqrt{16-t^4}}\,dt$

Likewise we obtain the integral formulas for the inverse hyperbolic functions.

3.3 Standard Integrals Involving Inverse Hyperbolic Functions

Derivatives	Integrals
$\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{x^2+1}}$	$\int \frac{1}{\sqrt{x^2+1}}\, dx = \sinh^{-1} x + C$
$\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2-1}}$	$\int \frac{1}{\sqrt{x^2-1}}\, dx = \cosh^{-1} x + C$
$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$	$\int \frac{1}{1-x^2}\, dx = \tanh^{-1} x + C$

Alternatively,

we can evaluate $\int \frac{1}{\sqrt{x^2+1}}\, dx$ by using the substitution:

$$x = \sinh u \, , \, dx = \cosh u \, du$$

then

$$\sqrt{x^2+1} = \sqrt{1+\sinh^2 u} = \sqrt{\cosh^2 u} = \cosh u$$

Hence

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \int \frac{1}{\cosh u}(\cosh u\,du) = \int du$$

$$= u + C = \sinh^{-1} x + C$$

Recall that inverse hyperbolic functions can be given in logarithmic forms.

Indefinite Integral
$\int \frac{1}{\sqrt{x^2+1}}\,dx = \sinh^{-1} x + C = \ln\left(x+\sqrt{x^2+1}\right) + C$
$\int \frac{1}{\sqrt{x^2-1}}\,dx = \cosh^{-1} x + C = \ln\left(x+\sqrt{x^2-1}\right) + C$
$\int \frac{1}{1-x^2}\,dx = \tanh^{-1} x + C = \frac{1}{2}\ln\left\lvert\frac{1+x}{1-x}\right\rvert + C$

Completing the Square

- Often a good strategy when quadratic functions are involved in the integration

Example 6:

Evaluate $\int \frac{1}{\sqrt{x^2-6x+5}}\,dx.$

Solution

$$x^2-6x+5=(x-3)^2-4, \text{ complete the square}$$

$$\therefore \int \frac{1}{\sqrt{x^2-6x+5}}dx=\int \frac{dx}{\sqrt{(x-3)^2-4}}$$

$$=\int \frac{2dt}{\sqrt{4t^2-4}}, \text{ substitute } x-3=2t$$

$$=\int \frac{dt}{\sqrt{t^2-1}} \quad =\cosh^{-1}t+c$$

$$=\cosh^{-1}\left(\frac{x-3}{2}\right)+C$$

Example 7

Evaluate the integrals:

(a) $\int \frac{1}{x^2+2x+10}dx$ (b) $\int \frac{1}{\sqrt{5+4x-x^2}}dx$

(c) $\int \frac{1}{4x^2-12x+25}dx$ (d) $\int \frac{dx}{\sqrt{3x-x^2}}$

Recognising Integrals

- Some integrals may not appear to fit basic integration formulas:
  - May be possible to split the integrand into two portions, each more easily handled

$$\int \frac{4x+3}{\sqrt{1-x^2}}\,dx = \int \frac{4x}{\sqrt{1-x^2}}\,dx + \int \frac{3}{\sqrt{1-x^2}}\,dx$$

  - Use integration techniques/rules

$$\int \sin^{-1} x\,dx = x\sin^{-1} x - \int x\left(\frac{1}{\sqrt{1-x^2}}\right)dx$$

  - Change of variables

$$\int \frac{\sqrt{x-2}}{x+1}\,dx = \int \frac{u}{u^2+2\;\;+1}\,(2udu)$$, substitute $u=\sqrt{x-2}$

Complete the solution...

Then try evaluating these integrals:

(a) $\displaystyle\int \frac{e^{x}}{1+e^{2x}}\,dx$ (b) $\displaystyle\int \frac{1}{(x-1)(x^{2}+1)}\,dx$

(c) $\displaystyle\int \frac{2x-3}{\sqrt{4x-x^{2}}}\,dx$

If $u = g(x)$, we have

$$\int \frac{du}{\sqrt{a^{2}-u^{2}}} = \arcsin\frac{u}{a} + C$$

$$\int \frac{du}{a^{2}+u^{2}} = \frac{1}{a}\arctan\frac{u}{a} + C$$

$$\int \frac{du}{u\sqrt{u^{2}-a^{2}}} = \frac{1}{a}\operatorname{arcsec}\frac{u}{a} + C$$

Example 8

Evaluate the following integrals:

(a) $\int \frac{1}{\sqrt{x^2-16}}\,dx$ (b) $\int \frac{1}{25-4x^2}\,dx$

(c) $\int \frac{1}{x\sqrt{9-4x^2}}\,dx$

Example 9

(a) $\int \sqrt{1+x^2}\,dx$ (b) $\int \frac{\sqrt{1+x^2}}{x^2}\,dx$

(c) $\int_1^2 \cosh^{-1} x\,dx$

3.4 Applications of Integration

- Area under a curve
  - Areas under curved surfaces
- Volumes of solids of revolution
- Mass
- Work done
- Moments & Centers of mass
- Displacement & velocity
- fluid flow
- modelling the behaviour of objects under stress

etc.

Example 10

Find the area of the region bounded by

$$y = \frac{1}{x^2 - 2x + 5}, y = 0, x = 1, \text{ and } x = 3.$$

Example 11

The curve formed by a freely hanging chain supported at two points has the form of a hyperbolic cosine, and is called the catenary.

(a) Calculate the length of the arc of a catenary given by the equation

$$y = 2\cosh\left(\frac{x}{2}\right) - 1$$

between $x = -4$ and $x = 4$.

(b) Find the area of the surface of revolution, when the catenary given above is rotated about the x-axis. Use the same x limits. (This surface is called the catenoid.)