2.4 Differentiation of Parametric Functions

If a curve can be represented as parametric functions, $y = f(t)$ and $x = g(t)$ with both x and y differentiable with respect to t, then

$\frac{dy}{dx}$ can be found in terms of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ which are related by the chain rule

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{1}{dx/dt}$$

provided $\frac{dx}{dt} \neq 0$

Example 2.9

Given parametric functions:

$$y = \cos 2t, \; x = \sin t.$$

Find $\frac{dy}{dx}$.

Ans: $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{dx/dt} = (-2\sin 2t) \cdot \left(\frac{1}{\cos t}\right) = -4\sin t$

Example 2.10

Find $\dfrac{dy}{dx}$ from the parametric functions,

$x = t^2$, $y = t^3$.

Ans.: $\dfrac{dy}{dx} = 3t^2 \; . \; \dfrac{1}{2t} = \dfrac{3}{2}t$

Example 2.11

Find $\dfrac{dy}{dx}$ if $y = e^t \sin t$ and $x = \ln(t^2)$.

Ans: $\dfrac{dy}{dx} = \dfrac{dy}{dt} . \dfrac{1}{dx/dt} = \dfrac{te^t}{2}(\sin t + \text{kos } t)$

Example 2.12

a) The parametric functions of a curve are given by

$y = t^3 - 12t$, $x = 7t + 2$. Find $\frac{dy}{dx}$ if t = 3.

b) If $x = \cosh^{-1} t$, $y = \sinh^{-1} t$, find $\frac{dy}{dx}$.

2.5 Higher order derivatives

Let $y = 2x^4 - 5x^3 + 3x^2 - 2x + 4$. If *y* is differentiable wrt *x*, then

$$\frac{dy}{dx} = 8x^3 - 15x^2 + 6x - 2$$

Note:

- $\frac{dy}{dx}$ is also a polynomial of *x*. Thus, it is also a differentiable function of *x*. We can find its derivative by:
  $$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(8x^3 - 15x^2 + 6x - 2) = 24x^2 - 30x + 6$$
- This derivative $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is called the second (second order) derivative wrt *x* and is written as $\frac{d^2y}{dx^2}$.

If the $\frac{d^2y}{dx^2}$ is a differentiable function of *x* then we can obtain the third order derivative provided the derivative exist:

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = 48x - 30$$

Other notation:

$$f'(x) = \frac{d}{dx}\left[f(x) \right]$$

$$f''(x) = \frac{d}{dx}\left[\frac{d}{dx}\left[f(x) \right] \right] = \frac{d^2}{dx^2}\left[f(x) \right]$$

$$f''(x) = \frac{d}{dx}\left[\frac{d^2}{dx^2}\left[f(x) \right] \right] = \frac{d^3}{dx^3}\left[f(x) \right]$$

The differentiation can continue as long as the derivative exists.

Thus if $y = f(x)$ then the first n derivatives are given by: $\frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \frac{d^3y}{dx^3}, \quad \frac{d^4y}{dx^4}, \ldots, \quad \frac{d^ny}{dx^n}$

or

$$y', \quad y'', \quad y''', \quad y^{(4)}, \ \ldots, \ y^{(n)}$$

or

$$f'(x), \ f''(x), \ f'''(x), \ f^{(4)}(x), \ldots, \ f^{(n)}(x)$$

To summarise, if n is a positive integer then

$f^{(n)}(x) = \frac{d^n}{dx^n}[f(x)]$ is the n^{th} derivative of f, which is obtained by differentiating the function, n times consecutively. n is called the order of the derivative.

Example 2.13

If $y = 2x^3 - 11x^2 + 12x - 5$, find the first four derivatives of y.

Example 2.14

If $y = x^5 - 3x^3 - 2x + 1$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$. Hence, evaluate $\dfrac{d^2y}{dx^2}$ at $x = 2$.

Example 2.15: Higher order derivatives and parametric functions

If $x = 7t + 2, \ \ y = t^3 - 12t$, find $\dfrac{d^2y}{dx^2}$.

Solution:

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{1}{dx/dt} = \frac{3t^2 - 12}{7}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{3t^2 - 12}{7}\right)$$

$$= \frac{d}{dt}\left(\frac{3t^2 - 12}{7}\right) \cdot \frac{dt}{dx} = \frac{3}{7}(2t) \cdot \frac{1}{7}$$

$$= \frac{6t}{49}$$

Example 2.16

A parametrized curve is given by the equations $x = 3\cosh\theta, \ \ y = 4\sinh\theta$. Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

Ans: $\frac{4}{3}\coth\theta; -\frac{4}{9}\operatorname{cosech}^3\theta$

Example 2.17: Higher order derivatives and implicit differentiation

(a) $y^2 = x^2 + 2x$, find $\dfrac{d^2y}{dx^2}$.

(b) If $xy + y^2 = 1$, find the value of $\dfrac{d^2y}{dx^2}$ at the point $(0,-1)$.

(c) Use implicit differentiation to find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ for $2\sqrt{y} = x - y$.

Solution:

(a) Given $y^2 = x^2 + 2x$

To find $\dfrac{dy}{dx}$:

$$\frac{d}{dx}y^2 = \frac{d}{dx}\left(x^2 + 2x\right)$$

$$2y\frac{dy}{dx} = 2x + 2$$

$$\therefore \frac{dy}{dx} = \frac{x+1}{y}$$

To find $\frac{d^2 y}{dx^2}$:

$$2y\frac{dy}{dx} = 2x + 2$$

$$\frac{d}{dx}\left(y\frac{dy}{dx}\right) = \frac{d}{dx}(x+1)$$

$$y\frac{d^2 y}{dx^2} + \frac{dy}{dx} = 1$$

$$\frac{d^2 y}{dx^2} = \frac{1-\frac{dy}{dx}}{y} = \frac{1-\frac{x+1}{y}}{y} = \frac{y-x-1}{y^2}$$

Example 2.18

You are observing a rocket launch from a point 4000 feet from the launch pad. When the observation angle is $\pi/3$, the angle is increasing at $\pi/12$ feet per second. How fast is the rocket travelling?